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Modified Proximal Point Algorithms for Solving Fixed Point Problem and Convex Minimization Problem in Non-Positive Curvature Metric Spaces

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Abstract : In this paper, we aim to introduce new four steps of proximal point algorithm for nonexpansive mappings in non-positive curvature metric spaces, namely CAT(0) spaces and also prove that the sequence generated by the proposed algorithm converges to a minimizer of a convex function and common fixed point of such mappings.

Keywords : CAT(0) space; convex minimization problem; fixed point problem; proximal point algorithm; iteration process

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1 Introduction

Let E be a uniformly convex Banach space, C be a nonempty closed convex set and $C \subseteq E$. A self-mapping T in C is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|\ \forall x, y \in C$. In this article, $\mathbb{N}$ denotes the set of all positive integers and $F(T) := \{x : Tx = x\}$.

The iteration process for approximating fixed points were studied by many authors as follows.

In this kind of iteration, we choose $x_1 \in X$ arbitrarily and $\{x_n\}_{n=1}^{\infty}$ was introduced iteratively by the following successive iteration method:

$$x_{n+1} = Tx_n,\ \forall n \geq 1. \tag{1.1}$$

We called the iteration method (1.1) as Picard iteration.

The iterative scheme of $\{x_n\}_{n=1}^{\infty}$ was given by

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n,\ \forall n \geq 1, \tag{1.2}$$

where $\lambda \in (0, 1)$. We called the iteration method (1.2) as Krasnoselskij iteration.

In 1953, Mann introduced the well-known iteration process, called Mann iteration, which start from $x_1 \in E$ and defined the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n,\ \forall n \geq 1, \tag{1.3}$$

where the sequence $\{\alpha_n\}$ is in $[0, 1]$.

In 1974, Ishikawa introduced the iteration as follows: the sequences $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(y_n) \\ y_n = (1-\beta_n)x_n + \beta_n T(x_n),\ \forall n \geq 1, \end{cases} \tag{1.4}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$. This iteration can reduce to the iteration (1.3) when $\beta_n = 0,\ \forall n \geq 1$.

In 2000, Noor introduced the following iteration process, by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(y_n) \\ y_n = (1-\beta_n)x_n + \beta_n T(z_n) \\ z_n = (1-\gamma_n)x_n + \gamma_n T(x_n),\ \forall n \geq 1, \end{cases} \tag{1.5}$$

where $x_1 \in C$ is arbitrary and the real sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $[0, 1]$.

In 2007, Agarwal et al. [1] defined the iteration process, namely S-iteration in a Banach space as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T(y_n) \\ y_n = (1-\beta_n)x_n + \beta_n T(x_n), \ \forall n \geq 1, \end{cases} \tag{1.6}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$. They proved that this iteration process converges to a fixed point of contractive mapping T and showed rate of convergence faster than iteration (1.3) and (1.4) respectively.

In 2014, Abbas et al. [2] defined the following iteration, where $\{x_n\}_{n=1}^{\infty}$ was constructed from arbitrary $x_1 \in C$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \ \forall n \geq 1, \end{cases} \tag{1.7}$$

where the real sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$, are in $[0, 1]$.

Very recently, Thakur et al. [3] modified iteration for finding fixed points of nonexpansive mappings, the sequence $\{x_n\}$ is generated by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tz_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \ \forall n \geq 1, \end{cases} \tag{1.8}$$

where the real sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, are in $[0, 1]$.

Moreover, the initials of CAT are in honor for three mathematicians include E. Cartan, A. D. Alexanderov and V. A. Toponogov, who have made important contributions to the understanding of curvature via inequalities for the distance function. A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as *thin* as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Kirk [4] first studied the theory of fixed point in CAT(κ) spaces. Later on, many authors generalized the notion of CAT(κ) given in [4, 5, 6], mainly focusing on CAT(0) spaces (see e.g., [7, 8, 9]). Various results for solving a fixed point problem of some nonlinear mappings in the CAT(0) spaces can also be found for examples, in [10, 11, 12].

On the other hand, let (X, d) be a geodesic metric space and f be a proper and convex function from the set X to $(-\infty, \infty]$. Some major problems in optimization is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

The set of minimizers of f was denoted by $\arg\min_{y\in X} f(y)$. In 1970, Martinet [13] first introduced the effective tool for solving this problem which is the proximal point algorithm (for short term, the PPA). Later in 1976, Rockafellar [14] found that the PPA converges to the solution of the convex minimization problem in Hilbert space.

