Domination Game Played on a Graph
Constructed from 1-Sum of Paths

Chutchawon Weeranukujit\(^{†}\) and Chanun Lewchalermvongs\(^{†, \dagger}\)

\(^{†}\)Department of Mathematics, Faculty of Science, and
Centre of Excellence in Mathematics,
Mahidol University, Bangkok 10400, Thailand

\(^{\dagger}\)Corresponding author

Abstract : The domination game consists of two players, Dominator and Staller, who construct a dominating set in a given graph \(G\) by alternately choosing a vertex from \(G\), with the restriction that in each turn at least one new vertex must be dominated. Dominator wants to minimize the size of the dominating set, while Staller wants to maximize it. In the game, both play optimally. The game domination number \(\gamma_\text{g}(G)\) is the number of vertices chosen in the game which Dominator starts, and \(\gamma'_\text{g}(G)\) is the number of vertices chosen in the game which Staller starts. In this paper these two numbers are analyzed when the game is played on a graph constructed from paths on \(n\) vertices, \(P_n\), and on two vertices, \(P_2\), by gluing them together at a vertex. This type of operation is called 1-sum. The motivation behind our research is to study the game domination number of a tree that can be constructed from 1-sum of paths.

Keywords : domination game; dominating set; game domination number; 1-sum; paths.

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1 Introduction

In this paper, the domination game is played on a finite simple graph $G$. The domination game was first introduced by Brešar, Klavžar and Rall in 2010 [1]. It is basically different from the domination number of a graph $G$ (the minimum size of its dominating set), $\gamma(G)$, although $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$, see [1]. The game domination numbers, $\gamma_g$ and $\gamma'_g$, of some simple graphs such as paths and cycles are determined in [2, 3]. For a tree $T$, a connected graph with no cycles, the problem of determining its game domination numbers are non-trivial and the lower bound of $\gamma_g(T)$ is given in terms of the number of vertices and maximum degree of $T$ [4]. To explain the relationship between $\gamma_g(G)$ and $\gamma'_g(G)$ of a graph $G$, they use imagination strategy, which compares the moves in a real game with an imaginary game both played on $G$. It is showed in [7] that these two numbers can differ only by 1, $|\gamma_g(G) - \gamma'_g(G)| \leq 1$. We call a pair $(k, l)$ is realized by $G$ if $\gamma_g(G) = k$ and $\gamma'_g(G) = l$. Some possible realizable pairs are studied in [1, 4]. All possible realizable pairs are given in [5]. For example, for every $k$, $(k, k + 1)$ can be realized by a tree [4], and for all $k \geq 2$, $(2k, 2k - 1)$ can be realized by a class of 2-connected graphs[6]. One way to study the game domination numbers of a graph is by considering graph operations such as deletion of a vertex or of an edge. As proved in [6], for a graph $G$ and an edge $e$ in $G$, the game domination numbers of $G$ and $G$ deleted $e$ can differ only by 2, $|\gamma_g(G) - \gamma_g(G - e)| \leq 2$ and $|\gamma'_g(G) - \gamma'_g(G - e)| \leq 2$. The same result holds for deleting a vertex in $G$.

We can think of a tree as joining paths together at vertices. The operation of combining two graphs by identifying a vertex of one graph with a vertex of another is called the 1-sum. Then a tree can be constructed from 1-sum of paths. In our paper, we consider the game domination numbers of a tree constructed from 1-sum of a path on $n$ vertices, $P_n$, and a path on two vertices, $P_2$. To state our main result we need to define a few graphs. Let $x_1, x_2, ..., x_n$ be vertices of $P_n$, and let $v_1, v_2$ be vertices of $P_2$. We define a graph $Q_{n+1}$, $n \geq 4$, to be a 1-sum of $P_{n \geq 4}$ and $P_2$ at $x_2$ and $v_1$, see Figure 1.

