Convergence Theorems for $G$-Nonexpansive Mappings in CAT(0) Spaces Endowed with Graphs

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Abstract: In this paper, we introduce the concept of a CAT(0) space endowed with a graph. Browder’s convergence theorem and Halpern iteration process for $G$-nonexpansive mappings in an underlying space will be presented. This result extends and generalizes the result of Tiammee, Kaewkhao and Suantai (2015).

Keywords: CAT(0) space; directed graph; nonexpansive mapping; Browder’s convergence theorem; Halpern iteration process.

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1 Introduction

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a nonexpansive mapping if there is $k \in (0, 1]$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. If in the case of $k < 1$, we call $T$ a contraction. A point $x \in X$ is called a fixed point of $T$ if $Tx = x$. In this paper, we use the notation $F(T)$ stand for the set of all fixed...
points of $T$.

A geodesic joining points $x$ and $y$ in a metric space $X$ is a mapping $c$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\gamma$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique this geodesic is denoted by $[x, y]$. We write $ax \oplus (1 - a)y$ stand for the point $c(\alpha_0 + (1 - \alpha_1)l) \in X$. The space $X$ is said to be a (uniquely) geodesic space if every two points of $X$ are joined by a (unique) geodesic. A geodesic space $X$ is said to be a CAT(0) space if every geodesic triangle in $X$ is at least as thin as its comparison triangle in the Euclidean plane, i.e.,

$$d(a, b) \leq d_{\mathbb{R}^2}(\overline{a}, \overline{b}),$$

for any $a, b \in \Delta(x, y, z)$ and $\overline{a}, \overline{b} \in \overline{\Delta}(x, y, z)$.

The first famous fixed point theorem in a metric space is established by Stefan Banach [1] in 1922. They investigated the theorem, called Banach contraction principle, which telling that a self mapping on a complete metric space $X$ has a unique fixed point. Browder [2] used the Banach’s result to prove the convergence theorem for the implicit iterative in a Hilbert space, called the Browder’s convergence theorem.

In 2008, Jachymski combined the knowledge of the original fixed point theory and graph theory. First of all, they introduced a concept of a metric space endowed with a graph as the following: For any metric space $(X, d)$ and a directed graph $G = (V(G), E(G))$, if $V(G) = X$ and $E(G)$ contains all loops, i.e., $\Delta = \{(x, x) : x \in X\} \subseteq E(G)$, the triple $(X, d, G)$ is called a metric space endowed with a graph. Let $C$ be a nonempty subset of a metric space endowed with graph $(X, d, G)$. Suppose $T : C \to C$ preserves edges of $G$ and satisfy $d(Tx, Ty) \leq kd(x, y)$ for any $x, y \in X$ for some $k \in \mathbb{R}^+$. Then

1. if $k < 1$, we call $T$ a $G$-contraction, and
2. if $k \leq 1$, we call $T$ a $G$-nonexpansive mapping.

The following theorem, a generalization of Banach contraction principle, has been presented in [3]:

**Theorem 1.1** ([3]). Suppose that a metric space endowed with graph $(X, d, G)$ have the Property P:

- for any $\{x_n\}_{n \in \mathbb{N}}$ if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$,
- then there is a subsequence $\{x_{k_n}\}_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let $T$ be a $G$-contraction, and $X_T = \{x \in X : (x, T(x)) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_T \neq \emptyset$.

In 2015, Tiammee et al. [4] extended the Browder’s convergence theorem for $G$-nonexpansive mappings in Hilbert spaces endowed with graphs. In the prove of their theorem, they have to replace the Property P to the stronger one, called the Property G: for every sequence $\{x_n\}$ in $C$ converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.
Theorem 1.2 [H]. Let $C$ be a bounded closed convex subset of Hilbert space $H$ and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose $C$ has Property $G$. Let $T : C \to C$ be $G$-nonexpansive. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Define $T_n : C \to C$ by

$$T_n x = (1 - \alpha_n)Tx + \alpha_n x_0$$

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \to 0$ as $n \to \infty$. Then the following hold:

(i) $T_n$ has a fixed point $u_n \in C$;

(ii) $F(T) \neq \emptyset$.

(iii) if $F(T) \times F(T) \subseteq E(G)$ and $Px_0$ is dominated by $\{u_n\}$, then the sequence $\{u_n\}$ converges strongly to $Px_0$ where $P$ is the metric projection onto $F(T)$.