Let f be a proper, convex, and lower semi-continuous function on a Hilbert space H which attains its minimum. The PPA is defined by $x_1 \in H$ and

$$x_{n+1} = \arg\min_{y\in H}[f(y) + \frac{1}{2\lambda_n} \parallel y - x_n \parallel^2]$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\sum_{n=1}^{\infty} \lambda_n = \infty$. However, as shown by Guler [15], the PPA does not necessarily converges strongly in general. In 2000, Kamimura and Takahashi [16] combined the PPA with Halpern algorithm [17] so that the strong convergence is guaranteed (see also [18, 19]).

In 2013, Bačák [20] introduced the PPA in a CAT(0) space (X, d) as follows: let $x_1 \in X$ and

$$x_{n+1} = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)]$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. Based on the concept of the Fejér monotonicity, it was shown that, if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer (see also [20]). Recently, in 2014, Bačák [21] employed a split version of the PPA for minimizing a sum of convex functions in complete CAT(0) spaces. Other interesting results can also be found in [20, 22].

Recently, many convergence results by the PPA for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds [22]. The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds (see in, [23, 24]).

Very recently, Cholamjiak et al [25] introduced a modified PPA combining with S-iteration for two nonexpansive mappings in CAT(0) spaces as follows:

$$\begin{cases} z_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)] \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T_1 z_n \\ x_{n+1} = (1 - \alpha_n)T_1 x_n \oplus \alpha_n T_2 y_n \end{cases} \tag{1.9}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$.

Motivated and inspired by (1.8) and (1.9), we introduce a new iterative scheme by modified PPA combining with iteration (1.8) which is defined by the following manner:

$$\begin{cases} z_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ w_n = (1-\alpha_n)x_n \oplus \alpha_n S z_n, \\ y_n = (1-\beta_n)w_n \oplus \beta_n T w_n, \\ x_{n+1} = (1-\gamma_n)S w_n \oplus \gamma_n T y_n \end{cases} \tag{1.10}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$.

The propose in this paper, we introduce new iterative scheme combining with PPA for two nonexpansive mapping in non-positive curvature metric spaces, namely CAT(0) spaces and under suitable conditions, we also prove that the sequence generated by (1.10) converges to a minimizer of a convex function and common fixed point of such mappings.

2 Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a mapping γ from $[0, l] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(l) = y$, and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Especially, γ is an isometry and $d(x, y) = l$. The image $\gamma([0, l])$ of γ is called a geodesic segment joining x and y.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X and a geodesic segment between each pair of vertices. A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ is Euclidean space $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for each $i, j \in \{1, 2, 3\}$. A geodesic space is called a CAT(0) space if, for each geodesic triangle $\Delta(x_1, x_2, x_3)$ in X and its comparison triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$, the CAT(0) inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is satisfied for all $x, y \in \Delta$ and comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$. A subset C of a CAT(0) space is called convex if $[x, y] \subset C$ for all $x, y \in C$. For more details, the readers may consult [26]. A geodesic space X is a CAT(0) space if and only if

$$d^2((1-\alpha))x \oplus \alpha y, z) \leq (1-\alpha)d^2(x, z) + \alpha d^2(y, z) - t(1-\alpha)d^2(x, y) \tag{2.1}$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$ (see in, [27]). In particular, if x, y, z are points in X and $\alpha \in [0, 1]$, then we have

$$d((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)d(x, z) + \alpha d(y, z). \tag{2.2}$$

The examples of CAT(0) spaces via Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature, Hyperbolic spaces and trees.

Let C be a nonempty closed and convex subset of a complete CAT(0) space. For each point $x \in X$, there exists a unique point of C denoted by $P_C x$, such that

$$d(x, P_C x) = \inf_{y \in C} d(x, y).$$

A mapping P_C is said to be the metric projection from X onto C.

Let $\{x_n\}$ be a bounded sequence in the set C. For any $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(\{x_n\}) = r(x, \{x_n\})\}.$$

In a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see [28]).

Definition 2.1. *A sequence $\{x_n\}$ in a CAT(0) space X is called Δ-converge to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.*

We can write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the $\Delta - limit$ of $\{x_n\}$. We denote $w_\Delta(x_n) := \cup\{A(\{u_n\})\}$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is called regular if $r(\{u_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. Every bounded sequence in X has a Δ-convergent subsequence [7].