![Figure 1: Graph $Q_{n+1}$](image)

In a graph $G$, vertices $u$ and $v$ in $G$ are neighbours if $uv$ is an edge in $G$. Let $N[u]$ be the set consisting of $u$ and all its neighbours. Note that a vertex in a graph is called dominated if it is chosen or it is a neighbour of the vertex chosen. Let $S$ be a subset of the vertex set of $G$, $V(G)$. Then a partially dominated graph $G|S$ is a subgraph of $G$ where the vertices of $S$ are already dominated. So these vertices do not need to be dominated in the course of the game. The residual graph corresponding to $G|S$ is a graph obtained from $G$ by deleting all edges between
dominated vertices and all vertices $u$ that cannot be chosen any more, $N[u] \subseteq S$. Our main results are as follows.

**Theorem 1.1.** $\gamma(Q_{n+1}) \leq 1 + \gamma'_g(Q_{n+1}|N[x_2]) < 1 + \gamma'_g(Q_{n+1}|N[x_3])$.

**Theorem 1.2.** $\gamma'_g(Q_{n+1}) \geq 1 + \gamma_g(Q_{n+1}|N[x_3])$.

**Theorem 1.3.** In a Staller-start game played on $Q_{n+1}$, for $n \equiv 3 \pmod{4}$, if the Staller first move is $x_2$, then Dominator cannot choose $x_4$.

For the rest of the paper, we start with introducing our tools used in our proofs. Then we analyze domination games played on $Q_{n+1}$. Finally, we consider a Dominator-start game played on 1-sum of $P_n$ and $P_2$.

## 2 Basic Lemmas

In this section, we introduce our main tools, which are the continuation principal, properties of realization, and formulas involving the game domination numbers of a path $P_n$.

**Theorem 2.1 (Continuation Principle).** Let $G$ be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.

The next theorem shows the relation between the game domination numbers.

**Theorem 2.2.** For any graph $G$, $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.

Suppose that $\gamma_g(G) = k$ and $\gamma'_g(G) = m$. Theorem 2.1 implies that the realization of $G$ is $(k,k), (k,k+1), (k,k-1)$, where $m = \{k-1, k, k+1\}$. We call equal for the case $(k,k)$, plus for the case $(k,k+1)$, and minus for the case $(k,k-1)$. If $G$ is a family of forests, then the realization is $(k,k)$ or $(k,k+1)$.

**Theorem 2.3.** Forests are no-minus graphs.

If the disjoint union of no-minus graphs has at least one equal graph (component), then the following holds.

**Theorem 2.4.** Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs. If $G_1|S_1$ realizes $(k,k)$ and $G_2|S_2$ realizes $(l,m)$ (where $m \in l,l+1$), then the disjoint union $(G_1 \cup G_2)|(S_1 \cup S_2)$ realizes $(k+l,k+m)$.

In the case that both components of a no-minus graph are plus, the following statement holds.

**Theorem 2.5.** Let $G_1|S_1$ and $G_2|S_2$ be partially dominated no-minus graphs such that $G_1|S_1$ realizes $(k,k+1)$ and $G_2|S_2$ realizes $(l,l+1)$. Then

\[ k + l \leq \gamma_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k + l + 1, \]

\[ k + l + 1 \leq \gamma'_g((G_1 \cup G_2)|(S_1 \cup S_2)) \leq k + l + 2. \]
Let $P''_n$ denote the partially dominated path of order $n+2$, which its endpoints are dominated, see Figure 2 and let $P'_n$ denote the partially dominated path of order $n+1$, which only one of its endpoint is dominated, see Figure 2. In both cases, $n$ vertices are not dominated. The following is an important lemma involving the proof of the game domination numbers of a path.

![Figure 2: Partially dominated paths of $P''_n$ (left) and $P'_n$ (right)](image)

**Lemma 2.6.** For every $n \geq 0$, we have

$$
\gamma_g(P''_n) = \begin{cases} 
\left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod{4}, \\
\left\lceil \frac{n}{2} \right\rceil; & \text{otherwise}, 
\end{cases}
$$

$$
\gamma'_g(P''_n) = \begin{cases} 
\left\lceil \frac{n}{2} \right\rceil + 1; & n \equiv 2 \pmod{4}, \\
\left\lceil \frac{n}{2} \right\rceil; & \text{otherwise}.
\end{cases}
$$