Motivating by the above results, in this paper, we will present the Brower convergence theorem in the framework of CAT(0) spaces endowed with graph. The conditions on the set $C$ in our result has been relaxed. The convergence theorems of the Halpern’s iteration scheme for a family of $G$-nonexpansive mappings are also presented.

2 Preliminaries

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if every $v \in V(G) \setminus X$ there exists $x \in X$ such that $(x, v) \in E(G)$ and we say that $x$ dominates $v$ or $v$ is dominated by $x$. Let $v \in V(G)$, a set $X \subseteq V(G)$ is dominated by $v$ if $(v, x) \in E(G)$ for any $x \in X$ and we say that $X$ dominates $v$ if $(x, v) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops. Let $G$ be a directed graph, and $E(G)$ the set of edges of $G$. We say $E(G)$ is a convex set if, for any $\alpha \in [0, 1]$,

$$(\alpha x + (1 - \alpha)y, \alpha u + (1 - \alpha)v) \in E(G)$$

for all $(x, y), (u, v) \in E(G)$.

Let $X$ be a metric space. The following statements are equivalent for a uniquely geodesic space $X$:

(i) $X$ is a CAT(0) space;

(ii) $X$ satisfies the (CN)-inequality: If $x, y \in X$ and $\alpha \in (0, 1)$, then

$$d^2(z, \alpha x + (1 - \alpha)y) \leq \alpha d^2(z, x) + (1 - \alpha)d^2(z, y) - \alpha(1 - \alpha)d^2(x, y),$$

for all $z \in X$;

(iii) $X$ satisfies the law of cosine: If $a = d(x, z), b = d(y, z), c = d(x, y)$ and $\xi$ is the Alexandrov angle at $z$ between $[x, z]$ and $[y, z]$, then

$$c^2 \geq a^2 + b^2 - 2ab \cos \xi.$$
Lemma 2.1. Let $X$ be a CAT(0) space. Then for each $p, q, x, y \in X$ and $\alpha \in [0, 1]$, we have
\[
d(\alpha p \oplus (1 - \alpha)q, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(p, x) + (1 - \alpha)d(q, y).
\]

For any nonempty subset $C$ of $X$, let $\pi = \pi_C$ be the projection mapping from $X$ to $C$. It is known that if $C$ is closed and convex, the mapping $\pi$ is well-defined, nonexpansive and satisfies
\[
d^2(x, y) \geq d^2(x, \pi x) + d^2(\pi x, y) \quad \text{for all } x \in X \text{ and } y \in C.
\]

In 2011, Dhompongsa et al. [6] introduced the following concepts of convex combination in CAT(0) spaces. Let $v_1, v_2, \ldots, v_n \subset X$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in (0, 1)$ with $\sum_{i=1}^{n} \lambda_i = 1$. Using the result in [7], the partial sum of $\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n$ can be written by:
\[
\bigoplus_{i=1}^{n} \lambda_i v_i := (1 - \lambda_n) \left( \frac{\lambda_1}{1 - \lambda_n} v_1 \oplus \frac{\lambda_2}{1 - \lambda_n} v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} v_{n-1} \right) \oplus \lambda_n v_n. \tag{2.2}
\]

Let $\{\lambda_n\} \subset (0, 1)$ be such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Let $\{v_n\} \subset X$ be bounded and $v_0$ be an arbitrary point in $X$. Suppose $\lambda'_n = \sum_{i=n+1}^{\infty} \lambda_i$ and $\sum_{i=n}^{\infty} \lambda'_i \to 0$ as $n \to \infty$. Set
\[
s_n := \left( \sum_{i=1}^{n} \lambda_i \right) w_n \oplus \lambda'_n v_0, \tag{2.3}
\]
where $w_1 = v_1$ and for each $n \geq 2$,
\[
w_n = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} v_1 \oplus \frac{\lambda_2}{\sum_{i=1}^{n} \lambda_i} v_2 \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} v_n.
\]

Then $s_n \to x$ as $n \to \infty$ for some $x \in X$. In [6], they use the element $x$ for representing the infinite summation of $\lambda_1 v_1, \lambda_2 v_2, \ldots$, i.e.,
\[
x = \bigoplus_{n=1}^{\infty} \lambda_n v_n.
\]

By (2.3), $d(s_n, w_n) \leq \lambda'_n d(w_n, v_0)$, it follows that $\lim_{n \to \infty} s_n = \lim_{n \to \infty} w_n$. Thus the limit $x$ is independent of the choice of $v_0$.

