Viscosity Approximation Methods
for Generalized Equilibrium Problems
and Fixed Point Problems of Finite Family
of Nonexpansive Mappings in Hilbert Spaces

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1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the projection of $H$ onto $C.$ A mapping $T$ of $H$ into itself is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

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for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of $T$. Goebel and Kirk [7] showed that $F(T)$ is always closed convex and nonempty provided $T$ has a bounded trajectory. Recall that a mapping $A : C \to H$ is called monotone if for all $x, y \in C$,

$$\langle x - y, Ax - Ay \rangle \geq 0.$$ 

The mapping $A$ is called $\alpha$-inverse-strongly monotone if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$; see [4, 11, 14]. We know that if $T : C \to C$ is nonexpansive, then $A = I - T$ is $\frac{1}{2}$-inverse strongly monotone; see [15, 16, 17] for more details.

The classical variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0 \quad \forall \quad v \in C.$$ 

The set of solutions of variational inequality is denoted by $VI(C, A)$.

The variational inequality has been extensively studied in the literature; see [3, 5, 14]. Let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and a nonlinear mapping $A : C \to H$. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z \rangle \geq 0, \quad \forall \quad y \in C.$$ 

(1.1)

The set of such $z \in C$ is denoted by $EP$ i.e.,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z \rangle \geq 0, \quad \forall \quad y \in C\}.$$

In the case of $A \equiv 0$, $EP$ is denoted by $EP(F)$. In the case of $F \equiv 0$, $EP$ is denoted by $VI(C, A)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see [2, 12].

For $r > 0$, let $T_r : H \to C$ be defined by

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall \quad y \in C \right\}.$$ 

(1.2)

Combettes and Hirstoaga [6] showed that under some suitable conditions of $F$, $T_r$ is single-valued and firmly nonexpansive and satisfies $F(T_r) = EP(F)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. In 1999, Atsushiba and Takahashi [1] defined the mapping $W_n$ as follows:

$$U_{n,1} = \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I,$$

$$U_{n,2} = \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I,$$

$$U_{n,3} = \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I,$$

$$\vdots$$

$$U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I,$$

$$W_n = U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I,$$
where \( \{ \lambda_n \} \) is \( \subseteq [0, 1] \). This mapping is called the \( W \)-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \). In 2000, Takahashi and Shimoji [18] proved that if \( X \) is a strictly convex Banach space, then \( F(W_n) = \bigcap_{i=1}^N F(T_i) \), where 

\[ 0 \leq \lambda_{n,i} < 1, i = 1, 2, \ldots, N. \]

In 2007, Takahashi and Takahashi [19] introduced the following iterative scheme by the viscosity approximation method in a real Hilbert space \( H \). They defined the iterative sequences \( \{ x_n \} \) and \( \{ u_n \} \) as follows: \( x_1 \in H \) and

\[
\begin{align*}
F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) &\geq 0, \quad \forall y \in C, \\
x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) Tu_n, \quad \forall n \in \mathbb{N},
\end{align*}
\]

(1.3)

where \( f : H \to H \) is a contraction mapping with \( \{ \alpha_n \} \subseteq [0, 1], \{ r_n \} \subseteq (0, \infty) \). They proved under some suitable conditions on the sequences \( \{ \alpha_n \}, \{ r_n \} \) and bifunction \( F \), that \( \{ x_n \} \) and \( \{ u_n \} \) converge strongly to \( z \in \bigcap_{i=1}^N F(T_i) \cap EP(F) \), where \( z = P_{\bigcap_{i=1}^N F(T_i) \cap EP} u \).

In 2008, Takahashi and Takahashi [20] introduced a hybrid iterative method for finding a common element of \( EP \) and \( F(T) \). They defined \( \{ x_n \} \) as follows: \( u, x_1 \in C \) and

\[
\begin{align*}
F(z_n, y) + (Ax_n, y - z_n) + \frac{1}{\lambda_n} (y - z_n, z_n - x_n) &\geq 0, \quad \forall y \in C, \\
x_{n+1} &= \beta_n + (1 - \beta_n) Tu_n, \quad \forall n \in \mathbb{N},
\end{align*}
\]

(1.4)

where \( \{ \alpha_n \} \subseteq [0, 1], \{ \beta_n \} \subseteq [0, 1], \{ \lambda_n \} \subseteq [0, 2\alpha] \) and proved strong convergence of the scheme (1.4) to \( z \in \bigcap_{i=1}^N F(T_i) \cap EP \), where \( z = P_{\bigcap_{i=1}^N F(T_i) \cap EP} u \).

In 2009, Kangtunyakarn and Suantai [10] defined the mappings \( S_n \) as follows:

\[
\begin{align*}
U_{n,0} &= I, \\
U_{n,1} &= \alpha_1^{n,1} T_1 U_{n,0} + \alpha_2^{n,1} U_{n,0} + \alpha_3^{n,1} I, \\
U_{n,2} &= \alpha_1^{n,2} T_2 U_{n,1} + \alpha_2^{n,2} U_{n,1} + \alpha_3^{n,2} I, \\
U_{n,3} &= \alpha_1^{n,3} T_3 U_{n,2} + \alpha_2^{n,3} U_{n,2} + \alpha_3^{n,3} I, \\
&\vdots \\
U_{n,N-1} &= \alpha_1^{n,N-1} T_{N-1} U_{n,N-2} + \alpha_2^{n,N-1} U_{n,N-2} + \alpha_3^{n,N-1} I, \\
S_n &= U_{n,N} = \alpha_1^{n,N} T_N U_{n,N-1} + \alpha_2^{n,N} U_{n,N-1} + \alpha_3^{n,N} I
\end{align*}
\]

where \( \alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1] \) with \( \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1 \). The mapping \( S_n \) is called the \( S \)-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_3^{(n)} \).

If \( \alpha_2^{n,j} = 0, j = 1, 2, \ldots, N-1 \), then the mapping \( S \) reduces to the \( W \)-mapping defined by Atsushiba and Takahashi [1] and if \( \alpha_3^{n,j} = 0, j = 1, 2, \ldots, N \), then the mapping \( S \) reduces to the \( K \)-mapping defined by Kangtunyakarn and Suantai [9].

In this paper, we introduce the iterative scheme as follows. For given \( x_1 \in C \), let \( \{ z_n \} \) and \( \{ x_n \} \subseteq C \) be sequences generated by

\[
\begin{align*}
F(z_n, y) + (Ax_n, y - z_n) + \frac{1}{\lambda_n} (y - z_n, z_n - x_n) &\geq 0, \quad \forall y \in C, \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n z_n, \quad \forall n \in \mathbb{N},
\end{align*}
\]

(1.5)
where \( \{ \alpha_n \} \), \( \{ \beta_n \} \subset [0, 1] \) and \( \{ \lambda_n \} \subset [0, 2\alpha] \). Using the viscosity approximation method we will find a common element of the set of solutions of the equilibrium problem and the set of fixed points of a finite family of nonexpansive mappings in a Hilbert space. Then, we shall prove a strong convergence theorem which is connected with Kangtunyakarn and Suantai [10] and Takahashi and Takahashi’s results [19].

