Some Extensions of Univalence Conditions for a General Integral Operator

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Abstract: In [4], Breaz and G"uney considered the subclasses $T_2, T_{2,\mu}$, and $S(p)$ of analytic functions $f$ in the open unit disk $U$ and proved some univalent conditions for an integral operator $F_{\alpha_1,\alpha_2,...,\alpha_n,\beta}$ of $f$ belonging to the classes $T_2, T_{2,\mu}$ and $S(p)$. In this note, we consider the subclasses $T_j, T_{j,\mu}$ and $S_j(p)$ ($j = 2, 3, \ldots$) and generalize the results of Breaz and G"uney.

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1 Introduction

Let $A$ be the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk

$$U := \{z \in \mathbb{C} : |z| < 1\}.$$

Also, let $S$ denote the subclass of $A$ consisting of functions $f$ which are univalent in $U$.

Let $A_j$ be the subclass of $A$ consisting of functions $f$ given by

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N}_1 := \mathbb{N} \setminus \{0, 1\} = \{2, 3, \ldots\}).$$
Let $T$ be the univalent subclass of $A$ consisting of functions $f$ which satisfy

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1 \quad (z \in U).$$

Let $T_j$ be the subclass of $T$ for which $f^{(k)}(0) = 0 \,(k = 2, 3, \ldots, j)$. Let $T_{j,\mu}$ be the subclass of $T_j$ consisting of functions $f$ of the form (1.1) which satisfy

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu \quad (z \in U) \quad (1.2)$$

for some $\mu \,(0 < \mu \leq 1)$, and let us denote by $T_{j,1} \equiv T_j$ when $\mu = 1$.

For some real number $p$ with $0 < p \leq 2$, we define the subclass $S(p)$ of $A$ consisting of all functions $f$ which satisfy

$$\left| \left( \frac{z}{f(z)} \right)'' \right| \leq p \quad (z \in U). \quad (1.3)$$

Singh [9] has shown that if $f \in S(p)$, then $f$ satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq p |z|^2 \quad (z \in U). \quad (1.4)$$

Let $S_j(p)$ be the subclass of $A$ consisting of functions $f \in A_j$ which satisfy (1.3) and

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq p |z|^j \quad (z \in U, \; j \in \mathbb{N}_1), \quad (1.5)$$

and let us denote by $S_2(p) \equiv S(p)$.

The subclasses $T_j, T_{j,\mu}$ and $S_j(p)$ are introduced by Seenivasagan [7].

To discuss our problems, we have to recall here the following results.

**General Schwarz Lemma.** (15) Let the function $f$ be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed $M$. If $f$ has one zero with multiplicity order bigger than $m$ for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in U_R).$$

The equality can hold only if $f(z) = e^{i\theta}(M/R^m)z^m$, where $\theta$ is constant.

**Theorem A.** (1, 2) Let $c$ be a complex number, $|c| \leq 1$, $c \neq -1$. If $f(z) = z + a_2z^2 + \cdots$ is a regular function in $U$ and

$$|c| |z|^2 + (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$

for all $z \in U$, then the function $f$ is regular and univalent in $U$. 
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**Theorem B.** (6) Let \( \beta \) be a complex number, \( \Re(\beta) > 0 \), \( c \) a complex number, \( |c| \leq 1 \), \( c \neq -1 \), and \( h(z) = z + a_2z^2 + \cdots \) a regular function in \( U \). If

\[
|c|z^{2\beta} + (1 - |z|^{2\beta}) \frac{zh''(z)}{\beta h'(z)} \leq 1
\]

for all \( z \in U \), then the function

\[
F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} h'(t) dt \right\}^\beta = z + \cdots
\]
is regular and univalent in \( U \).

In [8], the authors considered the integral operator

\[
F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} n \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^\frac{1}{\beta}
\]

for \( f_i \in A_2 \) (\( i = 1, 2, \ldots, n \)) and \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C} \).

For \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \), \( F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta} \) becomes the integral operator \( F_{\alpha,\beta} \) considered in [3].