The followings are importance properties of the convex combination in CAT(0) spaces introduced in [6].

Lemma 2.2. If $y_0$ and $v_n$ belong to $X$, $d(v_n, y_0) = d(x, y_0)$ for all $n$ where $x = \bigoplus_{n=1}^{\infty} \lambda_n v_n$, then $v_n = x$ for all $n$. 

Lemma 2.3 ([8], Lemma 3.8). Let C be a nonempty closed convex subset of a complete CAT(0) space X, let \( \{T_n : n \in \mathbb{N} \} \) be a family of single-valued nonexpansive mappings on C. Suppose \( \bigcap_{n=1}^{\infty} F(T_n) \) is nonempty. Define \( T : C \to C \) by

\[
Tx = \bigoplus_{n=1}^{\infty} \lambda_n t_n x
\]

for all \( x \in C \) where \( \{\lambda_n\} \subset (0, 1) \) with \( \sum_{n=1}^{\infty} \lambda_n = 1 \) and \( \sum_{i=n}^{\infty} \lambda_i' \to 0 \) as \( n \to \infty \). Then \( T \) is nonexpansive and \( F(T) = \bigcap_{n=1}^{\infty} F(T_n) \).

The following results are very useful in the proof of our main results.

Lemma 2.4 ([8]). Let \( (a_1, a_2, \ldots) \in l^\infty \) be such that \( \mu_n(a_n) \leq 0 \) for all Banach limits \( \mu \) and \( \limsup_{n \to \infty} (a_{n+1} - a_n) \leq 0 \). Then \( \limsup_{n \to \infty} a_n = 0 \).

Lemma 2.5 ([9]). Let \( \{s_n\} \) be a sequence of nonnegative real numbers, \( \{\alpha_n\} \) a sequence of real numbers in \([0, 1]\) with \( \sum_{n=1}^{\infty} \alpha_n = \infty \), \( \{u_n\} \) a sequence of nonnegative real numbers with \( \sum_{n=1}^{\infty} u_n < \infty \), and \( \{t_n\} \) a sequence of real numbers with \( \limsup_{n \to \infty} t_n \leq 0 \). Suppose that

\[
s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n t_n + u_n \quad \forall n \in \mathbb{N}.
\]

Then \( \lim_{n \to \infty} s_n = 0 \).

Lemma 2.6 ([10]). Let \( C \) be a closed convex subset of a complete CAT(0) space X and let \( T : C \to C \) be a nonexpansive mapping. Let \( u \in C \) be fixed. For each \( k \in (0, 1) \), the mapping \( S_k : C \to C \) defined by

\[
S_k x = ku \oplus (1 - k)Tx \quad \text{for } x \in C
\]

has a unique fixed point \( x_k \in C \), that is,

\[
x_k = S_k x_k = ku \oplus (1 - k)Tx_k.
\]

Then \( F(T) \neq \emptyset \) if and only if \( \{x_k\} \) given by (2.6) is bounded as \( k \to 0 \). In this case, the following statements hold:

(i) \( \{x_k\} \) converges to the unique fixed point \( z_0 \) of \( T \) which is nearest to \( u \);

(ii) \( d^2(u, z_0) \leq \mu_n d^2(u, x_n) \), for all Banach limits \( \mu \) and all bounded sequences \( \{x_n\} \) with \( d(x_n, Tx_n) \to 0 \).

3 Main Results

3.1 Browder’s Convergence Theorem

Lemma 3.1. Let \( (X, d, G) \) be a CAT(0) space endowed with graph. Suppose \( T : X \to X \) is a G-nonexpansive mapping. If X has a Property P, then T is continuous.
Proof. Let \( \{x_n\} \) be a sequence in \( X \) converging to some \( x \in X \). Let \( \{Tx_{n_k}\} \) be any subsequence of \( \{Tx_n\} \). Since \( x_{n_k} \to x \) as \( k \to 0 \), by Property \( P \), there exists a subsequence \( \{x_{m_k}\} \) such that \( (x_{m_k}, x) \in E(G) \) for each \( k \in \mathbb{N} \). Since \( T \) is \( G \)-nonexpansive and \( (x_{m_k}, x) \in E(G) \) we obtain
\[
d(Tx_{m_k}, Tx) \leq d(x_{m_k}, x) \to 0 \quad \text{as} \quad k \to \infty.
\]
Hence \( Tx_{m_k} \to Tx \). By the double extract subsequence principle, we conclude that \( Tx_n \to Tx \). Therefore \( T \) is continuous.