Moreover, for every $i, j \geq 0$ such that $i + j = n$, $i_r = (i \pmod{4})$ and $j_r = (j \pmod{4})$, we also have

$$
\gamma_g(P'_i \cup P'_j) = \begin{cases} 
\gamma_g(P'_i) + \gamma_g(P'_j); & (i_r, j_r) \in \{0, 1\} \times \{0, 1, 2, 3\} \cup \\
\{0, 1, 2, 3\} \times \{0, 1\}, \\
\gamma_g(P'_i) + \gamma_g(P'_j) + 1; & (i_r, j_r) \in \{2, 3\} \times \{0, 1\}, 
\end{cases}
$$

$$
\gamma'_g(P'_i \cup P'_j) = \begin{cases} 
\gamma_g(P'_i) + \gamma_g(P'_j); & (i_r, j_r) \in \{0, 1\} \times \{0, 1\}, \\
\gamma_g(P'_i) + \gamma_g(P'_j) + 1; & (i_r, j_r) \in \{0, 1\} \times \{2, 3\} \cup \\
\{2, 3\} \times \{0, 1\} \cup \{(2, 2)\}, \\
\gamma_g(P'_i) + \gamma_g(P'_j) + 2; & (i_r, j_r) \in \{(2, 3), (3, 2), (3, 3)\}. 
\end{cases}
$$

This lemma shows the optimal first move of both players playing on a partially dominated graph $P''_n$. Dominator always chooses a vertex distance two from the dominated endpoint, but Staller always choose dominated endpoint. Hence, both players play the same way in $P''_n$. The following statement holds.

**Lemma 2.7.** For every $n, m \geq 0$, we have

$$
\gamma_g(P'_n \cup P'_m) = \gamma_g(P''_n \cup P''_m) = \gamma_g(P''_n \cup P''_m) \text{ and}
$$

$$
\gamma'_g(P'_n \cup P'_m) = \gamma'_g(P''_n \cup P''_m) = \gamma'_g(P''_n \cup P''_m).
$$

We can apply Lemmas 2.6 and 2.7 to determine the game domination number of paths.
Theorem 2.8. [3] For every \( n \geq 0 \), we have
\[
\gamma_g(P_n) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 1; & n \equiv 3 \pmod{4}, \\
\left\lfloor \frac{n}{2} \right\rfloor; & \text{otherwise},
\end{cases}
\]
\[
\gamma'_g(P_n) = \left\lfloor \frac{n}{2} \right\rfloor.
\]

3 A Dominator-Start Game Played on \( Q_{n+1} \)

In this section, we analyze \( \gamma_g(Q_{n+1}) \). First, we study the game when the Dominator first move is vertex \( x_3 \).

Lemma 3.1. Suppose the Dominator first move is \( x_3 \). Then
\[
\gamma'_g(Q_{n+1}|N[x_3]) \geq \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor - 1; & n \equiv 3 \pmod{4}, \\
\left\lfloor \frac{n}{2} \right\rfloor; & \text{otherwise}.
\end{cases}
\]

Proof. After the Dominator first move at \( x_3 \), the residual graph is a disjoint union between graph \( P_{x_1x_2v_2} \) and \( P'_{n-4} \), where \( P_{x_1x_2v_2} \) is a path in \( P_n \) with the vertex set \( \{x_1, x_2, v_2\} \). Notice that
\[
\gamma'_g(Q_{n+1}|N[x_3]) = \gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4}).
\]
We calculate the game domination number directly, and obtain that
\[
\gamma_g(P_{x_1x_2v_2}) = 1 \quad \text{and} \quad \gamma'_g(P_{x_1x_2v_2}) = 2.
\]
So \( P_{x_1x_2v_2} \) is a plus graph. We now consider \( \gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4}) \).