The purpose of this paper is to generalize the main results of Breaz and Güney [4].

In the sequel, by \( \mathbb{N}^* \) we denote the set of strictly positive integers.

### 2 Main Results

**Theorem 2.1.** Let \( M_i \geq 1 \), let the functions \( f_i \in S_j(p_i) \) (\( i = 1, 2, \ldots, n \); \( n \in \mathbb{N}^* \); \( j \in \mathbb{N}_1^* \)), let \( \alpha_i, \beta \in \mathbb{C} \), \( \Re(\beta) \geq \sum_{i=1}^n \frac{(1+p_i)M_i+1}{|\alpha_i|} \), and let \( c \in \mathbb{C} \). If

\[
|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^n \frac{(1+p_i)M_i+1}{|\alpha_i|},
\]

\[
|f_i(z)| \leq M_i,
\]

for all \( z \in U \), then \( F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta} \) defined in (1.6) is in the class \( S \).

**Proof.** Let us define the function \( h \) by

\[
h(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt
\]

for \( f_i \in S_j(p_i) \) (\( i = 1, 2, \ldots, n \); \( n \in \mathbb{N}^* \); \( j \in \mathbb{N}_1^* \)). Since

\[
h'(z) = \prod_{i=1}^n \left( \frac{f_i(z)}{z} \right)^{\frac{1}{\alpha_i}},
\]
we see that \( h(0) = 0 \) and \( h'(0) = 1 \). Also, after some calculation, we obtain

\[
\frac{z h''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left( \frac{z f_i'(z)}{f_i(z)} - 1 \right).
\]  

(2.1)

It follows from (2.1) that

\[
\left| \frac{z h''(z)}{h'(z)} \right| \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left( \left| \frac{z^2 f_i'(z)}{(f_i(z))^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right).
\]  

(2.2)

Since \(|f_i(z)| \leq M_i (i = 1, 2, \ldots, n; z \in U)\), applying the general Schwarz lemma, we know that

\[
|f_i(z)| \leq M_i |z| (i = 1, 2, \ldots, n; z \in U).
\]

Using this inequality in (2.2), we find that

\[
\left| \frac{z h''(z)}{h'(z)} \right| \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left( \left| \frac{z^2 f_i'(z)}{(f_i(z))^2} \right| |f_i(z)| + 1 \right) \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left( p_i M_i |z|^j + M_i + 1 \right).
\]

(2.3)

Thus, it follows from (2.3) that

\[
\left| c |z|^{2\beta} + (1 - |z|^{2\beta}) \frac{z h''(z)}{\beta h'(z)} \right| \leq |c| + \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1 + p_i) M_i + 1}{|\alpha_i|} \leq 1
\]

because \(|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1 + p_i) M_i + 1}{|\alpha_i|} \). Finally, applying Theorem B for the function \( h \), we prove that \( F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta} \in S \).

Corollary 2.2. Let \( M_i \geq 1 \), let the functions \( f_i \in S_j(p_i) (i = 1, 2, \ldots, n; n \in \mathbb{N}^*; j \in \mathbb{N}_1) \), let \( \alpha, \beta \in \mathbb{C} \), \( \Re(\beta) \geq \sum_{i=1}^{n} \frac{(1 + p_i) M_i + 1}{|\alpha_i|} \), and let \( c \in \mathbb{C} \). If

\[
|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{(1 + p_i) M_i + 1}{|\alpha_i|},
\]

\(|f_i(z)| \leq M_i\),
for all \( z \in U \), then the function

\[
F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}}
\]  
(2.4)

is in the class \( S \).

**Proof.** In Theorem 2.1, we consider \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \).

**Corollary 2.3.** Let \( M \geq 1 \), let the functions \( f_i \in S_j(p) \) \( (i = 1, 2, \ldots, n; n \in \mathbb{N}^*; j \in \mathbb{N}_1^*) \), let \( \alpha_i, \beta \in \mathbb{C} \), \( \Re(\beta) \geq \sum_{i=1}^n \frac{(1+p)M_i+1}{|\alpha_i|} \), and let \( c \in \mathbb{C} \). If

\[
|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^n \frac{(1+p)M_i+1}{|\alpha_i|},
\]

for all \( z \in U \), then \( F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta} \) defined in (1.6) is in the class \( S \).