In what follows, we will prove the Brower’s convergence theorem for a \( G \)-nonexpansive mapping on a bounded closed and star-shaped subset \( C \) of a CAT(0) space \( X \) under the hypothesis that \( X \) satisfies the property \( P \). We first present the definition of a star-shaped set in a CAT(0) space.

**Definition 3.2** ([11]). Let \( X \) be a CAT(0) space. A subset \( C \) is said to be star-shaped if there exists \( p \in C \) such that \((1-t)p \oplus tx \in C \) for any \( x \in C \) and \( t \in [0,1] \). In this case, \( C \) is also called \( p \)-star-shaped, where \( p \) is the center of the star.

**Remark 3.1.** The assumption “\( C \) is \( p \)-star-shape” is weaker than the convexity of \( C \).

**Theorem 3.3.** Let \((X,d,G)\) be a CAT(0) space endowed with graph having Property \( P \) and \( C \) be a nonempty subset of \( X \). Suppose \( T : C \to C \) is a \( G \)-nonexpansive mapping and \( F(T) \times F(T) \subseteq E(G) \). If \( E(G) \) is convex, then \( F(T) \) is closed and convex.

**Proof.** Suppose \( F(T) \neq \emptyset \). Let \( \{x_n\} \) be a sequence in \( F(T) \) such that \( x_n \to x \) as \( n \to \infty \). By Property \( P \), there is a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( (x_{n_k}, x) \in E(G) \) for all \( k \in \mathbb{N} \). Since \( T \) is \( G \)-nonexpansive, we obtain
\[
d(x, Tx) \leq d(x, x_{n_k}) + d(x_{n_k}, Tx) \\
= d(x, x_{n_k}) + d(Tx_{n_k}, Tx) \\
\leq d(x, x_{n_k}) + d(x_{n_k}, x) \to 0.
\]
Therefore \( x = Tx \), i.e., \( x \in F(T) \). This shows that \( F(T) \) is closed.

Let \( x, y \in F(T) \) and \( \lambda \in [0,1] \). Denote \( z = \lambda x + (1-\lambda)y \). By the convexity of \( E(G) \), we obtain
\[
(x, z) = (\lambda x + (1-\lambda)x, \lambda x + (1-\lambda)y) \in E(G).
\]
Similarly, we also have \((y, z) \in E(G) \). Finally, we will show by contradiction, that \( z \in F(T) \). Suppose the contrary i.e., \( z \neq Tz \). Using the \((CN)\)-inequality and the
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$G$-nonexpansiveness of $T$, we have

\[
d_2\left(\frac{z \oplus Tz}{2}, x\right) \leq \frac{d_2^2(z, x)}{2} + \frac{d_2^2(Tz, x)}{2} - \frac{d_2^2(z, Tz)}{4}
\]

\[
= \frac{d_2(z, x)}{2} + \frac{d_2^2(Tx, Tz)}{2} - \frac{d_2^2(z, Tz)}{4}
\]

\[
\leq \frac{d_2(z, x)}{2} + \frac{d_2^2(z, x)}{2} - \frac{d_2^2(z, Tz)}{4}
\]

\[
= d_2(z, x) - \frac{d_2^2(z, Tz)}{4}
\]

Therefore $d\left(\frac{z \oplus Tz}{2}, x\right) < d(z, x)$ and, by the similar argument, we also get $d\left(\frac{z \oplus Tz}{2}, y\right) \leq d(z, y)$. Hence

\[
d(x, y) \leq d\left(x, \frac{z \oplus Tz}{2}\right) + d\left(y, \frac{z \oplus Tz}{2}\right)
\]

\[
< d(x, z) + d(y, z)
\]

\[
= d(x, y).
\]

Which lead us a contradiction. Thus $F(T)$ is convex. \qed

Now, we already to prove our first main result.

**Theorem 3.4.** Let $(X, d, G)$ be a complete CAT(0) space endowed with graph. Assume that there exists $p \in C$ such that $(p, Tp) \in E(G)$. Let $C$ be a bounded closed $p$-star-shaped of $X$ which has Property $P$ and $E(G)$ is convex. Let $T : C \to C$ be a $G$-nonexpansive mapping. Define $T_n : C \to C$ by

\[
T_n x = (1 - \alpha_n)Tx \oplus \alpha_n p
\]

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \to 0$. Then all of the followings hold:

(i) $T_n$ has a fixed point $u_n \in C$;

(ii) $F(T) \neq \emptyset$; and

(iii) if $F(T) \times F(T) \subseteq E(G)$ and $(u_n, u_k) \in E(G)$ for all $n, k \in \mathbb{N}$, then the sequence $\{u_n\}$ converges strongly to $v^* \in F(T)$ which is nearest to $p$.