Lemma 2.2. *[12] Let C be a closed and convex subset of a complete CAT(0) space X and $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x$. Then $x = Tx$.*

Lemma 2.3. *[12] If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}, \{u_n\}$ is a sequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Recall that a function $f : C \to (-\infty, \infty]$ define on the set C is convex. For any geodesic $\gamma : [a, b] \to C$, the function $f \circ \gamma$ is convex. We say that a function f defined on C is lower semi-continuous at a point $x \in C$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n)$$

for each sequence $x_n \to x$. A function f is called lower semi-continuous on C if it is lower semi-continuous at any point in C.

For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in CAT(0) spaces as follows:

$$J_\lambda(x) = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda}d^2(y,x)] \tag{2.3}$$

for all $x \in X$. The mapping J_λ is well define (see in [29]).

Let $f : X \to (-\infty,\infty]$ be a proper convex and lower semi-continuous function. It was shown in [21] that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $\arg\min_{y\in X} f(y)$ of minimizers of f.

Lemma 2.4. *[29] Let (X,d) be a complete CAT(0) space and $f : X \to (-\infty,\infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive.*

Lemma 2.5. *[30] Let (X,d) be a complete CAT(0) space and $f : X \to (-\infty,\infty]$ be proper convex and lower semi-continuous. Then, for all $x,y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d^2(J_\lambda x,y) - \frac{1}{2\lambda}d^2(x,y) + \frac{1}{2\lambda}d^2(x,J_\lambda x) + f(J_\lambda x) \le f(y).$$

Proposition 2.6. *[29] (The resolvent identity) Let (X,d) be a complete CAT(0) space and $f : X \to (-\infty,\infty]$ be proper convex and lower semi-continuous. Then the following identity holds:*

$$J_\lambda x = J_\mu(\frac{\lambda-\mu}{\lambda}J_\lambda x \oplus \frac{\mu}{\lambda}x)$$

for all $x \in X$ and $\lambda > \mu > 0$.

3 Main Results

We now construct and prove useful lemma to prove our main results.

Lemma 3.1. *Let (X,d) be a complete CAT(0) space and $f : X \to (\infty,\infty]$ be a proper convex and lower semi-continuous function. Let S and T are two nonexpansive mappings on X such that $\omega = F(S) \cap F(T) \cap \arg\min_{y\in X} f(y)$ is nonempty. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences such that $0 < a \le \alpha_n, \beta_n, \gamma_n \le b < 1$ $\forall n \in \mathbb{N}$ and for some a,b and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ $\forall n \in \mathbb{N}$ and for some λ. Let the sequence $\{x_n\}$ is defined by*

$$\begin{cases} z_n = \arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}d^2(y,x_n)] \\ w_n = (1-\alpha_n)x_n \oplus \alpha_n S z_n \\ y_n = (1-\beta_n)w_n \oplus \beta_n T w_n \\ x_{n+1} = (1-\gamma_n)Sw_n \oplus \gamma_n T y_n, \end{cases} \tag{3.1}$$

for all $n \ge 1$. Then we have the statements hold:

(1) $\lim_{n\to\infty} d(x_n, \tilde{q})$ *exists for all* $\tilde{q} \in \omega$,

(2) $\lim_{n\to\infty} d(x_n, z_n) = 0$,

(3) $\lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Let $\tilde{q} \in \omega$. Then $\tilde{q} = S\tilde{q} = T\tilde{q}$ and $f(\tilde{q}) \leq f(y)$ for all $y \in X$. It follows that

$$f(\tilde{q}) + \frac{1}{2\lambda_n} d^2(\tilde{q}, \tilde{q}) \leq f(y) + \frac{1}{2\lambda_n} d^2(y, \tilde{q})$$

for all $y \in X$ and hence $\tilde{q} = J_{\lambda_n}\tilde{q}$ for all $n \in \mathbb{N}$.