2 Preliminaries

Let \( C \) be a nonempty closed convex subset of \( H \). Then, for any \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_C(x) \), such that

\[
\| x - P_C(x) \| \leq \| x - y \|, \quad \forall y \in C.
\]

Such a \( P_C \) is called the metric projection of \( H \) onto \( C \).

Lemma 2.1. [15] Let \( x \in H \) and \( z \in C \). Then \( P_C(x) = z \) if and only if \( \langle x - z, z - y \rangle \geq 0 \) for all \( y \in C \).

Lemma 2.2. [21] Let \( \{ a_n \} \subset [0, \infty) \), \( \{ b_n \} \subset [0, \infty) \) and \( \{ c_n \} \subset (0, 1) \) be sequences of real numbers such that

\[
a_{n+1} \leq (1 - c_n)a_n + b_n \quad \text{for all } n \in \mathbb{N},
\]

\[
\sum_{n=1}^{\infty} c_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} b_n < \infty.
\]

Then \( \lim_{n \to \infty} a_n = 0 \).

For solving the equilibrium problem for a bifunction \( F : C \times C \to \mathbb{R} \), let us assume that \( F \) satisfies the following conditions:

(A1) \( F(x, x) = 0 \) for all \( x \in C \);

(A2) \( F \) is monotone, i.e., \( F(x, y) + F(y, x) \leq 0 \) for all \( x, y \in C \);

(A3) for each \( x, y, z \in C \), \( \lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y) \);

(A4) for each \( x \in C \), \( y \mapsto F(x, y) \) is convex and lower semicontinuous.

Lemma 2.3. [2] Let \( C \) be a nonempty closed convex subset of \( H \) and let \( F \) be a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1)-(A4). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in C \) such that

\[
F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.
\]

Lemma 2.4. [6] Assume that \( F : C \times C \to \mathbb{R} \) satisfies (A1)-(A4). For \( r > 0 \) and \( x \in H \), define a mapping \( T_r : H \to C \) as follows:

\[
T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}
\]
for all $x \in H$. Then, the following hold:

1. $T_r$ is single-valued;
2. $T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$,
   \[ \|T_r(x) - T_r(y)\|^2 \leq (T_r(x), T_r(y), x - y); \]
3. $F(T_r) = EP(F)$;
4. $EP(F)$ is closed and convex.

**Definition 2.5.** Let $C$ be a nonempty convex subset of a real Banach space. Let \( \{T_i\}_{i=1}^N \) be a finite family of nonexpansive mappings of $C$ into itself. For each $j = 1, 2, \ldots, N$, let $\alpha_j = (\alpha_{1j}, \alpha_{2j}, \alpha_{3j})$, where $\alpha_{1j}, \alpha_{2j}, \alpha_{3j} \in [0, 1]$ and $\alpha_{1j} + \alpha_{2j} + \alpha_{3j} = 1$. Kangtunyakarn and Suantai [10] defined the mapping $S: C \to C$ as follows:

\[
\begin{align*}
U_0 &= I \\
U_1 &= \alpha_{11}T_1U_0 + \alpha_{21}U_0 + \alpha_{31}I \\
U_2 &= \alpha_{12}T_2U_1 + \alpha_{22}U_1 + \alpha_{32}I \\
U_3 &= \alpha_{13}T_3U_2 + \alpha_{23}U_2 + \alpha_{33}I \\
& \vdots \\
U_{N-1} &= \alpha_{1(N-1)}T_{N-1}U_{N-2} + \alpha_{2(N-1)}U_{N-2} + \alpha_{3(N-1)}I \\
S &= U_N = \alpha_{1N}T_NU_{N-1} + \alpha_{2N}U_{N-1} + \alpha_{3N}I.
\end{align*}
\]

This mapping is called S-mapping generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

**Lemma 2.6.** [10] Let $C$ be a nonempty closed convex subset of a strictly convex Banach space. Let \( \{T_i\}_{i=1}^N \) be a finite family of nonexpansive mappings of $C$ into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_{1j}, \alpha_{2j}, \alpha_{3j}), j = 1, 2, 3, \ldots, N$, where $\alpha_{1j}, \alpha_{2j}, \alpha_{3j} \in [0, 1]$, $\alpha_{1j} + \alpha_{2j} + \alpha_{3j} = 1, \alpha_{3j} \in (0, 1)$ for all $j = 1, 2, \ldots, N$. Let $S$ be the S-mapping generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.

**Lemma 2.7.** [10] Let $C$ be a nonempty closed convex subset of Banach space. Let \( \{T_i\}_{i=1}^N \) be a finite family of nonexpansive mappings of $C$ into itself and for each $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, N\}$, let $\alpha_{ij}^{(n)} = (\alpha_{1ij}^{(n)}, \alpha_{2ij}^{(n)}, \alpha_{3ij}^{(n)})$, $\alpha_j = (\alpha_{1j}, \alpha_{2j}, \alpha_{3j})$, where $\alpha_{1ij}, \alpha_{2ij}, \alpha_{3ij} \in [0, 1], \alpha_{1ij}, \alpha_{2ij}, \alpha_{3ij} \in [0, 1], \alpha_{1ij} + \alpha_{2ij} + \alpha_{3ij} = 1$ and $\alpha_{1ij} + \alpha_{2ij} + \alpha_{3ij} = 1$. Suppose $\alpha_{1ij}^{(n)} \to \alpha_{1ij}$ as $n \to \infty$ for $i = 1, 2, 3$ and $j = 1, 2, 3, \ldots, N$. Let $S$ and $S_n$ be the S-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. Then $\lim_{n \to \infty} \|S_nx - Sx\| = 0$ for all $x \in C$.

3 Main Results

**Theorem 3.1.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $F: C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let $A$
be an $\alpha$-inverse strongly monotone mapping of $C$ into $H$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself with $\cap_{i=1}^N F(T_i) \cap EP \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_i^{n,j}\}_{j=1}^N \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_i^{n,N}\}_{i=1}^N \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_i^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \leq \theta_3 < 1$. Let $f$ be a contraction of $H$ into itself and let $S_n$ be the $S$-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{z_n\}, \{x_n\} \subset C$ be sequences generated by

$$
\begin{cases}
F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n z_n, \quad \forall n \in \mathbb{N},
\end{cases}
$$

(3.1)

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions:

(i) $0 < a \leq \lambda_n \leq b < 2\alpha$, $0 < c \leq \beta_n \leq d < 1$;

(ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;

(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ and $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ as $n \to \infty$, for all $j \in \{1, 2, 3, \ldots, N\}$.