**Proof.** We first show that $T_n$ is $G$-contraction for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $x, y \in C$ such that $(x, y) \in E(G)$. Since $T$ is $G$-nonexpansive, we obtain $T_n$ is also nonexpansive. Since $T$ is edge-preserving, $(Tx, Ty) \in E(G)$. By the convexity of $E(G)$, we have $(T_n x, T_n y) = ((1 - \alpha_n)Tx \oplus \alpha_n p, (1 - \alpha_n)Ty \oplus \alpha_n p) \in E(G)$. Hence $T_n$ is $G$-contraction. For each sequence $\{x_n\}$ in $C$ such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, by Property $P$, there is a subsequence $\{x_{n_k}\}$ such that
$(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$. Since $E(G)$ is convex and $(p, p) \in E(G)$, so $(p, T_n p) = ((1 - \alpha_n) p \oplus \alpha_n p, (1 - \alpha_n) T p \oplus \alpha_n p) \in E(G)$. Then $T_n$ has a fixed point, i.e., $u_n = T_n u_n$, because $X_{T_n} = \{ x \in X : (x, T_n(x)) \in E(G) \} \neq \emptyset$.

To prove (ii) & (iii), let $\{ u_m \}$ be any subsequence of $\{ u_n \}$. Since $\alpha_n \to 0$ as $n \to \infty$, there exists a monotone decreasing subsequence $\{ \alpha_{m_k} \}$ of $\{ \alpha_m \}$. Let $\{ u_{m_k} \}$ be a subsequence of $\{ u_m \}$ corresponds with the coefficient $\{ \alpha_{m_k} \}$. We show that $\{ u_{m_k} \}$ is a Cauchy sequence. Indeed, let $l, k \in \mathbb{N}$ and suppose without loss of generality that $l < k$. So $\alpha_{m_l} > \alpha_{m_k}$. Consider $\Delta(p, Tu_{m_k}, Tu_{m_l})$, the comparison triangle of $\Delta(p, Tu_{m_l}, Tu_{m_k})$ in $\mathbb{R}^2$. For convenience, we take $\overline{p} = (0, 0)$

d = \overline{u_{m_l}} - \overline{u_{m_k}}, a = 1 - \alpha_{m_l}$ and $b = 1 - \alpha_{m_k}$. We have $\overline{u_{m_k}} = b Tu_{m_k}$ and $\overline{u_{m_l}} = a Tu_{m_l}$. Consider

$$\left\| \frac{1}{a}(\overline{u_{m_k}} + d) - \frac{1}{b} \overline{u_{m_k}} \right\|^2 = \left\| \left(\frac{1}{a} \overline{u_{m_k}} - \frac{1}{b} \overline{u_{m_k}}\right) + \frac{1}{a} d \right\|^2$$

$$= \left\| \frac{1}{a} \overline{u_{m_l}} - \frac{1}{b} \overline{u_{m_k}} \right\|^2$$

$$= \| Tu_{m_l} - Tu_{m_k} \|^2 \leq \| d \|^2.$$

Thus

$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 \| u_{m_k} \|^2 + \left(\frac{1}{a}\right)^2 \| d \|^2 + 2 \left(\frac{1}{a} - \frac{1}{b}\right) \left(\frac{1}{a}, \frac{1}{a}\right) \| u_{m_k} \| d \| \leq \| d \|^2.$$

Therefore

$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 \| u_{m_k} \|^2 + \left(\frac{1}{a} - 1\right)^2 \| d \|^2 \leq 2 \left(\frac{1}{b} - 1\right) \langle \overline{u_{m_k}}, d \rangle.$$