(1) To prove that $\lim_{n\to\infty} d(x_n, \tilde{q})$ exists. Noting that $z_n = J_{\lambda_n} x_n$ for all $n \in \mathbb{N}$. By Lemma 2.4, we have

$$d(z_n, \tilde{q}) = d(J_{\lambda_n} x_n, J_{\lambda_n}\tilde{q}) \leq d(x_n, \tilde{q}). \tag{3.2}$$

Also, we have, by (2.2) and (3.1)

$$\begin{aligned} d(w_n, \tilde{q}) &= d((1-\alpha_n)x_n \oplus \alpha_n Sz_n, \tilde{q}) \\ &\leq (1-\alpha_n)d(x_n, \tilde{q}) + \alpha_n d(Sz_n, \tilde{q}) \\ &\leq (1-\alpha_n)d(x_n, \tilde{q}) + \alpha_n d(z_n, \tilde{q}) \\ &\leq d(x_n, \tilde{q}), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(y_n, \tilde{q}) &= d((1-\beta_n)w_n \oplus \beta_n Tw_n, \tilde{q}) \\ &\leq (1-\beta_n)d(w_n, \tilde{q}) + \beta_n d(Tw_n, \tilde{q}) \\ &\leq (1-\beta_n)d(w_n, \tilde{q}) + \beta_n d(w_n, \tilde{q}) \\ &= d(w_n, \tilde{q}) \\ &\leq d(x_n, \tilde{q}). \end{aligned} \tag{3.4}$$

So, it follows from (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned} d(x_{n+1}, \tilde{q}) &= d((1-\gamma_n)Sw_n \oplus \gamma_n Ty_n, \tilde{q}) \\ &\leq (1-\gamma_n)d(Sw_n, \tilde{q}) + \gamma_n d(Ty_n, \tilde{q}) \\ &\leq (1-\gamma_n)d(w_n, \tilde{q}) + \gamma_n d(y_n, \tilde{q}) \\ &\leq (1-\gamma_n)d(x_n, \tilde{q}) + \gamma_n d(x_n, \tilde{q}) \\ &= d(x_n, \tilde{q}). \end{aligned} \tag{3.5}$$

This shows that $\lim_{n\to\infty} d(x_n, \tilde{q})$ exists. Hence $\lim_{n\to\infty} d(x_n, \tilde{q}) = k$ for some k.

(2) To show that $\lim_{n\to\infty} d(x_n, z_n) = 0$. By Lemma 2.5, we see that

$$\frac{1}{2\lambda_n} d^2(z_n, \tilde{q}) - \frac{1}{2\lambda_n} d^2(x_n, \tilde{q}) + \frac{1}{2\lambda_n} d^2(x_n, z_n) \leq f(\tilde{q}) - f(z_n).$$

Since $f(\tilde{q}) \leq f(z_n)$ for all $n \in \mathbb{N}$, it follows that

$$d^2(x_n, z_n) \leq d^2(x_n, \tilde{q}) - d^2(z_n, \tilde{q}).$$

In order to show that $\lim_{n\to\infty} d(x_n, z_n) = 0$, we need to prove show that

$$\lim_{n\to\infty} d(z_n, \tilde{q}) = k.$$

In fact, from (3.3) and (3.5), we have

$$\begin{aligned} d(x_{n+1}, \tilde{q}) &\leq (1-\gamma_n) d(w_n, \tilde{q}) + \gamma_n d(y_n, \tilde{q}) \\ &\leq (1-\gamma_n) d(x_n, \tilde{q}) + \gamma_n d(y_n, \tilde{q}) \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(x_n, \tilde{q}) &\leq \frac{1}{\gamma_n}(d(x_n, \tilde{q}) - d(x_{n+1}, \tilde{q})) + d(y_n, \tilde{q}) \\ &\leq \frac{1}{a}(d(x_n, \tilde{q}) - d(x_{n+1}, \tilde{q})) + d(y_n, \tilde{q}) \end{aligned}$$

since $d(x_{n+1}, \tilde{q}) \leq d(x_n, \tilde{q})$ and $\gamma_n \geq a > 0$ for all $n \in \mathbb{N}$. Hence we have

$$k = \liminf_{n\to\infty} d(x_n, \tilde{q}) \leq \liminf_{n\to\infty} d(y_n, \tilde{q}).$$

It follows from (3.4) , we see that

$$\limsup_{n\to\infty} d(y_n, \tilde{q}) \leq \limsup_{n\to\infty} d(x_n, \tilde{q}) = k.$$