Then $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^N F(T_i) \cap EP$, where $z = P_{\cap_{i=1}^N F(T_i) \cap EP} f(z)$.

Proof. Let $Q = P_{\cap_{i=1}^N F(T_i) \cap EP}$. Note that $f$ is a contraction with coefficient $k \in [0, 1)$. Then

$$
\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha \|x - y\|
$$

for all $x, y \in H$. Thus $Qf$ is a contraction of $H$ into itself. Since $H$ is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$. Such a $z \in H$ is an element of $C$.

Next, we show that $(I - \lambda_n A)$ is nonexpansive. Let $x, y \in C$. Since $A$ is $\alpha$-inverse strongly monotone and $\lambda_n < 2\alpha$ for all $n \in \mathbb{N}$, we obtain

$$
\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 = \|x - y - \lambda_n (Ax - Ay)\|^2
$$

$$
= \|x - y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle + \lambda_n^2 \|Ax - Ay\|^2
$$

$$
\leq \|x - y\|^2 - 2\alpha \lambda_n \|Ax - Ay\|^2 + \lambda_n^2 \|Ax - Ay\|^2
$$

$$
= \|x - y\|^2 + \lambda_n (\lambda_n - 2\alpha) \|Ax - Ay\|^2
$$

$$
\leq \|x - y\|^2.
$$

Therefore $\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \leq \|x - y\|^2$ for all $x, y \in C$. Thus $(I - \lambda_n A)$ is nonexpansive. Since

$$
F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C,
$$

we obtain

$$
F(z_n, y) + \frac{1}{\lambda_n} \langle y - z_n, z_n - (I - \lambda_n A)x_n \rangle \geq 0, \quad \forall y \in C.
$$
By Lemma 2.4, we have $z_n = T_{\lambda_n}(x_n - \lambda_n Ax_n)$ for all $n \in \mathbb{N}$.

Let $z \in \cap_{i=1}^{N} F(T_i) \cap EP$. Then $F(z, y) + \langle y - z, Az \rangle \geq 0$ for all $y \in C$. So $F(z, y) + \frac{1}{\lambda_n}(y - z, z - z + \lambda_n Az) \geq 0$ for all $y \in C$. Again by Lemma 2.4, we obtain $z = T_{\lambda_n}(z - \lambda_n Az)$. Since $I - \lambda_n A$ and $T_{\lambda_n}$ are nonexpansive, we have

$$\|z_n - z\|^2 = \|T_{\lambda_n}(x_n - \lambda_n Ax_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2 \leq \|x_n - z\|^2$$

and hence $\|z_n - z\| \leq \|x_n - z\|$. This implies that

$$\|x_{n+1} - z\|^2 = \|\alpha_n(f(x_n) - z) + \beta_n(x_n - z) + (1 - \alpha_n - \beta_n)(S_n z_n - z)\| \leq \alpha_n \|f(x_n) - z\| + \beta_n \|x_n - z\| + (1 - \alpha_n - \beta_n) \|z_n - z\| \leq \alpha_n \|x_n - z\| + \beta_n \|f(z) - z\| + (1 - \alpha_n) \|x_n - z\|$$

Putting $K = \max\{\|x_1 - z\|, \frac{1}{1-k} \|f(z) - z\|\}$. By (3.2), we can show by induction that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded. Hence $\{Ax_n\}, \{S_n z_n\}, \{f(x_n)\}$ are also bounded.

Next, we shall show that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. Putting $u_n = x_n - \lambda_n Ax_n$. Then, we have $z_{n+1} = T_{\lambda_n+1}(x_{n+1} - \lambda_{n+1} Ax_{n+1}) = T_{\lambda_n+1}u_{n+1}$, and $z_n = T_{\lambda_n}(x_n - \lambda_n Ax_n) = T_{\lambda_n}u_n$. Thus

$$\|z_{n+1} - z_n\| = \|T_{\lambda_n+1}u_{n+1} - T_{\lambda_n}u_n\| \leq \|T_{\lambda_n+1}u_{n+1} - T_{\lambda_n+1}u_n\| + \|T_{\lambda_n+1}u_n - T_{\lambda_n}u_n\| \leq \|u_{n+1} - u_n\| + \|T_{\lambda_n+1}u_n - T_{\lambda_n}u_n\|. \quad (3.3)$$

Since $I - \lambda_{n+1} A$ is nonexpansive, we obtain

$$\|u_{n+1} - u_n\| = \|x_{n+1} - \lambda_{n+1} Ax_{n+1} - x_n + \lambda_n Ax_n\| = \|(I - \lambda_{n+1} A)x_{n+1} - (I - \lambda_n A)x_n + (\lambda_n - \lambda_{n+1})Ax_n\| \leq \|x_{n+1} - x_n + |\lambda_n - \lambda_{n+1}|Ax_n\|. \quad (3.4)$$

By Lemma 2.4, we obtain

$$F(T_{\lambda_n}u_n, y) + \frac{1}{\lambda_n} \langle y - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n \rangle \geq 0, \quad \forall y \in C$$

and

$$F(T_{\lambda_{n+1}}u_n, y) + \frac{1}{\lambda_{n+1}} \langle y - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n \rangle \geq 0, \quad \forall y \in C.$$
and
\[ F(T_{\lambda_{n+1}}u_n, T_{\lambda_n}u_n) + \frac{1}{\lambda_{n+1}} (T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n) \geq 0. \] (3.6)

Summing up (3.5) and (3.6) and using (A2), we obtain
\[ \frac{1}{\lambda_{n+1}} (T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, T_{\lambda_{n+1}}u_n - u_n) + \frac{1}{\lambda_n} (T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n) \geq 0, \]
for all \( y \in C \). It follows that
\[ \left\langle T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n, \frac{T_{\lambda_{n+1}}u_n - u_n}{\lambda_{n+1}} - \frac{T_{\lambda_n}u_n - u_n}{\lambda_n} \right\rangle \geq 0. \]

This implies
\[ 0 \leq \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - u_n - \frac{\lambda_n}{\lambda_{n+1}} (T_{\lambda_{n+1}}u_n - u_n) \right\rangle \]
\[ = \left\langle T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n, T_{\lambda_n}u_n - T_{\lambda_{n+1}}u_n + \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right) (T_{\lambda_{n+1}}u_n - u_n) \right\rangle. \]

It follows that
\[ \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|^2 \leq \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| (\|T_{\lambda_{n+1}}u_n\| + \|u_n\|). \]

Hence, we obtain
\[ \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\|^2 \leq \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| L, \] (3.7)
where \( L = \sup\{\|u_n\| + \|T_{\lambda_{n+1}}u_n\| : n \in \mathbb{N}\} \). By (3.3), (3.4) and (3.7), we obtain
\[ \|z_{n+1} - z_n\| \leq \|u_{n+1} - u_n\| + \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| \]
\[ \leq \|x_{n+1} - x_n + b \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| A\|x_n\| + \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| L \] (3.8)

Next, we show that
\[ \lim_{n \to \infty} \|S_{n+1}z_n - S_nz_n\| = 0. \]
Viscosity Approximation Methods for Generalized Equilibrium Problems ...