If \( P'_{n-4} \) is a plus graph where \( n - 4 \equiv 2, 3 \pmod{4} \), then the residual graph is a disjoint union between plus graphs \( P_{x_1x_2v_2} \) and \( P'_{n-4} \). By Theorem 2.8, we have
\[
\gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4}) \geq \gamma_g(P_{x_1x_2v_2}) + \gamma'_g(P'_{n-4}) + 1 \\
\geq 2 + \gamma_g(P'_{n-4}).
\]

If \( P'_{n-4} \) is an equal graph where \( n - 4 \equiv 0, 1 \pmod{4} \), then the residual graph is a disjoint union between plus and equal graphs. By Theorem 2.4, we have
\[
\gamma'_g(P_{x_1x_2v_2} \cup P'_{n-4}) = \gamma'_g(P_{x_1x_2v_2}) + \gamma'_g(P'_{n-4}) \\
= \gamma_g(P_{x_1x_2v_2}) + 1 + \gamma_g(P'_{n-4}) \\
= 2 + \gamma_g(P'_{n-4}).
\]

We can easily check by hand for the case \( n = 4 \). Suppose that \( n \geq 5 \), we consider four cases according to the value of \( n \pmod{4} \). We apply Lemmas 2.7 and 2.7 to obtain the solution for all \( k \geq 1 \) as follows.
\[
\gamma'_g(P_{x_1x_2v_2} \cup P'_{4(k-1)}) = 2 + \gamma_g(P'_{4(k-1)}) \\
= 2 + \gamma_g(P'_{4(k-1)}) \\
= 2 + 2k - 2 = 2k,
\]
Notice that $\gamma_g(Q_{n+1}) = 1 + \min_{x \in Q_{n+1}} \{\gamma'_g(Q_{n+1}|N[x])\}$. We obtain this equality when $x$ is the Dominator first move in an optimal strategy. Since it does not guarantee that the Dominator first move at $x_3$ is an optimal strategy, we obtain the following corollary.

**Corollary 3.1.** $\gamma(Q_{n+1}) \leq 1 + \gamma'_g(Q_{n+1}|N[x_3])$.

Next, we consider the game domination number on graph $Q_{n+1}$ after the Dominator first move choosing vertex $x_2$.

**Lemma 3.2.** If the Dominator first move is $x_2$, then

$$\gamma'_g(Q_{n+1}|N[x_2]) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 2; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil - 1; & \text{otherwise}. \end{cases}$$

**Proof.** Suppose that the Dominator first move is $x_2$. Then vertices $x_1, x_2, x_3, v_2$ are all dominated, and the residual graph is $P'_{n-3}$. We consider four cases according to the value of $n \mod 4$. Let $k \geq 1$. By Lemmas 2.6 and 2.7, we obtain that

$$\gamma'_g(P'_{4(k-1)+1}) = \gamma'_g(P''_{4(k-1)+1})$$

$$= \gamma_g(P''_{4(k-1)+1})$$

$$= 2(k-1) + 1 = 2k - 1,$$

$$\gamma'_g(P'_{4(k-1)+2}) = \gamma'_g(P''_{4(k-1)+2})$$

$$= \gamma_g(P''_{4(k-1)+2}) + 1 = 2k,$$

$$\gamma'_g(P'_{4(k-1)+3}) = \gamma'_g(P''_{4(k-1)+3})$$

$$= \gamma_g(P''_{4(k-1)+3}) + 1 = 2k,$$
Proof of Theorem 4.1. We compare $\gamma'(Q_{n+1} \mid N[x_2])$ and $\gamma'(Q_{n+1} \mid N[x_3])$. Since $\gamma'(Q_{n+1} \mid N[x_2]) < \gamma'(Q_{n+1} \mid N[x_3])$, we obtain the result.

We now consider some vertices which are not the Dominator first move in an optimal strategy.

Lemma 3.3. In an optimal strategy of the Dominator-start game played on $Q_{n+1}$, the Dominator first move cannot be $x_1, x_3, x_n$ and $v_2$.

Proof. We know that $N[x_1]$ and $N[v_2]$ are subsets of $N[x_2]$, and $\{x_n\}$ is a subset of $N[x_{n-1}]$. By the continuation principle and Theorem 4.1, the result follows.

From our analysis, we propose the following conjecture. In an optimal strategy of the Dominator-start game played on $Q_{n+1}$, the Dominator first move is $x_2$. Then

$$\gamma_g(Q_{n+1}) = 1 + \gamma'(Q_{n+1} \mid N[x_2])$$

$$= \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise}. \end{cases}$$

4 A Staller-Start Game Played on $Q_{n+1}$

In this part, we consider the Staller-start game domination number on graph $Q_{n+1}$.