This means $\langle \overline{u_{m_k}}, d \rangle \geq 0$. Since $\overline{u_{m_l}} = \overline{u_{m_k}} + d$, we have

$$\| u_{m_l} \|^2 = \langle \overline{u_{m_l}} + d, \overline{u_{m_l}} + d \rangle$$

$$= \| u_{m_k} \|^2 + \| d \|^2 + 2\langle u_{m_k}, d \rangle$$

$$\geq \| u_{m_k} \|^2 + \| u_{m_l} - u_{m_k} \|^2 \geq \| u_{m_k} \|^2.$$

This shows that the sequence $\{ ||u_{m_k}||^2 \}$ is monotone decreasing. By the boundedness of $C$, we can conclude that $||u_{m_k}||^2 \to M$, for some $M > 0$ as $k \to \infty$. From [1.1], we have

$$d^2(u_{m_k}, u_{m_l}) \leq ||u_{m_k} - u_{m_l}||^2 \leq ||u_{m_l}||^2 - ||u_{m_k}||^2 \to 0$$

as $k, l \to \infty$. Hence $\{ u_{m_k} \}$ is a Cauchy sequence. By the completeness of $X$, it converges to some $v^* \in C$. From the continuity of the metric $d$, we can say that $d(v^*, Tu^*) = \lim_{k \to \infty} \alpha_{m_k} d(p, Tu_{m_k}) = 0$. Therefore $v^* \in F(T)$ and (ii) has been proved.
The sequence \( G \), Halpern Iteration Process for Convergence Theorems for Theorem 3.5. Let using the Halpern iteration process \( G \), that the sequence \( \{x_n\} \) is transitive and \( \{u_n\} \) are in \( E \). Therefore, by induction, we can conclude that \( \{x_n\} \) converges to \( v^* \in F(T) \).

3.2 Halpern Iteration Process for G-Nonexpansive Mappings

In this section, we will prove the strong convergence theorem for a family of G–nonexpansive mappings in a complete CAT(0) space endowed with graph by using the Halpern iteration process

**Theorem 3.5.** Let \( C \) be a convex subset of a complete CAT(0) space endowed with graph \( (X,d,G) \). Suppose that \( G \) is transitive and \( E(G) \) is convex. Let \( T : C \rightarrow C \) be edge-preserving and \( \{a_n\} \) be a sequence in \([0,1]\). Let \( \{x_n\} \) be a sequence defined by \( x_1 \in C \) and

\[
x_{n+1} = a_n u + (1 - a_n)Tx_n \quad \forall n \geq 2,
\]

where \( u \in C \) such that \((u,Tu) \in E(G)\). If \( \{x_n\} \) dominates \( u \), then \( \{x_n,x_{n+1}\} \), \((u,x_n)\) and \((x_n,Tx_n)\) are in \( E(G) \) for any \( n \in \mathbb{N} \).

**Proof.** We prove by induction. Since \( E(G) \) is convex, \((u,u)\) and \((u,Tu)\) are in \( E(G) \), we have \((u,x_1) \in E(G)\). Then \((Tu,Tx_1) \in E(G)\), since \( T \) is edge-preserving. Because \( G \) is transitive, we have \((u,Tx_1) \in E(G)\). By convexity of \( E(G) \) and \((u,Tx_1)\), \((Tu,Tx_1) \in E(G)\), we get \((x_1,Tx_1) \in E(G)\). By assumption, \((x_1,u) \in E(G)\). So, by convexity of \( E(G) \), we get \((x_1,x_2) \in E(G)\).

Next, assume that \((x_k,x_{k+1})\), \((u,Tx_k)\) and \((x_k,Tx_k)\) are in \( E(G)\). Then \((Tx_k,Tx_{k+1}) \in E(G)\), since \( T \) is edge-preserving. By transitivity of \( G \), we have \((u,Tx_{k+1}) \in E(G)\). By convexity of \( E(G) \) and \((u,Tx_{k+1})\), \((Tx_k,Tx_{k+1}) \in E(G)\), we get \((x_{k+1},Tx_{k+1}) \in E(G)\). Since \( u \) is dominated by \( \{x_n\} \), we have \((x_{k+1},u) \in E(G)\). By convexity of \( E(G) \), we get \((x_{k+1},x_{k+2}) \in E(G)\).

Therefore, by induction, we can conclude that \((x_n,x_{n+1})\), \((u,x_n)\) and \((x_n,Tx_n)\) are in \( E(G)\), for all \( n \in \mathbb{N} \).