For $k \in \{2, 3, \ldots, N\}$, we have

$$
\|U_{n+1,k}z_n - U_{n,k}z_n\| = \|\alpha_1^{n+1,k}T_kU_{n+1,k-1}z_n + \alpha_2^{n+1,k}U_{n+1,k-1}z_n + \alpha_3^{n+1,k}z_n
$$

$$-
\alpha_1^{n,k}T_kU_{n,k-1}z_n - \alpha_2^{n,k}U_{n,k-1}z_n - \alpha_3^{n,k}z_n\|
$$

$$
= \|\alpha_1^{n+1,k}(T_kU_{n+1,k-1}z_n - T_kU_{n,k-1}z_n) + (\alpha_1^{n+1,k} - \alpha_1^{n,k})T_kU_{n,k-1}z_n
$$

$$+
(\alpha_3^{n+1,k} - \alpha_3^{n,k})z_n + \alpha_2^{n+1,k}(U_{n+1,k-1}z_n - U_{n,k-1}z_n)
$$

$$+
(\alpha_2^{n+1,k} - \alpha_2^{n,k})U_{n,k-1}z_n\|
$$

$$\leq \alpha_1^{n+1,k}\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\|
$$

$$+
|\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\|
$$

$$+
|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|z_n\| + \alpha_2^{n+1,k}\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\|
$$

$$+|\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\|
$$

$$= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\|
$$

$$+
|\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\|
$$

$$+
|\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\|
$$

$$\leq \|U_{n+1,k-1}z_n - U_{n,k-1}z_n\| + |\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\|
$$

$$+
|\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\|
$$

$$+
|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|z_n\| + (\alpha_1^{n,k} - \alpha_1^{n+1,k})
$$

$$+
(\alpha_3^{n,k} - \alpha_3^{n+1,k})\|z_n\| + \|U_{n,k-1}z_n\|
$$

$$= (\alpha_1^{n+1,k} + \alpha_2^{n+1,k})\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\|
$$

$$+
|\alpha_1^{n+1,k} - \alpha_1^{n,k}|\|T_kU_{n,k-1}z_n\|
$$

$$+
|\alpha_2^{n+1,k} - \alpha_2^{n,k}|\|U_{n+1,k-1}z_n - U_{n,k-1}z_n\|
$$

$$+
|\alpha_3^{n+1,k} - \alpha_3^{n,k}|\|z_n\|
$$

$$+ (\alpha_1^{n,k} - \alpha_1^{n+1,k})(\|U_{n,k-1}z_n\|
$$

$$+ \|z_n\| + \|U_{n,k-1}z_n\|).
$$

By (3.9), we obtain that for each $n \in \mathbb{N}$,

$$
\|S_{n+1}z_n - S_nz_n\| = \|U_{n+1,N}z_n - U_{n,N}z_n\|
$$

$$\leq \|U_{n+1,z_n} - U_{n,z_n}\| + \sum_{j=2}^{N} |\alpha_1^{n+1,j} - \alpha_1^{n,j}|\|T_jU_{n,j-1}z_n\|
$$

$$+
\|U_{n,j-1}z_n\| + \sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}|\|z_n\| + \|U_{n,j-1}z_n\|
$$

$$= |\alpha_1^{n+1,1} - \alpha_1^{n,1}|\|T_1z_n - z_n\| + \sum_{j=2}^{N} |\alpha_1^{n+1,j} - \alpha_1^{n,j}|\|T_jU_{n,j-1}z_n\|
$$

$$+
\|U_{n,j-1}z_n\| + \sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}|\|z_n\| + \|U_{n,j-1}z_n\|).
This together with condition (iv), we obtain

$$\lim_{n \to \infty} ||S_{n+1} z_n - S_n z_n|| = 0. \quad (3.10)$$

Next, we show that

$$\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0.$$  

By (3.1), we obtain

$$||x_{n+1} - x_n|| = ||\alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n z_n - \alpha_{n-1} f(x_{n-1}) - \beta_{n-1} x_{n-1}$$

$$- (1 - \alpha_{n-1} - \beta_{n-1}) S_{n-1} z_{n-1}||$$

$$= ||\alpha_n (f(x_n) - f(x_{n-1})) + \beta_n (x_n - x_{n-1}) + (1 - \alpha_n - \beta_n)$$

$$(S_n z_n - S_{n-1} z_{n-1}) + (\alpha_n - \alpha_{n-1}) f(x_{n-1}) + (\alpha_{n-1} - \alpha_n + \beta_n - \beta_{n-1})$$

$$S_{n-1} z_{n-1} + (\beta_n - \beta_{n-1}) x_{n-1}||$$

$$\leq \alpha_n ||f(x_n) - f(x_{n-1})|| + \beta_n ||x_n - x_{n-1}|| + |1 - \alpha_n - \beta_n|$$

$$||S_n z_n - S_{n-1} z_{n-1}|| + |\alpha_n - \alpha_{n-1}||f(x_{n-1})|| + |\alpha_{n-1} - \alpha_n + \beta_n - \beta_{n-1}||$$

$$||S_{n-1} z_{n-1}|| + |\beta_n - \beta_{n-1}||x_{n-1}||$$

$$\leq (\alpha_n k + \beta_n) ||x_n - x_{n-1}|| + |1 - \alpha_n - \beta_n||S_n z_n - S_{n-1} z_{n-1}||$$

$$+ |\alpha_n - \alpha_{n-1}| K + |\alpha_{n-1} - \alpha_n + \beta_n - \beta_{n-1}| K$$

$$+ |\beta_n - \beta_{n-1}| ||x_{n-1}||,$$  

(3.11)

where $K = \sup\{||f(x_n)|| + ||S_n z_n||, n \in \mathbb{N}\}$. By (3.11) and since

$$||S_n z_n - S_{n-1} z_{n-1}|| \leq ||z_n - z_{n-1}|| + ||S_n z_{n-1} - S_{n-1} z_{n-1}||$$