Hence, we have $\lim_{n\to\infty} d(y_n, \tilde{q}) = k$. Since

$$\begin{aligned} d(y_n, \tilde{q}) &\leq (1-\beta_n) d(w_n, \tilde{q}) + \beta_n d(Tw_n, \tilde{q}) \\ &\leq (1-\beta_n) d(x_n, \tilde{q}) + \beta_n d(w_n, \tilde{q}) \end{aligned}$$

simplifying

$$\begin{aligned} d(x_n, \tilde{q}) &\leq \frac{1}{\beta_n}(d(x_n, \tilde{q}) - d(y_n, \tilde{q})) + d(w_n, \tilde{q}) \\ &\leq \frac{1}{a}(d(x_n, \tilde{q}) - d(y_n, \tilde{q})) + d(w_n, \tilde{q}), \end{aligned}$$

which yields

$$k = \liminf_{n\to\infty} d(x_n, \tilde{q}) \leq \liminf_{n\to\infty} d(w_n, \tilde{q}).$$

On the other hand, by (3.3), we observe that

$$\limsup_{n\to\infty} d(w_n,\tilde{q}) \le \limsup_{n\to\infty} d(x_n,\tilde{q}) = k.$$

Hence, we obtain

$$\lim_{n\to\infty} d(w_n,\tilde{q}) = k.$$

From (3.3), we have

$$d(w_n,\tilde{q}) \le (1-\alpha_n)d(x_n,\tilde{q}) + \alpha_n d(z_n,\tilde{q}),$$

which is equivalent to

$$d(x_n,\tilde{q}) \le \frac{1}{\alpha_n}(d(x_n,\tilde{q}) - d(w_n,\tilde{q})) + d(z_n,\tilde{q}).$$

In the same way, we get

$$k = \liminf_{n\to\infty} d(x_n,\tilde{q}) \le \liminf_{n\to\infty} d(z_n,\tilde{q}),$$

and by (3.2), we see that

$$\limsup_{n\to\infty} d(z_n,\tilde{q}) \le \limsup_{n\to\infty} d(x_n,\tilde{q}) = k.$$

Therefore, we conclude that

$$\lim_{n\to\infty} d(z_n,\tilde{q}) = k.$$

This shows that

$$\lim_{n\to\infty} d(x_n,z_n) = 0. \tag{3.6}$$

(3) To show that

$$\lim_{n\to\infty} d(x_n,Sx_n) = \lim_{n\to\infty} d(x_n,Tx_n) = 0.$$

We observe that

$$\begin{aligned} d^2(w_n,\tilde{q}) &= d^2((1-\alpha_n)x_n \oplus \alpha_n Sz_n,\tilde{q}) \\ &\le (1-\alpha_n)d^2(x_n,\tilde{q}) + \alpha_n d^2(Sz_n,\tilde{q}) - \alpha_n(1-\alpha_n)d^2(x_n,Sz_n) \\ &\le (1-\alpha_n)d^2(x_n,\tilde{q}) + \alpha_n d^2(z_n,\tilde{q}) - \alpha_n(1-\alpha_n)d^2(x_n,Sz_n) \\ &\le (1-\alpha_n)d^2(x_n,\tilde{q}) + \alpha_n d^2(x_n,\tilde{q}) - \alpha_n(1-\alpha_n)d^2(x_n,Sz_n) \\ &\le d^2(x_n,\tilde{q}) - a(1-b)d^2(x_n,Sz_n). \end{aligned}$$

This implies that

$$\begin{aligned} d^2(x_n,Sz_n) &\le \frac{1}{a(1-b)}(d^2(x_n,\tilde{q}) - d^2(w_n,\tilde{q})) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Hence we have

$$\lim_{n\to\infty} d(x_n, Sz_n) = 0. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} d(x_n, Sx_n) &\leq d(x_n, Sz_n) + d(Sz_n, Sx_n) \\ &\leq d(x_n, Sz_n) + d(z_n, x_n) \\ &\to 0 \end{aligned} \tag{3.8}$$

as $n \to \infty$. Similarly, we obtain

$$\begin{aligned} d^2(y_n, \tilde{q}) &= d^2((1-\beta_n)w_n \oplus \beta_n Tw_n, \tilde{q}) \\ &\leq (1-\beta_n)d^2(w_n, \tilde{q}) + \beta_n d^2(Tw_n, \tilde{q}) - \beta_n(1-\beta_n)d^2(w_n, Tw_n) \\ &\leq (1-\beta_n)d^2(w_n, \tilde{q}) + \beta_n d^2(w_n, \tilde{q}) - \beta_n(1-\beta_n)d^2(w_n, Tw_n) \\ &\leq d^2(w_n, \tilde{q}) - a(1-b)d^2(w_n, Tw_n), \end{aligned}$$