Lemma 4.1. If the Staller first move is $x_3$, then

$$\gamma_g(Q_{n+1} \mid N[x_3]) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 1; & n \equiv 0, 1 \pmod{4}, \\ \left\lceil \frac{n}{2} \right\rceil - 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil; & n \equiv 2 \pmod{4}, \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ or } \left\lceil \frac{n}{2} \right\rceil - 2; & n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Suppose that the Staller first move is $x_3$. Then the residual graph is a disjoint union between $P_{x_1x_2v_2}$ and $P'_{n-4 \geq 0}$ where $P_{x_1x_2v_2}$ is a path in $P_n$ with the vertex set $\{x_1, x_2, v_2\}$. Notice that $\gamma_g(Q_{n+1} \mid N[x_3]) = \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4})$. We can find the game domination number directly from the graph: $\gamma_g(P_{x_1x_2v_2}) = 1$ and $\gamma_g(P_{x_1x_2v_2}) = 2$. So $P_{x_1x_2v_2}$ is a plus graph. Next we find $\gamma_g(P_{x_1x_2v_2} \cup P'_{n-4})$. If $P'_{n-4}$ is a plus graph, where $n - 4 \equiv 2, 3 \pmod{4}$, then the residual graph is a disjoint union of plus graphs $P_{x_1x_2v_2}$ and $P'_{n-4}$. By Theorem 4.1, we have that

$$\gamma_g(P_{x_1x_2v_2}) = 1 + \gamma_g(P'_{n-4}) \leq \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) \leq 1 + \gamma_g(P_{x_1x_2v_2} \cup P'_{n-4}) + 1 \leq 2 + \gamma_g(P'_{n-4}).$$
If $P'_{n-4}$ is an equal graph, where $n - 4 \equiv 0, 1 \pmod{4}$, then the residual graph is a disjoint union between plus graph and equal graph. By Theorem 1.3, we have that

$$
\gamma_g(P_{x_1}x_{2v_2} \cup P'_{n-4}) = \gamma_g(P_{x_1}x_{2v_2}) + \gamma_g(P'_{n-4}) = 1 + \gamma_g(P'_{n-4}).
$$

It can be easily checked for $n = 4$. Assume that $n \geq 5$. There are four cases according to the value of $n \mod 4$. Then we apply Lemmas 2.6 and 2.7 to obtain the solution for all $k \geq 1$.

$$
\gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)}) = 1 + \gamma_g(P'_{4(k-1)}) = 1 + 2k - 2 = 2k - 1,
$$

$$
\gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+1}) = 1 + \gamma_g(P'_{4(k-1)+1}) = 1 + 2k - 2 + 1 = 2k,
$$

$$
1 + \gamma_g(P'_{4(k-1)+2}) \leq \gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+2}) \leq 2 + \gamma_g(P'_{4(k-1)+2})
$$

$$
1 + 2k - 2 + 1 \leq \gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+2}) \leq 2 + 2k - 2 + 1
$$

$$
2k \leq \gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+2}) \leq 2k + 1.
$$

$$
\gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+3}) \leq \gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+3}) \leq 2 + \gamma_g(P'_{4(k-1)+3})
$$

$$
1 + 2k - 2 + 1 \leq \gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+3}) \leq 2 + 2k - 2 + 1
$$

$$
2k \leq \gamma_g(P_{x_1}x_{2v_2} \cup P'_{4(k-1)+3}) \leq 2k + 1.
$$

Proof of Theorem 1.2. We know that

$$
\gamma_g(Q_{n+1}) = 1 + \max_{x \in V(Q_{n+1})} \{\gamma'_g(Q_{n+1}|N[x])\}.
$$

We can obtain this equality when $x$ is the Staller first move in an optimal strategy. Since it does not guarantee that the Staller first move at $x_3$ is an optimal strategy, we obtain the result.

We next consider the Staller-start game domination number when Staller chooses $v_2$ and Dominator chooses $x_2$. 