and (3.8), (3.10), Lemma 2.2, conditions (ii) and (iii), we obtain

$$||x_{n+1} - x_n|| \leq (\alpha_n k + \beta_n) ||x_n - x_{n-1}|| + |1 - \alpha_n - \beta_n||(x_{n-1} - x_n||$$

$$+ b \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| ||Ax_n|| + \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| L + ||S_n z_{n-1} - S_{n-1} z_{n-1}||$$

$$+ |\alpha_n - \alpha_{n-1}| K + |\alpha_{n-1} - \alpha_n + \beta_n - \beta_{n-1}| K + |\beta_n - \beta_{n-1}| ||x_{n-1}||$$

$$= (1 - \alpha_n (1 - k)) ||x_{n-1} - x_n|| + (|1 - \alpha_n - \beta_n| b \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| ||Ax_n||$$

$$+ |1 - \alpha_n - \beta_n| \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| L + |1 - \alpha_n - \beta_n||S_n z_{n-1} - S_{n-1} z_{n-1}||$$

$$+ |\alpha_n - \alpha_{n-1}| K + |\alpha_{n-1} - \alpha_n + \beta_n - \beta_{n-1}| K + |\beta_n - \beta_{n-1}| ||x_{n-1}||$$

It follows that

$$\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0. \quad (3.12)$$
Viscosity Approximation Methods for Generalized Equilibrium Problems ...

Next, we shall show that

$$\lim_{n \to \infty} \| S_n z_n - x_n \| = 0.$$  

By (3.1), we obtain

$$\| S_n z_n - x_n \| = \| S_n z_n - S_n^{-1} z_{n-1} + S_n^{-1} z_{n-1} - \alpha_{n-1} f(x_{n-1})$$

$$- \beta_{n-1} x_{n-1} - S_n^{-1} z_{n-1} + \alpha_{n-1} S_n^{-1} z_{n-1} + \beta_{n-1} S_n^{-1} z_{n-1} \|$$

$$= \| S_n z_n - S_n^{-1} z_{n-1} + \alpha_{n-1} (S_n^{-1} z_{n-1} - f(x_{n-1}))$$

$$+ \beta_{n-1} (S_n^{-1} z_{n-1} - x_{n-1}) \|$$

$$\leq \| S_n z_n - S_n^{-1} z_{n-1} \| + \alpha_{n-1} \| S_n^{-1} z_{n-1} - f(x_{n-1}) \|$$

$$+ \beta_{n-1} \| S_n^{-1} z_{n-1} - x_{n-1} \|$$

By (3.10), we obtain

$$\lim_{n \to \infty} \| S_n z_n - x_n \| = 0. \quad (3.13)$$

Next, we want to show

$$\lim_{n \to \infty} \| x_n - z_n \| = 0.$$  

By monotonicity of $A$ and nonexpansiveness of $T_{\lambda_n}$, we obtain

$$\| x_{n+1} - z \|^2 = \| \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n z_n - z \|^2$$

$$\leq \alpha_n \| f(x_n) - z \|^2 + \beta_n \| x_n - z \|^2 + (1 - \alpha_n - \beta_n) \| S_n z_n - z \|^2$$

$$\leq \alpha_n k^2 \| x_n - z \|^2 + \alpha_n \| f(z) - z \|^2 + \beta_n \| x_n - z \|^2$$

$$+ (1 - \alpha_n - \beta_n) \| (x_n - \lambda_n A x_n) - (z - \lambda_n A z) \|^2$$

$$= \alpha_n k^2 \| x_n - z \|^2 + \alpha_n \| f(z) - z \|^2 + \beta_n \| x_n - z \|^2$$

$$+ (1 - \alpha_n - \beta_n) \| (x_n - z) - \lambda_n (A x_n - A z) \|^2$$

$$= \alpha_n k^2 \| x_n - z \|^2 + \alpha_n \| f(z) - z \|^2 + \beta_n \| x_n - z \|^2 + (1 - \alpha_n - \beta_n)$$

$$\left( \| x_n - z \|^2 - 2 \lambda_n \langle x_n - z, A x_n - A z \rangle + \lambda_n^2 \| A x_n - A z \|^2 \right)$$

$$\leq \alpha_n k^2 \| x_n - z \|^2 + \alpha_n \| f(z) - z \|^2 + \beta_n \| x_n - z \|^2 + (1 - \alpha_n - \beta_n)$$

$$\left( \| x_n - z \|^2 - 2 \alpha \lambda_n \| A x_n - A z \|^2 + \lambda_n^2 \| A x_n - A z \|^2 \right)$$

$$= \alpha_n k^2 \| x_n - z \|^2 + \alpha_n \| f(z) - z \|^2 + \beta_n \| x_n - z \|^2 + (1 - \alpha_n - \beta_n)$$

$$\left( \| x_n - z \|^2 + \lambda_n (\lambda_n - 2 \alpha) \| A x_n - A z \|^2 \right)$$

$$= \alpha_n k^2 \| x_n - z \|^2 + \alpha_n \| f(z) - z \|^2 + (1 - \alpha_n \| x_n - z \|^2$$

$$+ (1 - \alpha_n - \beta_n) \lambda_n (\lambda_n - 2 \alpha) \| A x_n - A z \|^2$$

$$\leq \alpha_n k^2 \| x_n - z \|^2 + \alpha_n \| f(z) - z \|^2 + \| x_n - z \|^2$$

