Hyers-Ulam Stability of Linear Differential Equations $y'' = \lambda^2 y$

Y. Li

Abstract: The aim of this paper is to prove the stability in the sense of Hyers-Ulam of differential equation of second order $y'' = \lambda^2 y$.

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1 Introduction and preliminaries

In 1940, S. M. Ulam [18] posed the following problem concerning the stability of functional equations: Give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by D. H. Hyers [2] when $G_1$ and $G_2$ are Banach spaces and the result of Hyers was generalized by Th. M. Rassias (see [14]). Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [3], [4], [5], [13] and [14]).

C. Alsina and R. Ger [1] remarked that the differential equation $y' = y$ has the Hyers-Ulam stability. More explicitly, they proved that if a differentiable function $y : I \to \mathbb{R}$ satisfies $|y'(t) - y(t)| \leq \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \leq 3\varepsilon$ for every $t \in I$.

The above result of C. Alsina and R. Ger has been generalized by T. Miura, S.-E. Takahasi and H. Choda [12], by T. Miura [9], and also by S.-E. Takahasi, T. Miura and S. Miyajima [16]. Indeed, they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(t)$, while C. Alsina and R. Ger investigated the differential equation $y'(t) = y(t)$.

Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized by T. Miura, S. Miyajima and S.-E. Takahasi [11], by S.-E. Takahasi, H. Takagi, T. Miura and S. Miyajima [17], and also by S.-M. Jung ([4], [5], [8]). They dealt with the nonhomogeneous linear differential equations...
equation of first order
\[ y' + p(t)y + q(t) = 0. \]


Motivated by the works of [16] and [19], in this paper, we will investigate the Hyers-Ulam stability of the following linear differential equations of second order:

\[ y'' = \lambda^2 y \]  

where \( y \in C^2(I) = C^2(a, b), -\infty < a < b < +\infty, \lambda > 0. \)

We say that Eq. (1.1) has the Hyers-Ulam stability if there exists a constant \( K > 0 \) with the following property: for every \( \varepsilon > 0, y \in C^2(I) \), if

\[ |y'' - \lambda^2 y| \leq \varepsilon, \]

then there exists some \( z \in C^2(I) \) satisfying

\[ z'' - \lambda^2 z = 0 \]

such that \( |y(x) - z(x)| \leq K\varepsilon \). We call such \( K \) a Hyers-Ulam stability constant for Eq. (1.1).

\section{Main Results}

Now, the main result of this work is given in the following theorem.

\textbf{Theorem 2.1.} If a twice continuously differentiable function \( y : I \to \mathbb{R} \) satisfies the differential inequality

\[ |y'' - \lambda^2 y| \leq \varepsilon \]

for all \( t \in I \) and for some \( \varepsilon > 0 \), then there exists a solution \( v : I \to \mathbb{R} \) of the Eq. (1) such that

\[ |y(x) - v(x)| \leq K\varepsilon \]

Where \( K > 0 \) is a constant.

\textbf{Proof.} Let \( \varepsilon > 0 \) and \( y : I \to \mathbb{R} \) be a twice continuously differentiable function such that

\[ |y'' - \lambda^2 y| \leq \varepsilon \]

We will show that there exists a constant \( K \) independent of \( \varepsilon \) and \( v \) such that \( |y - v| \leq K\varepsilon \) for some \( v \in C^2(I) \) satisfying \( v'' - \lambda^2 v = 0. \)

If we set

\[ g(x) = y'(x) - \lambda y(x), \]

then...
then
\[ g'(x) = y''(x) - \lambda y'(x) \]
thus
\[ |g'(x) + \lambda g(x)| = |y''(x) - \lambda y'(x) + \lambda(y'(x) - \lambda y(x))| = |y'' - \lambda^2 y| \leq \varepsilon \]
Equivalently, \( g \) satisfies
\[ -\varepsilon \leq g'(x) + \lambda g(x) \leq \varepsilon \]
Multiplying the formula by the function \( e^{\lambda(x-a)} \), we obtain
\[ -\varepsilon e^{\lambda(x-a)} \leq g'(x)e^{\lambda(x-a)} - \lambda g(x)e^{\lambda(x-a)} \leq \varepsilon e^{\lambda(x-a)} \]
For the case \( 0 < \lambda \leq 1 \), there exists \( M > 0 \) such that \( M\lambda > 1 \), so without loss of generality, we may assume that \( \lambda > 1 \), thus
\[ -\lambda \varepsilon e^{\lambda(x-a)} \leq g'(x)e^{\lambda(x-a)} - \lambda g(x)e^{\lambda(x-a)} \leq \lambda \varepsilon e^{\lambda(x-a)} \quad (2.1) \]
For some fixed \( c \in (a, b) \) with \( g(c) < \infty \) and any \( x \in (c, b) \), integrating (2.1) from \( c \) to \( x \), we get
\[ -\varepsilon (e^{\lambda(x-a)} - e^{\lambda(c-a)}) \leq g(x)e^{\lambda(x-a)} - g(c)e^{\lambda(c-a)} \leq \varepsilon (e^{\lambda(x-a)} - e^{\lambda(c-a)}) \]
so
\[ -\varepsilon e^{\lambda(x-a)} \leq g(x)e^{\lambda(x-a)} - (g(c) - \varepsilon)e^{\lambda(c-a)} \leq \varepsilon e^{\lambda(x-a)} \]
Multiplying the formula by the function \( e^{-\lambda(x-a)} \), we get
\[ -\varepsilon \leq g(x) - (g(c) - \varepsilon)e^{\lambda(c-a)}e^{-\lambda(x-a)} \leq \varepsilon \]
\[ -\varepsilon \leq g(x) - (g(c) - \varepsilon)e^{\lambda(c-x)} \leq \varepsilon \]
Let \( z(x) = (g(c) - \varepsilon)e^{\lambda(c-x)} \), then \( z(x) \) satisfies
\[ z'(x) + \lambda z(x) = 0 \]
and
\[ |g(x) - z(x)| \leq \varepsilon \]
For any \( x \in (x, c) \), the proof is very similar to the above, so we omit it.

Since \( g(x) = y'(x) - \lambda y(x) \), we have
\[-\epsilon \leq y'(x) - \lambda y(x) - z(x) \leq \epsilon \quad (2.2)\]

By an argument similar to the above, we can show that there exists \( u(x) = (g(c) - \epsilon)e^{\lambda(x-c)} - e^{\lambda(x-a)} \int_x^b z(s)e^{-\lambda(s-a)}ds \) such that

\[|y(x) - u(x)| \leq \epsilon\]

and \( u \in C^2(I) \) satisfying

\[u'(x) - \lambda u(x) - z(x) = 0\]

so

\[z(x) = u'(x) - \lambda u(x)\]

by

\[z'(x) + \lambda z(x) = 0\]

We obtain

\[u''(x) - \lambda u'(x) + \lambda(u'(x) - \lambda u(x)) = 0\]

Hence

\[u''(x) - \lambda^2 u(x) = 0\]

which completes the proof. \( \square \)

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**References**


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Yongjin Li
Department of Mathematics,
Sun Yat-Sen University,
Guangzhou, 510275 P. R. China
E-mail: stslyj@mail.sysu.edu.cn