**Remark 3.2.** The sequence \( \{x_n\} \) generated by (3.1) is called the Halpern iteration process.

**Theorem 3.6.** Let \( C \) be a nonempty convex subset of a complete CAT(0) space endowed with graph \( (X,d,G) \). Suppose \( G \) is transitive and \( E(G) \) is convex. Let \( T : C \rightarrow C \) be a G-nonexpansive mapping with nonempty fixed point set \( F(T) \) and...
From Lemma 2.6, let \( z \) be an arbitrary chosen and \( \{x_n\} \) is iteratively generated by

\[
x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)Tx_n \quad \forall n \geq 2,
\]

where \( \{\alpha_n\} \) is a sequence in \((0,1)\) satisfying

\[\text{(C1)} \lim_{n \to \infty} \alpha_n = 0; \]
\[\text{(C2)} \sum_{n=1}^{\infty} \alpha_n = \infty; \]
\[\text{(C3)} \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty \text{ or } \lim_{n \to \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1.\]

If \( \{x_n\} \) is dominated by \( p \) for some \( p \in F(T) \) and \( \{x_n\} \) dominates \( u \), then \( \{x_n\} \) converges strongly to \( z \in F(T) \) which is nearest to \( u \).

**Proof.** We first show that the sequence \( \{x_n\} \) is bounded. Let \( p \) be any point in \( F(T) \). Consider

\[
d(x_{n+1}, p) = d(\alpha_n u \oplus (1 - \alpha_n)Tx_n, p)
\leq \alpha_n d(u, p) + (1 - \alpha_n)d(x_n, p)
\leq \max\{d(u, p), d(x_n, p)\}.
\]

This implies that \( \{x_n\} \) is bounded. By the nonexpansiveness of \( T \) and \((x_n, z) \in E(G)\), we have \( d(Tx_n, Tp) \leq d(x_n, p) \leq \max\{d(u, p), d(x_1, p)\} \). This shows that \( \{Tx_n\} \) is also bounded. Consider the following calculation:

\[
d(x_{n+1}, x_n) = d(\alpha_n u \oplus (1 - \alpha_n)Tx_n, \alpha_{n-1} u \oplus (1 - \alpha_{n-1})Tx_{n-1})
\leq d(\alpha_n u \oplus (1 - \alpha_n)Tx_n, \alpha_n u \oplus (1 - \alpha_n)Tx_{n-1})
\leq d(\alpha_n u \oplus (1 - \alpha_n)Tx_n, x_n - x_{n-1}) + d(x_n, x_{n-1})
\leq (1 - \alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|M.
\]

for some \( M \geq 0 \). By (C2), (C3) and Lemma 2.5, we can conclude that \( d(x_n, x_{n+1}) \to 0 \) as \( n \to \infty \). Consequently, by (C1),

\[
d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n)
= d(x_n, x_{n+1}) + d(\alpha_n u \oplus (1 - \alpha_n)Tx_n, Tx_n)
\leq d(x_n, x_{n+1}) + \alpha_n d(u, Tx_n) \to 0.
\]

From Lemma 2.6, let \( z = \lim_{k \to \infty} x_k \), \( x_k \) given by (2.6), is the nearest point to \( u \).

Consider

\[
d^2(x_{n+1}, z) = d^2(\alpha_n u \oplus (1 - \alpha_n)Tx_n, z)
\leq \alpha_n d^2(u, z) + (1 - \alpha_n)d^2(Tx_n, z) - \alpha_n(1 - \alpha_n)d^2(u, Tx_n)
\leq (1 - \alpha_n)d^2(x_n, z) + \alpha_n d^2(u, z) - (1 - \alpha_n)d^2(u, Tx_n).
\]

Lemma 2.4 guarantee that

\[
\limsup_{n \to \infty} (d^2(u, z) - d^2(u, x_n)) \leq 0.
\]
Moreover, by law of cosine, we get that
\[
d^2(x_n, Tx_n) \geq d^2(x_n, u) + d^2(Tx_n, u) - 2d(x_n, u)d(Tx_n, u)
\]
\[
= (d(x_n, u) - d(Tx_n, u))^2 \geq 0.
\]
Since \(d^2(x_n, Tx_n) \to 0\), this implies that \(d(x_n, u) - d(Tx_n, u) \to 0\), and
\[
\lim_{n \to \infty} d(u, x_n) = \lim_{n \to \infty} d(u, Tx_n) = \lim_{n \to \infty} (1 - \alpha_n)d(u, Tx_n).
\] (3.5)
From (3.4) and (3.5),
\[
\lim_{n \to \infty} (d^2(u, z) - (1 - \alpha_n)d^2(u, Tx_n)) = d^2(u, z) - \lim_{n \to \infty} (1 - \alpha_n)d^2(u, x_n)
\]
\[
= d^2(u, z) - \lim_{n \to \infty} d^2(u, x_n)
\]
\[
= \lim_{n \to \infty} (d^2(u, z) - d^2(u, x_n)) \leq 0
\]
Hence, from (3.3) and Lemma 2.5, we get \(\lim_{n \to \infty} d^2(x_n, z) = 0\). The proof has been completed.