Lemma 4.2. If the Staller first move is $v_2$ and the next move by Dominator is $x_2$, then

$$\gamma'_g(Q_{n+1}|N[v_2,x_2]) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil - 2; & n \equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{4} \right\rceil - 1; & \text{otherwise}. \end{cases}$$

Proof. Suppose that the first move of Staller is $v_2$, then $x_2$ is dominated. If Dominator chooses $x_2$, then $\gamma_g(Q_{n+1}|N[v_2]) = 1 + \gamma'_g(Q_{n+1}|N[v_2,x_2])$. The corresponding residual graph is $P'_r$, where $r \geq 0$ and $r + 1 = n - 3$. We have that $\gamma'_g(P'_r) = \gamma'_g(Q_{n+1}|N[v_2,x_2])$. There are four cases according to the value of $n \mod{4}$. For $k \geq 1$, by Lemmas 2.6 and 2.7, we have that

\[
g'_g(P'_{4(k-1)+1}) = \gamma'_g(P'_{4(k-1)+1}) = \gamma_g(P'_r) = 2(k-1) + 1 = 2k - 1,
\]

\[
g'_g(P'_{4(k-1)+2}) = \gamma'_g(P'_{4(k-1)+2}) = \gamma_g(P'_r) + 1 = 2k,
\]

\[
g'_g(P'_{4(k-1)+3}) = \gamma'_g(P'_{4(k-1)+3}) = \gamma_g(P'_r) + 1 = 2k,
\]

\[
g'_g(P'_{4k}) = \gamma'_g(P'_{4k}) = \gamma_g(P'_r) = 2k.
\]

We assume that the Dominator first move is $x_4$ in the Staller-start game.

Lemma 4.3. In the Staller-start game, if the Staller first move is $v_2$ and the Dominator first move is $x_4$, then

$$\gamma'_g(Q_{n+1}|N[v_2,x_4]) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil - 2; & n \equiv 1 \pmod{4}, \\ \left\lceil \frac{n}{4} \right\rceil - 1; & \text{otherwise}. \end{cases}$$

Proof. Suppose that the Staller first move is $v_2$ and the Dominator first move is $x_4$. Then $\gamma_g(Q_{n+1}|N[x_4]) = 1 + \gamma'_g(Q_{n+1}|N[v_2,x_4])$. The corresponding residual graph is $P'_r \cup P'_r$, where $r \geq 0$ and $r + 1 = n - 4$. We have that $\gamma'_g(P'_r \cup P'_{n-3}) = \gamma'_g(Q_{n+1}|N[v_2,x_4])$. There are four cases according to the value of $n \mod{4}$. For $k \geq 1$, by Lemmas 2.6 and 2.7, we have that

\[
g'_g(P'_1 \cup P'_{4(k-2)+3}) = \gamma_g(P'_{4k}) + \gamma_g(P'_{4(k-2)+3}) + 1 = 2 + \gamma_g(P'_{4(k-2)+3}) = 2 + 2(k-2) + 2 - 1 = 2 + 2k - 4 + 1 = 2k - 1,
\]
\[ g'(P_1' \cup P_{4(k-1)}) = g(P_1'') + g(P_{4(k-1)}) \\
= 1 + g(P_{4(k-1)}) \\
= 1 + 2k - 2 = 2k - 1, \]

\[ g'(P_1' \cup P_{4(k-1)+1}) = g(P_1'') + g(P_{4(k-1)+1}) \\
= 1 + g(P_{4(k-1)+1}) \\
= 1 + 2k - 2 + 1 = 2k, \]

\[ g'(P_1' \cup P_{4(k-1)+2}) = g(P_1'') + g(P_{4(k-1)+2}) + 1 \\
= 2 + g(P_{4(k-1)+2}) \\
= 2 + 2k - 2 + 1 = 2k + 1. \]

\[ \square \]

**Proof of Theorem 1.3.** Note that for \( u \in Q_{n+1}, \)

\[ g(Q_{n+1}|N[u]) = 1 + \min_{v \in V(Q_{n+1})-u} \left\{ g'(Q_{n+1}|N[u,v]) \right\}. \]

From Lemmas 4.2 and 4.3, for \( n \equiv 3 \pmod{4}, \)

\[ g'(Q_{n-1}|N[v_2, x_2]) = \left\lceil \frac{n}{2} \right\rceil - 2 < \left\lceil \frac{n}{2} \right\rceil - 1 = g'(Q_{n-1}|N[v_2, x_4]). \]