which implies that

$$\begin{aligned} d^2(w_n, Tw_n) &\leq \frac{1}{a(1-b)}(d^2(w_n, \tilde{q}) - d^2(y_n, \tilde{q})) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Hence we have

$$\lim_{n\to\infty} d(w_n, Tw_n) = 0. \tag{3.9}$$

And also, we see that

$$\begin{aligned} d^2(x_{n+1}, \tilde{q}) &= d^2((1-\gamma_n)Sw_n \oplus \gamma_n Ty_n, \tilde{q}) \\ &\leq (1-\gamma_n)d^2(Sw_n, \tilde{q}) + \gamma_n d^2(Ty_n, \tilde{q}) - \gamma_n(1-\gamma_n)d^2(Sw_n, Ty_n) \\ &\leq (1-\gamma_n)d^2(w_n, \tilde{q}) + \gamma_n d^2(y_n, \tilde{q}) - \gamma_n(1-\gamma_n)d^2(Sw_n, Ty_n) \\ &\leq (1-\gamma_n)d^2(x_n, \tilde{q}) + \gamma_n d^2(x_n, \tilde{q}) - \gamma_n(1-\gamma_n)d^2(Sw_n, Ty_n) \\ &\leq d^2(x_n, \tilde{q}) - \gamma_n(1-\gamma_n)d^2(Sw_n, Ty_n) \\ &\leq d^2(x_n, \tilde{q}) - a(1-b)d^2(Sw_n, Ty_n). \end{aligned}$$

This implies that

$$\begin{aligned} d^2(Sw_n, Ty_n) &\leq \frac{1}{a(1-b)}(d^2(x_n, \tilde{q}) - d^2(x_{n+1}, \tilde{q})) \\ &\to 0 \end{aligned} \tag{3.10}$$

as $n \to \infty$. From (3.7) we have

$$\begin{aligned} d(w_n, x_n) &= d((1-\alpha_n)x_n \oplus \alpha_n Sz_n, x_n) \\ &\leq (1-\alpha_n)d(x_n, x_n) + \alpha_n d(Sz_n, x_n) \\ &= \alpha_n d(Sz_n, x_n) \\ &\to 0 \end{aligned} \tag{3.11}$$

as $n \to \infty$. It follows from (3.9) and (3.11) that

$$\begin{aligned} d(y_n, x_n) &= d((1-\beta_n)w_n \oplus \beta_n Tw_n, x_n) \\ &\leq (1-\beta_n)d(w_n, x_n) + \beta_n d(Tw_n, x_n) \\ &\leq (1-\beta_n)d(w_n, x_n) + \beta_n (d(Tw_n, w_n) + d(w_n, x_n)) \\ &= d(w_n, x_n) + \beta_n d(Tw_n, w_n) \\ &\to 0 \end{aligned} \tag{3.12}$$

as $n \to \infty$. From (3.6), (3.7), (3.10), (3.11) and (3.12), we get

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, Sz_n) + d(Sz_n, Sx_n) + d(Sx_n, Sw_n) + d(Sw_n, Ty_n) + d(Ty_n, Tx_n) \\ &\leq d(x_n, Sz_n) + d(z_n, x_n) + d(x_n, w_n) + d(Sw_n, Ty_n) + d(y_n, x_n) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. This means that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.13}$$

It follows from (3.8) and (3.13), we can conclude that

$$\lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.14}$$

This completes the proof.

□

Next, we prove Δ-convergence theorem of the proposed algorithm.

Theorem 3.2. *Let (X, d) be a complete CAT(0) space and $f : X \to (\infty, \infty]$ be a proper convex and lower semi-continuous function. Let S and T are two nonexpansive mappings on X such that $\omega = F(S) \cap F(T) \cap \arg\min_{y\in X} f(y)$ is nonempty. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ $\forall n \in \mathbb{N}$ and for some a, b and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ $\forall n \in \mathbb{N}$ and for some λ. Let the sequence $\{x_n\}$ is generated by* (3.1). *Then the sequence $\{x_n\}$ Δ-converges to a common element of ω.*