This following lemma, the extension of Lemma 2.3, will be used for proving the last main theorem.

**Lemma 3.7.** Let \(C\) be a nonempty closed convex subset of a complete CAT(0) space endowed with graph \((X, d, G)\), and let \(\{T_n : n \in \mathbb{N}\}\) be a family of single-valued \(G\)-nonexpansive mappings on \(C\). Suppose that \(\bigcap_{n=1}^{\infty} F(T_n)\) is nonempty. Define \(T : C \to C\) by
\[
T_x = \bigoplus_{n=1}^{\infty} \lambda_n T_n x
\]
for all \(x \in C\), where \(\{\lambda_n\} \subset (0, 1)\) with \(\sum_{n=1}^{\infty} \lambda_n = 1\) and \(\sum_{i=n}^{\infty} \lambda_i' \to 0\) as \(n \to \infty\). Then \(T \) is \(G\)-nonexpansive and \(F(T) = \bigcap_{n=1}^{\infty} F(T_n)\).

**Proof.** Let \(y_0 \in \bigcap_{n=1}^{\infty} F(T_n)\) be arbitrary given. Since \(d(T_n(x), y_0) \leq d(x, y_0)\), for all \(n \in \mathbb{N}\), \(\{T_n(x)\}\) is bounded. For each \(n \in \mathbb{N}\), let \(w_n : C \to C\) given by
\[
w_n x = \sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} T_i x + \sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} T_i x + \cdots + \sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} T_i x.
\] (3.6)
Since \(w_1 = T_1\), so \(w_1\) is \(G\)-nonexpansive. Suppose that \(w_k\) is \(G\)-nonexpansive and let \(x, y \in X\) be such that \((x, y) \in E(G)\). Consider
\[
d(w_{k+1} x, w_{k+1} y) = d\left(\sum_{i=1}^{k+1} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} w_k x \oplus \sum_{i=1}^{k+1} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} T_{k+1} x, \sum_{i=1}^{k} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} w_k y \oplus \sum_{i=1}^{k+1} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} T_{k+1} y\right)
\]
\[
\leq \sum_{i=1}^{k+1} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} d(w_k x, w_k y) + \sum_{i=1}^{k+1} \frac{\lambda_i}{\sum_{i=1}^{\infty} \lambda_i} d(T_{k+1} x, T_{k+1} y)
\]
\[
\leq d(x, y).
\]
Let \( G \) be a nonempty convex subset of a complete CAT(0) space endowed with graph \( (X, d, G) \). Suppose \( G \) is transitive and \( E(G) \) is convex. Let \( \{ T_n : C \to C \} \) be a countable family of \( G \)-nonexpansive mappings with \( \bigcap_{n=1}^{\infty} F(T_n) \). Let \( \{ \lambda_n \} \subset (0, 1) \) such that \( \sum_{n=1}^{\infty} \lambda_n = 1 \) and \( \sum_{n=1}^{\infty} \lambda'_n \to 0 \) as \( n \to 0 \). Suppose that \( u, x_1 \in C \) are arbitrary chosen and \( x_n \) is defined by

\[
x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) w_n x_n \quad \forall n \geq 2,
\]

where \( w_n \) defined by (3.6) and \( \{ \alpha_n \} \in (0, 1) \) satisfying

\begin{enumerate}
  \item \( \lim_{n \to \infty} \alpha_n = 0 \);
  \item \( \sum_{n=1}^{\infty} \alpha_n = \infty \);
  \item \( \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty \) or \( \lim_{n \to \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1 \).
\end{enumerate}

If \( \{ x_n \} \) is dominated by \( p \) for some \( p \in \bigcap_{n=1}^{\infty} F(T_n) \) and \( \{ x_n \} \) dominates \( u \). Then \( \{ x_n \} \) converges to \( z \in \bigcap_{n=1}^{\infty} F(T_n) \) which is nearest to \( u \).

**Proof.** Let \( \{ w_n \} \) and \( T \) be as in the proof of lemma [3.7] so \( w_n \) is \( G \)-nonexpansive and \( \bigcap_{n=1}^{\infty} F(w_n) = F(T) = \bigcap_{n=1}^{\infty} F(T_n) \) and \( w_n(p) = p \) for all \( p \in F(T) \). Then we follow the proof from Theorem [3.6] by replace \( w_n \) by \( T_n \). Then the proof is complete.

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References


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