$$+ (1 - \alpha_n - \beta_n) \lambda_n (\lambda_n - 2 \alpha) \| A x_n - A z \|^2. \quad (3.14)$$
By (3.14), we obtain
\[
(1 - \alpha_n - \beta_n)\lambda_n (2\alpha - \lambda_n)\|Ax_n - Az\|^2 
\leq \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 
+ \|x_n - z\|^2 - \|x_{n+1} - z\|^2.
\]
Since \(0 < a \leq \lambda_n \leq b < 2\alpha\) and \(0 < c \leq \beta_n \leq d < 1\), we obtain
\[
(1 - \alpha_n - \beta_n)\lambda_n (2\alpha - \lambda_n)\|Ax_n - Az\|^2 
\leq (1 - \alpha_n - \beta_n)\lambda_n (2\alpha - \lambda_n)\|Ax_n - Az\|^2.
\]
Thus
\[
(1 - \alpha_n - \beta_n)\lambda_n (2\alpha - \lambda_n)\|Ax_n - Az\|^2 
\leq \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 
+ \|x_n - z\|^2 - \|x_{n+1} - z\|^2 
\leq \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 
+ \|x_{n+1} - x_n\| \|\|x_n - z\| + \|x_{n+1} - z\|\|.
\]
This implies, by (3.12) and condition (i), that
\[
\lim_{n \to \infty} \|Ax_n - Az\| = 0.
\] (3.15)
Since \(T_{\lambda_n}\) is a firmly nonexpansive, we obtain
\[
\|z_n - z\|^2 = \|T_{\lambda_n} (x_n - \lambda_n Ax_n) - T_{\lambda_n} (z - \lambda_n Az)\|^2 
\leq \langle (x_n - \lambda_n Ax_n) - (z - \lambda_n Az), z_n - z \rangle 
= \frac{1}{2} \| (x_n - \lambda_n Ax_n) - (z - \lambda_n Az) \|^2 + \| z_n - z \|^2 - \| (x_n - \lambda_n Ax_n) 
- (z - \lambda_n Az) - (z_n - z) \|^2 
\leq \frac{1}{2} \|x_n - z\|^2 + \| z_n - z \|^2 - \| (x_n - z_n) - \lambda_n (Ax_n - Az) \|^2 
= \frac{1}{2} \|x_n - z\|^2 + \| z_n - z \|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle x_n - z_n, Ax_n - Az\rangle 
- \lambda_n^2 \|Ax_n - Az\|^2.
\]
It follows that
\[
\|z_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\| \] (3.16)
Since
\[
\|x_{n+1} - z\|^2 \leq \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 + \beta_n \|x_n - z\|^2 
+ (1 - \alpha_n - \beta_n) \|z_n - z\|^2
\]
and by (3.16), we obtain
\[
\|x_{n+1} - z\|^2 \leq \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 + \beta_n \|x_n - z\|^2
\]
\[
+ (1 - \alpha_n - \beta_n)(\|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|)
\]
\[
\leq \alpha_n k^2 \|x_n - z\|^2 + \alpha_n \|f(z) - z\|^2 + \|x_n - z\|^2 - (1 - \beta_n) \|x_n - z_n\|^2
\]
\[
+ \alpha_n \|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|
\]
\[
= \alpha_n (k^2 \|x_n - z\|^2 + \|f(z) - z\|^2 + \|x_n - z_n\|^2) + \|x_n - z\|^2
\]
\[
- (1 - \beta_n) \|x_n - z_n\|^2 + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|
\]
This implies
\[
(1 - \beta_n) \|x_n - z_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n (k^2 \|x_n - z\|^2
\]
\[
+ \|f(z) - z\|^2 + \|x_n - z_n\|^2) + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|
\]
and by condition (i), we obtain
\[
(1 - d) \|x_n - z_n\|^2 \leq \|x_{n+1} - x_n\| (\|x_n - z\| + \|x_{n+1} - z\|) + \alpha_n (k^2 \|x_n - z\|^2
\]
\[
+ \|f(z) - z\|^2 + \|x_n - z_n\|^2) + 2\lambda_n \|x_n - z_n\| \|Ax_n - Az\|
\]
Thus
\[
\lim_{n \to \infty} \|x_n - z_n\| = 0. \tag{3.17}
\]
From (3.13) and (3.17), we obtain
\[
\|S_n z_n - z_n\| = \|S_n z_n - x_n + x_n - z_n\|
\]
\[
\leq \|S_n z_n - x_n\| + \|x_n - z_n\| \to 0 \text{ as } n \to \infty. \tag{3.18}
\]
We shall show that
\[
\limsup_{n \to \infty} (f(z_0) - z_0, z_n - z_0) \leq 0,
\]
where \(z_0 = P_{\bigcap_{i=1}^N F(T_i) \cap EP(f)}(z_0)\). To show this inequality, we choose a subsequence \(\{z_{n_k}\}\) of \(\{z_n\}\) such that
\[
\lim_{n \to \infty} (f(z_0) - z_0, z_n - z_0) = \lim_{k \to \infty} (f(z_0) - z_0, z_{n_k} - z_0).
\]
Without loss of generality, we may assume that \(z_{n_k} \to w\) as \(k \to \infty\) where \(w \in C\). We first show \(w \in EP\). Since \(z_n = T_{\lambda_n}(x_n - \lambda_n Ax_n)\), we obtain
\[
F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C.
\]
From (A2), we have
\[
\langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq F(y, z_n).
\]
Thus
\[ \langle Ax_{nk}, y - z_{nk} \rangle + \frac{1}{\lambda_{nk}}(y - z_{nk}, z_{nk} - x_{nk}) \geq F(y, z_{nk}), \quad \forall y \in C. \] (3.19)

Putting \( z_t = ty + (1 - t)w \) for all \( t \in (0, 1] \) and \( y \in C \). Then, we have \( z_t \in C \). So, from (3.19), we obtain
\[
\langle z_t - z_{nk}, A z_t \rangle \geq \langle z_t - z_{nk}, A z_{nk} \rangle - \langle z_t - z_{nk}, \frac{z_{nk} - x_{nk}}{\lambda_{nk}} \rangle + F(z_t, z_{nk})
\]
\[ = \langle z_t - z_{nk}, A z_t - A z_{nk} \rangle + \langle z_t - z_{nk}, A z_{nk} - A x_{nk} \rangle \]
\[ - \langle z_t - z_{nk}, \frac{z_{nk} - x_{nk}}{\lambda_{nk}} \rangle + F(z_t, z_{nk}). \]

Since \( \| z_{nk} - x_{nk} \| \to 0 \) as \( k \to \infty \), we obtain \( \| A z_{nk} - A x_{nk} \| \to 0 \) as \( k \to \infty \).

Further, from the monotonicity of \( A \), we have \( \langle z_t - z_{nk}, A z_t - A z_{nk} \rangle \geq 0 \). So, from (A4), we obtain
\[ \langle z_t - w, A z_t \rangle \geq F(z_t, w). \] (3.20)

From (A1), (A4) and (3.20), we also have
\[ 0 = F(z_t, z_t) \leq tF(z_t, y) + (1 - t)F(z_t, w) \]
\[ \leq tF(z_t, y) + (1 - t)\langle z_t - w, A z_t \rangle \]
\[ = tF(z_t, y) + (1 - t)t\langle y - w, A z_t \rangle \]
and hence
\[ 0 \leq F(z_t, y) + (1 - t)\langle y - w, A z_t \rangle \]

Letting \( t \to 0 \), we obtain
\[ 0 \leq F(w, y) + \langle y - w, A w \rangle, \quad \forall y \in C. \]

Therefore \( w \in EP \). Next, we show that \( w \in \cap_{i=1}^N F(T_i) \). We assume that
\[ \alpha_{1k} \to \alpha_1^1, \quad \alpha_{1n}^k \to \alpha_1^N \in (0, 1] \] as \( k \to \infty \)

for \( j = 1, 2, \ldots, N - 1 \) and \( \alpha_{nk}^j \to \alpha_3^j \in [0, 1) \) as \( k \to \infty \) for \( j = 1, 2, \ldots, N \).