Proof. Since $\lambda_n \geq \lambda > 0$, by Proposition 2.6 and Lemma 3.1 (2),

$$\begin{aligned} d(J_\lambda x_n, J_{\lambda_n} x_n) &= d(J_\lambda x_n, J_\lambda(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n)) \\ &\leq d(x_n, (1 - \frac{\lambda}{\lambda_n}) J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n) \\ &= (1 - \frac{\lambda}{\lambda_n}) d(x_n, z_n) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. So, we obtain

$$\begin{aligned} d(x_n, J_\lambda x_n) &\leq d(x_n, z_n) + d(z_n, J_\lambda x_n) \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Lemma 3.1 (1) shows that $\lim_{n\to\infty} d(x_n, \tilde{q})$ exists for all $\tilde{q} \in \omega$ and Lemma 3.1 (3) that is $\lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$

Next, we show that $w_\triangle(x_n) \subset \omega$. Let $u \in w_\triangle(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. From Lemma 2.2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$ for some $v \in \omega$. So, $u = v$ by Lemma 2.3. This shows that $w_\triangle(x_n) \subset \omega$.

Finally, we show that the sequence $\{x_n\}$ Δ-converges to a point in ω. To finish proving, it suffices to show that $w_\triangle(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in w_\triangle(x_n) \subset \omega$ and $\{d(x_n, u)\}$ converges, by Lemma 2.3, we have $x = u$. Hence $w_\triangle(x_n) = \{x\}$. This completes the proof. □

If $S = T$ in Theorem 3.2 we obtain the following result.

Corollary 3.3. *Let (X, d) be a complete CAT(0) space and $f : X \to (\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T be a nonexpansive mapping on X such that $\omega = F(T) \cap argmin_{y\in X} f(y)$ is nonempty. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1\ \ \forall n \in \mathbb{N}$ and for some a, b and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0\ \ \forall n \in \mathbb{N}$ and for some λ. Let the sequence $\{x_n\}$ is generated by* (3.1). *Then the sequence $\{x_n\}$ Δ-converges to a common element of ω.*

Since every Hilbert space is a complete CAT(0) space, we obtain following result immediately.

Corollary 3.4. *Let H be a Hilbert space and $f : H \to (\infty, \infty]$ be a proper, convex and lower semi-continuous function. Let S and T are two nonexpansive mappings on X such that $\omega = F(S) \cap F(T) \cap argmin_{y\in H} f(y)$ is nonempty. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1\ \forall n \in \mathbb{N}$ and for some a, b and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0\ \ \forall n \in \mathbb{N}$ and for some λ. Let the sequence $\{x_n\}$ is generated by*

$$\begin{cases} z_n = \underset{y\in H}{\arg\min}[f(y) + \dfrac{1}{2\lambda_n} \parallel y - x_n \parallel^2] \\ w_n = (1-\alpha_n)x_n + \alpha_n S z_n \\ y_n = (1-\beta_n)w_n + \beta_n T w_n \\ x_{n+1} = (1-\gamma_n)S w_n + \gamma_n T y_n, \end{cases}$$

for all $n \geq 1$. Then the sequence $\{x_n\}$ weakly converges to common element of ω.

Next, we establish strong convergence theorem under mild conditions.

A self mapping T is said to be *semi-compact* if any sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 3.5. *Let (X, d) be a complete CAT(0) space and $f : X \to (\infty, \infty]$ be a proper convex and lower semi-continuous function. Let S and T are two nonexpansive mappings on X such that $\omega = F(S) \cap F(T) \cap \arg\min_{y \in X} f(y)$ is nonempty. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1\ \forall n \in \mathbb{N}$ and for some a, b and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0\ \ \forall n \in \mathbb{N}$ and for some λ. If S or T, or J_λ is semi-compact, then the sequence $\{x_n\}$ generated by* (3.1) *strongly converges to a common element of ω.*

Proof. Suppose that S is semi-compact. By Lemma 3.1 (3), we have

$$d(x_n, Sx_n) \to 0$$

as $n \to \infty$. Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \hat{x} \in X$. Again by Lemma 3.1 and Theorem 3.2, we have $d(\hat{x}, T\hat{x}) = d(\hat{x}, S\hat{x}) = 0$ and $d(\hat{x}, J_\lambda \hat{x}) = 0$, which shows that $\hat{x} \in \omega$. For other cases, we can also prove the strong convergence of $\{x_n\}$ to a common element of ω. This completes the proof. □

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