**Proof.** In Theorem 2.1, we consider \( p_1 = p_2 = \cdots = p_n = p \) and \( M_1 = M_2 = \cdots = M_n = M \).

**Remark 2.4.** If we set \( j = 2 \) in Corollary 2.3, then we have Theorem 2.1 in [4].

**Corollary 2.5.** Let \( M \geq 1 \), let the functions \( f \in S_j(p) \) \( (j \in \mathbb{N}_1^*) \), let \( \alpha, \beta \in \mathbb{C} \), \( \Re(\beta) \geq \sum_{i=1}^n \frac{(1+p)M_i+1}{|\alpha_i|} \), and let \( c \in \mathbb{C} \). If

\[
|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^n \frac{(1+p)M_i+1}{|\alpha_i|},
\]

for all \( z \in U \), then the function

\[
G_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left( \frac{f(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}}
\]  
(2.5)

is in the class \( S \).

**Proof.** In Theorem 2.1, we consider \( n = 1 \).

**Theorem 2.6.** Let \( M_i \geq 1 \), let the functions \( f_i \in T_{j,\mu_i} \) \( (i = 1, 2, \ldots, n; n \in \mathbb{N}^*; j \in \mathbb{N}_1^*) \), let \( \alpha_i, \beta \in \mathbb{C} \), \( \Re(\beta) \geq \sum_{i=1}^n \frac{(1+p_i)M_i+1}{|\alpha_i|} \), and let \( c \in \mathbb{C} \). If

\[
|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^n \frac{(1+p_i)M_i+1}{|\alpha_i|},
\]

for all \( z \in U \), then \( F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta} \) defined in (1.6) is in the class \( S \).
Proof. Defining the function \( h \) by

\[
h(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} \, dt,
\]

we take the same steps as in the proof of Theorem 2.1. Then, we obtain that

\[
|c|, |z|, \beta^2 + (1 - |z|^2) \frac{zh''(z)}{h'(z)} < \left| c + \frac{1}{|\beta|} \right|, \quad \left| \frac{zh''(z)}{h'(z)} \right| \leq \left| c + \frac{1}{|\beta|} \right| + \frac{1}{|\beta|} \sum_{i=1}^n \frac{1}{|\alpha_i|} \left( |z|^2 f_i'(z)^2 - 1 \right) M_i + M_i + 1
\]

\[
\leq |c| + \frac{1}{|\beta|} \sum_{i=1}^n \frac{1}{|\alpha_i|} (\mu_i M_i + M_i + 1)
\]

\[
= |c| + \frac{1}{|\beta|} \sum_{i=1}^n \frac{1}{|\alpha_i|} (1 + \mu_i) M_i + 1
\]

for \( f_i \in T_{j, \mu_i} \), \( i = 1, 2, \ldots, n \); \( n \in \mathbb{N}^* \); \( j \in \mathbb{N}^1 \). In view of Theorem B, we have \( F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta} \in S \). □

Corollary 2.7. Let \( M_i \geq 1 \), let the functions \( f_i \in T_{j, \mu_i} \), \( i = 1, 2, \ldots, n \); \( n \in \mathbb{N}^* \); \( j \in \mathbb{N}_1 \), let \( \alpha, \beta \in \mathbb{C} \), \( \Re(\beta) \geq \sum_{i=1}^n \frac{(1 + \mu_i) M_i + 1}{|\alpha_i|} \), and let \( c \in \mathbb{C} \). If

\[
|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^n \frac{(1 + \mu_i) M_i + 1}{|\alpha_i|},
\]

\[
|f_i(z)| \leq M_i
\]

for all \( z \in U \), then \( F_{\alpha, \beta} \) defined in (2.4) is in the class \( S \).