Let \( S_n \) be the \( S \)-mappings generated by \( T_1, T_2, \ldots, T_N \) and \( \beta_1, \beta_2, \ldots, \beta_N \), where \( \beta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \) for \( j = 1, 2, \ldots, N \). By Lemma 2.7, we have
\[ \lim_{k \to \infty} \| S_{nk} x - S x \| = 0, \quad \forall x \in C. \] (3.21)

By Lemma 2.6, we have \( \cap_{i=1}^N F(T_i) = F(S) \). Assume that \( Sw \neq w \). By using the Opial property and (3.18) and (3.21), we obtain
\[
\lim_{k \to \infty} \| z_{nk} - w \| < \liminf_{k \to \infty} \| z_{nk} - Sw \|
\]
\[ \leq \liminf_{k \to \infty} (\| z_{nk} - S_{nk} z_{nk} \| + \| S_{nk} z_{nk} - S_{nk} w \| + \| S_{nk} w - Sw \|)
\]
\[ \leq \liminf_{k \to \infty} \| z_{nk} - w \| \]
which is a contradiction. Thus $Sw = w$ and $w \in F(S) = \cap_{i=1}^{N} F(T_i)$. Therefore $w \in \cap_{i=1}^{N} F(T_i) \cap EP$. Since $z_{n_k} \to w$ and $w \in \cap_{i=1}^{N} F(T_i) \cap EP$, we obtain
\[
\limsup_{n \to \infty} (f(z_0) - z_0, z_n - z_0) = \limsup_{k \to \infty} (f(z_0) - z_0, z_{n_k} - z_0)
\]
\[
= \langle f(z_0) - z_0, w - z_0 \rangle \leq 0. \quad (3.22)
\]
From $x_{n+1} - z_0 = \alpha_n(f(x_n) - z_0) + \beta_n(x_n - z_0) + (1 - \alpha_n - \beta_n)(S_n z_n - z_0)$, we obtain
\[
(1 - \alpha_n - \beta_n)^2 \|S_n z_n - z_0\|^2 \geq \|x_{n+1} - z_0\|^2 - 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0 \rangle - \beta_n \|x_n - z_0\|^2.
\]
So we have
\[
\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n - \beta_n)^2 \|z_n - z_0\|^2 + \beta_n \|x_n - z_0\|^2 \\
+ 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0 \rangle \\
\leq (1 - \alpha_n - \beta_n)^2 \|z_n - z_0\|^2 + \beta_n \|x_n - z_0\|^2 \\
+ 2\alpha_n \langle f(x_n) - f(z_0), x_{n+1} - z_0 \rangle + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\
\leq (1 - \alpha_n - \beta_n)^2 \|x_n - z_0\|^2 + \beta_n \|x_n - z_0\|^2 \\
+ 2\alpha_n \|f(x_n) - f(z_0), x_{n+1} - z_0 \rangle + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\
\leq \left( (1 - \alpha_n - \beta_n)^2 + \beta_n \right) \|x_n - z_0\|^2 + \alpha_n k \|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2 \\
+ 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle.
\]
So
\[
(1 - \alpha_n) \|x_{n+1} - z_0\|^2 \leq \left( (1 - \alpha_n - \beta_n)^2 + \beta_n + \alpha_n k \right) \|x_n - z_0\|^2 \\
+ 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle.
\]
Thus
\[
\|x_{n+1} - z_0\|^2 \leq \frac{(1 - \alpha_n - \beta_n)^2 + \beta_n + \alpha_n k}{1 - \alpha_n k} \|x_n - z_0\|^2 + \frac{2\alpha_n}{1 - \alpha_n k} \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\
\leq 1 - \frac{2\alpha_n + \alpha_n k}{1 - \alpha_n k} \|x_n - z_0\|^2 + \frac{\alpha_n^2}{1 - \alpha_n k} \|x_n - z_0\|^2 \\
+ \frac{2\alpha_n}{1 - \alpha_n k} \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \\
\leq \left( 1 - \frac{2(1 - k)\alpha_n}{1 - \alpha_n k} \right) \|x_n - z_0\|^2 \\
+ \frac{2(1 - k)\alpha_n}{1 - \alpha_n k} \left( \frac{\alpha_n M}{2(1 - k)} + \frac{1}{1 - k} \langle f(z_0) - z_0, x_{n+1} - z_0 \rangle \right),
\]
where $M = \sup \{ \|x_n - z_0\|^2 : n \in \mathbb{N} \}$. Put $\beta_n = \frac{2(1-k)\|x_n\|}{1-\alpha_n k}$. Then $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\lim_{n \to \infty} \beta_n = 0$. Let $\epsilon > 0$. From (3.22), there exists $m \in \mathbb{N}$ such that
\[
\frac{\alpha_n M}{2(1-k)} \leq \frac{\epsilon}{2} \quad \text{and} \quad \frac{1}{1-k} (f(z_0) - z_0, x_{n+1} - z_0) \leq \frac{\epsilon}{2}
\]
for all $n \geq m$. Then
\[
\|x_{n+1} - z_0\|^2 \leq (1-\beta_n)\|x_n - z_0\|^2 + (1 - (1-\beta_n))\epsilon.
\]
Similarly, we have
\[
\|x_{m+n} - z_0\|^2 \leq \prod_{k=m}^{m+n-1} (1-\beta_k)\|x_m - z_0\|^2 + \left(1 - \prod_{k=m}^{m+n-1} (1-\beta_k)\right)\epsilon.
\]
From $\sum_{k=m}^{\infty} \beta_k = \infty$, we have $\prod_{k=m}^{\infty} (1-\beta_k) = 0$. Therefore
\[
\limsup_{n \to \infty} \|x_n - z_0\|^2 = \limsup_{n \to \infty} \|x_{m+n} - z_0\|^2 \leq \epsilon.
\]
Since $\epsilon > 0$ is arbitrary, we obtain
\[
\limsup_{n \to \infty} \|x_n - z_0\|^2 \leq 0.
\]
Thus $\{x_n\}$ converges strongly to $z_0 \in \cap_{i=1}^{N} F(T_i) \cap EP$, where $z_0 = P_{\cap_{i=1}^{N} F(T_i) \cap EP}(0)$. \hfill $\square$

Using our main theorem (Theorem 3.1), we obtain strong convergence in a Hilbert space.