So for Dominator, choosing \( x_2 \) is better than choosing \( x_4. \) \( \square \)

From our analysis, we propose the following conjecture. In an optimal strategy of the Staller-start game, if the Staller first move is \( v_2 \) and the Dominator first move is \( x_{n-1}, \) then

\[ g'(Q_{n+1}) = 1 + \begin{cases} \left\lceil \frac{n}{2} \right\rceil; & n \equiv 1, 3 \pmod{4}, \\ \frac{n}{2} + 1; & \text{otherwise}. \end{cases} \]

5 A Dominator-Start Game Played on 1-sum of \( P_n \) and \( P_2 \)

In this section, we analyze the game domination number on a graph \( T_{n+1}, \) which is a graph constructed from 1-sum of \( P_n \) and \( P_2 \) at \( x_k, \) for some \( k = 2, ..., n-1, \) and \( v_1, \) see figure 3. Then we find the upper bound of \( g(T_{n+1}) \) by assuming that the Dominator first move is \( x_k. \) By applying Lemmas 4.4 and 4.7, we obtain the following lemma.
Lemma 5.1. If \( k \equiv 0 \pmod{4} \), then

\[
\gamma_g(T_{n+1}) \leq \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor ; & n \equiv 0 \pmod{4}, \\
\left\lfloor \frac{n}{2} \right\rfloor + 1; & n \equiv 1 \pmod{4}, \\
\left\lfloor \frac{n}{2} \right\rfloor + 2; & n \equiv 2, 3 \pmod{4}.
\end{cases}
\]

If \( k \equiv 1 \pmod{4} \), then

\[
\gamma_g(T_{n+1}) \leq \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor ; & n \equiv 0 \pmod{4}, \\
\left\lfloor \frac{n}{2} \right\rfloor + 1; & n \equiv 1, 2 \pmod{4}, \\
\left\lfloor \frac{n}{2} \right\rfloor + 2; & n \equiv 3 \pmod{4}.
\end{cases}
\]

If \( k \equiv 2 \pmod{4} \), then \( \gamma_g(T_{n+1}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

If \( k \equiv 3 \pmod{4} \), then

\[
\gamma_g(T_{n+1}) \leq \begin{cases} 
\left\lceil \frac{n}{2} \right\rceil + 1; & n \equiv 0, 1 \pmod{4}, \\
\left\lceil \frac{n}{2} \right\rceil + 2; & n \equiv 2, 3 \pmod{4}.
\end{cases}
\]

Proof. We can easily check by hand for the case \( n = 4 \). Assume that \( n \geq 5 \). Suppose that Dominator chooses \( x_k \) in the first move, then the residual graph is \( P'_r \cup P'_s \), where \( r + s = n - 3 \). We now consider the following cases of the residual graph according to the value of \( n \pmod{4} \).

If \( n \equiv 0 \pmod{4} \) or \( n = 4j \), where \( j > 0 \), there are two cases: 1) \( P'_r \cup P'_{4m+1} \) where \( l + m + 1 = j \) and \( l, m \geq 0 \); and 2) \( P'_{4l+2} \cup P'_{4m+3} \) where \( l + m + 2 = j \) and \( l, m \geq 0 \).

If \( n \equiv 1 \pmod{4} \) or \( n = 4j + 1 \), where \( j > 0 \), there are three cases: 1) \( P'_{4l+3} \cup P'_{4m+3} \) where \( l + m + 2 = j \) and \( l, m \geq 0 \); 2) \( P'_{4l} \cup P'_{4m+2} \) where \( l + m + 1 = j \) and \( l, m \geq 0 \); and 3) \( P'_{4l+1} \cup P'_{4m+1} \) where \( l + m + 1 = j \) and \( l, m \geq 0 \).

If \( n \equiv 2 \pmod{4} \) or \( n = 4j + 2 \), where \( j > 0 \), there are two cases: 1) \( P'_{4l} \cup P'_{4m+3} \) where \( l + m + 1 = j \) and \( l, m \geq 0 \); and 2) \( P'_{4l+1} \cup P'_{4m+2} \) where \( l + m + 1 = j \) and \( l, m \geq 0 \).

By applying Lemmas 2.6 and 2.7 to consider each cases, the result follows. □
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