Proof. In Theorem 2.6, we consider \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha \). □

Corollary 2.8. Let \( M \geq 1 \), let the functions \( f_i \in T_{j, \mu_i} \), \( i = 1, 2, \ldots, n \); \( n \in \mathbb{N}^* \); \( j \in \mathbb{N}_1 \), let \( \alpha_i, \beta \in \mathbb{C} \), \( \Re(\beta) \geq \sum_{i=1}^n \frac{(1 + \mu_i) M + 1}{|\alpha_i|} \), and let \( c \in \mathbb{C} \). If

\[
|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^n \frac{(1 + \mu_i) M + 1}{|\alpha_i|},
\]

\[
|f_i(z)| \leq M
\]

for all \( z \in U \), then \( F_{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta} \) defined in (1.6) is in the class \( S \).

Proof. In Theorem 2.6, we consider \( M_1 = M_2 = \cdots = M_n = M \). □

Remark 2.9. If we set \( j = 2 \) in Corollary 2.8, then we have Theorem 2.6 in [4].
Corollary 2.10. Let $M \geq 1$, let the functions $f \in T_{j,\mu}$ ($j \in \mathbb{N}_1^*$), let $\alpha, \beta \in \mathbb{C}$, $\Re(\beta) \geq \frac{(1+\mu)M+1}{\alpha}$, and let $c \in \mathbb{C}$. If

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \frac{(1 + \mu)M + 1}{|\alpha|},$$

$$|f(z)| \leq M,$$

for all $z \in U$, then $G_{\alpha,\beta}$ defined in (2.5) is in the class $S$.

Proof. In Theorem 2.6, we consider $n = 1$. \hfill $\square$

Theorem 2.11. Let $M_i \geq 1$, let the functions $f_i \in T_j$ ($i = 1, 2, \ldots, n; n \in \mathbb{N}^*$ ; $j \in \mathbb{N}_1^*$), let $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|}$, and let $c \in \mathbb{C}$. If

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|},$$

$$|f_i(z)| \leq M_i,$$

for all $z \in U$, then $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) is in the class $S$.

Proof. In Theorem 2.6, we consider $\mu_1 = \mu_2 = \cdots = \mu_n = 1$. \hfill $\square$

Corollary 2.12. Let $M_i \geq 1$, let the functions $f_i \in T_j$ ($i = 1, 2, \ldots, n; n \in \mathbb{N}^*$ ; $j \in \mathbb{N}_1^*$), let $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|}$, and let $c \in \mathbb{C}$. If

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|},$$

$$|f_i(z)| \leq M_i,$$

for all $z \in U$, then $F_{\alpha,\beta}$ defined in (2.4) is in the class $S$.

Proof. In Theorem 2.11, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. \hfill $\square$

Corollary 2.13. Let $M \geq 1$, let the functions $f_i \in T_j$ ($i = 1, 2, \ldots, n; n \in \mathbb{N}^*$ ; $j \in \mathbb{N}_1^*$), let $\alpha_i, \beta \in \mathbb{C}$, $\Re(\beta) \geq \sum_{i=1}^{n} \frac{2M+1}{|\alpha_i|}$, and let $c \in \mathbb{C}$. If

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \sum_{i=1}^{n} \frac{2M+1}{|\alpha_i|},$$

$$|f_i(z)| \leq M,$$

for all $z \in U$, then $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) is in the class $S$.

Proof. In Theorem 2.6, we consider $M_1 = M_2 = \cdots = M_n = M$. \hfill $\square$
Corollary 2.14. Let $M \geq 1$, let the functions $f \in T_j$ ($j \in \mathbb{N}_1$), let $\alpha, \beta \in \mathbb{C}$, $\Re(\beta) \geq \frac{2M+1}{|\alpha|}$, and let $c \in \mathbb{C}$. If

$$|c| \leq 1 - \frac{1}{\Re(\beta)} \cdot \frac{2M + 1}{|\alpha|},$$

$$|f(z)| \leq M,$$

for all $z \in U$, then $G_{\alpha,\beta}$ defined in (2.5) is in the class $S$.

Proof. In Theorem 2.11, we consider $n = 1$.

References


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