**Corollary 3.2.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of $C$ into itself with $\cap_{i=1}^{N} F(T_i) \cap EP \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_j^{(n)}, \alpha_j^{(n)}, \alpha_j^{(n)})$ be such that $\alpha_j^{(n)}, \alpha_j^{(n)} \in [0, 1], \alpha_j^{(n)} + \alpha_j^{(n)} + \alpha_j^{(n)} = 1, \{\alpha_j^{(n)}\}_{j=1}^{N-1} \subset [\eta_j, \theta_j]$ with $0 < \eta_j < \theta_j < 1$, $\{\alpha_j^{(n)}\}_{j=1}^{N} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_j^{(n)}\}_{j=1}^{N}, \{\alpha_j^{(n)}\}_{j=1}^{N} \subset [0, \theta]$ with $0 \leq \theta < 1$. Let $f$ be a contraction of $H$ into itself and let $S_n$ be the $S$-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{z_n\}, \{x_n\} \subset C$ be sequences generated by
\[
\begin{cases}
F(z_n, y) + \frac{1}{\lambda_n} (y - z_n, z_n - x_n) \geq 0, \quad \forall y \in C, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n z_n, \quad \forall n \in \mathbb{N},
\end{cases}
\]
where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the following conditions:
(i) $0 < a \leq \lambda_n \leq b < \infty$, $0 < c \leq \beta_n \leq d < 1$;
(ii) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$;
(iv) $|\alpha_j^{n+1} - \alpha_j^{n}| \to 0$ and $|\alpha_j^{n+1} - \alpha_j^{n}| \to 0$ as $n \to \infty$, for all $j \in \{1, 2, 3, \ldots, N\}$. Then $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^{N} F(T_i) \cap EP(F)$, where $z = P_{\cap_{i=1}^{N} F(T_i) \cap EP(F)}(z)$. 

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Proof. In Theorem 3.1, put $A \equiv 0$. Then, for all $\alpha \in (0, \infty)$, we have
$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$
for all $x, y \in C$. So, taking $a, b \in (0, \infty)$ with $0 < a \leq b < \infty$ and choosing a sequence $\{\lambda_n\}$ of real numbers with $a \leq \lambda_n \leq b$, we obtain the result from Theorem 3.1. \hfill \Box

Corollary 3.3. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $A$ be an $\alpha$-inverse strongly monotone mapping of $C$ into $H$ and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself with $\cap_{i=1}^N F(T_i) \cap VI(C, A) \neq \emptyset$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_j^{n,j}\}_{j=1}^{N-1} \subseteq [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{N,j}\} \subseteq [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{N,j}\}_{j=1}^{N-1}, \{\alpha_3^{N,j}\}_{j=1}^{N-1} \subseteq [0, \theta_3]$ with $0 \leq \theta_3 < 1$. Let $f$ be a contraction of $H$ into itself and let $S_n$ be the $S$-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{x_n\} \subseteq C$ be sequences generated by
$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)S_n(PC(x_n - \lambda_n Ax_n)), \quad \forall n \in \mathbb{N},$$
where $\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$ and $\{\lambda_n\} \subseteq [0, 2\alpha]$ satisfy the following conditions:
(i) $0 < a \leq \lambda_n \leq b < 2\alpha$, $0 < c \leq \beta_n \leq d < 1$;
(ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\lim_{n \to \infty} \lambda_n = \infty$;
(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ and $|\alpha_2^{n+1,j} - \alpha_2^{n,j}| \to 0$ as $n \to \infty$, for all $j \in \{1, 2, 3, \ldots, N\}$. Then $\{x_n\}$ converges strongly to $z \in \cap_{i=1}^N F(T_i) \cap VI(C, A)$, where
$$z = P_{\cap_{i=1}^N F(T_i) \cap VI(C, A)} f(z).$$

Proof. In Theorem 3.1, put $F \equiv 0$. Then, we obtain
$$\langle Ax_n, y - z_n \rangle + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \forall n \in \mathbb{N}.$$  
This implies that
$$\langle y - z_n, x_n - \lambda_n Ax_n - z_n \rangle \leq 0, \quad \forall y \in C.$$  
So, we obtain $PC(x - \lambda_n Ax_n) = z_n$ for all $n \in \mathbb{N}$. Then, we obtain the result from Theorem 3.1. \hfill \Box

A mapping $G : C \to C$ is called strictly pseudocontractive if there exists $g$ with $0 \leq g < 1$ such that
$$\|Gx - Gy\|^2 \leq \|x - y\|^2 + g\|(I - G)x - (I - G)y\|^2, \quad \forall x, y \in C.$$  
Such a mapping $G$ is called strictly $g$-pseudocontractive. Putting $A = I - G$, we know that
$$\langle x - y, Ax - Ay \rangle \geq \frac{1-g}{2} \|Ax - Ay\|^2, \quad \forall x, y \in C;$$  
see [8]. So, we have the following corollary.
**Corollary 3.4.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let $G$ be a strictly $g$-pseudocontractive mapping of $C$ into itself and let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself with $\cap_{n=1}^N F(T_n) \cap EP \neq \emptyset$, where $A = I - G$. For $j = 1, 2, \ldots, N$, let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$ be such that $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$, $\{\alpha_1^{n,j}\}_{j=1}^N \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1$, $\{\alpha_1^{n,\infty}\} \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_2^{n,j}\}_{j=1}^N \subset [0, \theta_3]$ with $0 \leq \theta_3 < 1$. Let $f$ be a contraction of $H$ into itself and let $S_n$ be the $S$-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Let $x_1 \in C$ and $\{x_n\}, \{x\} \subset C$ be sequences generated by

$$\begin{align*}
F(z_n, y) + (I - G)x_n, y - z_n) + \frac{1}{\lambda_n}(y - z_n, z_n - x_n) & \geq 0, \quad \forall y \in C, \\
\eta_n - n + 1 \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S_n x_n, \quad \forall n \in \mathbb{N},
\end{align*}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 1 - g]$ satisfy the following conditions:

(i) $0 < a \leq \lambda_n \leq b < 1 - g$, $0 < c \leq \beta_n \leq d < 1$;

(ii) $\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$, and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\lim_{n \to \infty} \frac{\lambda_{n+1}}{\lambda_n} = 1$;

(iv) $|\alpha_1^{n+1,j} - \alpha_1^{n,j}| \to 0$ and $|\alpha_3^{n+1,j} - \alpha_3^{n,j}| \to 0$ as $n \to \infty$, for all $j \in \{1, 2, 3, \ldots, N\}$.

Then $\{x_n\}$ converges strongly to $z \in \cap_{n=1}^N F(T_n) \cap EP$, where $z = P_{\cap_{n=1}^N F(T_n) \cap EP} f(z)$.

**Proof.** A strictly $g$-pseudocontractive mapping is $\frac{1-\theta}{2}$-inverse-strongly monotone. So, from Theorem 3.1, we obtain the desired result. \hfill \Box

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