Fine Spectrum of the Generalized Difference Operator $\Delta_v$ on Sequence Space $l_1$

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Abstract: The purpose of this paper is to determine spectrum and fine spectrum of the operator $\Delta_v$ on sequence space $l_1$. The operator $\Delta_v$ on $l_1$ is defined by $\Delta_v x = (v_n x_n - v_{n-1} x_{n-1})_{n=0}^\infty$ with $x_{-1} = 0$, where $x = (x_n) \in l_1$ and $v = (v_k)$ is either constant or strictly decreasing sequence of positive real numbers satisfying certain conditions. In this paper we have obtained the results on spectrum and point spectrum for the operator $\Delta_v$ over the sequence space $l_1$. Further, the results on continuous spectrum, residual spectrum and fine spectrum of the operator $\Delta_v$ on space $l_1$ are also derived.

Keywords: Spectrum of an operator, Generalized difference operator, Sequence spaces.

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1 Introduction

Let $v = (v_k)$ be either constant or strictly decreasing sequence of positive real numbers satisfying

\[
\lim_{k \to \infty} v_k = L > 0 \quad \text{and} \quad \sup_k v_k \leq 2L.
\]

We introduce the operator $\Delta_v$ on sequence space $l_1$ as follows;

$\Delta_v : l_1 \to l_1$ is defined by;

$\Delta_v x = \Delta_v (x_n) = (v_n x_n - v_{n-1} x_{n-1})_{n=0}^\infty$ with $x_{-1} = 0$, where $x \in l_1$. 

It is easy to verify that the operator $\Delta_v$ can be represented by the matrix

$$
\Delta_v = \begin{pmatrix}
v_0 & 0 & 0 & \ldots \\
v_0 & v_1 & 0 & \ldots \\
0 & -v_1 & v_2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The fine spectrum of the Cesaro operator on sequence space $l_p$ is studied by Gonzalez [5], where $1 < p < \infty$. The spectrum of the Cesaro operator on sequence spaces $bv_0$ and $bv$ is also investigated by Okutoyi [9] and Okutoyi [10], respectively. Spectrum and fine spectrum of the difference operator $\Delta$ over sequence spaces $l_1$ and $bv$ is determined by K. Kayaduman and H. Furkan [7]. The fine spectra of the difference operator $\Delta$ over sequence space $l_p$ is determined by Akhmedov and Basar [1], where $1 \leq p < \infty$. Furthermore, the fine spectrum of the operator $B(r,s)$ on the sequence spaces $l_1$ and $bv$ is examined by H. Furkan, H. Bilgic and K. Kayaduman [3]. Recently, H. Bilgic and H. Furkan [2] studied the spectrum and fine spectrum for the operator $B(r,s,t)$ over sequence spaces $l_1$ and $bv$.

In this paper we determine spectrum, point spectrum, continuous spectrum and residual spectrum of the operator $\Delta_v$ on sequence space $l_1$. The results of this paper not only generalize the corresponding results of [7] but also give results for some more operators.

## 2 Preliminaries and notation

Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. The set of all bounded linear operators on $X$ into itself is denoted by $B(X)$. The adjoint $T^* : X^* \to X^*$ of $T$ is defined by

$$(T^* \phi)(x) = \phi(T x) \text{ for all } \phi \in X^* \text{ and } x \in X.$$ 

Clearly, $T^*$ is a bounded linear operator on the dual space $X^*$.

Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With $T$, we associate the operator $T_\alpha = (T - \alpha I)$, where $\alpha$ is a complex number and $I$ is the identity operator on $D(T)$. The inverse of $T_\alpha$ (if exists) is denoted by $T_\alpha^{-1}$ and call it the resolvent operator of $T$. Many properties of $T_\alpha$ and $T_\alpha^{-1}$ depend on $\alpha$, and spectral theory is concerned with those properties. We are interested in the set of all $\alpha$ in the complex plane such that $T_\alpha^{-1}$ exists/ $T_\alpha^{-1}$ is bounded/ domain of $T_\alpha^{-1}$ is dense in $X$. We need some definitions and known results which will be used in the sequel.

**Definition 2.1.** ([6], pp. 371) Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A regular value of $T$ is a complex number $\alpha$ such that

(R1) $T_\alpha^{-1}$ exists,
(R2) $T_\alpha^{-1}$ is bounded,
(R3) $T_\alpha^{-1}$ is defined on a set which is dense in $X$. 

Resolvent set $\rho(T, X)$ of $T$ is the set of all regular values $\alpha$ of $T$. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the spectrum of $T$. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets namely point spectrum, continuous spectrum and residual spectrum as follows:

Point spectrum $\sigma_p(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that $T^{-1}_\alpha$ does not exist. The element of $\sigma_p(T, X)$ is called eigenvalue of $T$.

Continuous spectrum $\sigma_c(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that $T^{-1}_\alpha$ exists and satisfies (R3) but not (R2), i.e., range of $T_\alpha$ is dense in $X$ and $T^{-1}_\alpha$ is unbounded.

Residual spectrum $\sigma_r(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that $T^{-1}_\alpha$ exists but do not satisfy (R3), i.e., domain of $T^{-1}_\alpha$ is not dense in $X$. The condition (R2) may or may not holds good.

Goldberg’s classification of operator $T_\alpha$ (see [4], pp. 58): Let $X$ be a Banach space and $T_\alpha \in B(X)$, where $\alpha$ is a complex number. Again, let $R(T_\alpha)$ and $T^{-1}_\alpha$ be denote the range and inverse of the operator $T_\alpha$, respectively. Then following possibilities may occur:

(A) $R(T_\alpha) = X$,
(B) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(C) $R(T_\alpha) \neq X$,

and

(1) $T_\alpha$ is injective and $T^{-1}_\alpha$ is continuous,
(2) $T_\alpha$ is injective and $T^{-1}_\alpha$ is discontinuous,
(3) $T_\alpha$ is not injective.

Remark 2.1. Combining (A), (B), (C) and (1), (2), (3); we get nine different states. These are labeled by $A_1$, $A_2$, $A_3$, $B_1$, $B_2$, $B_3$, $C_1$, $C_2$ and $C_3$. We use $\alpha \in B_2\sigma(T, X)$ means the operator $T_\alpha \in B_2$, i.e., $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$ and $T_\alpha$ is injective but $T^{-1}_\alpha$ is discontinuous. Similarly others.

Remark 2.2. If $\alpha$ is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then $\alpha$ belongs to the resolvent set $\rho(T, X)$ of $T$ on $X$. The other classification gives rise to the fine spectrum of $T$.

Definition 2.2. ([8], pp. 220-221) Let $\lambda$, $\mu$ be two nonempty subsets of the space $w$ of all real or complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers $a_{nk}$, where $n, k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. For every $x = (x_k) \in \lambda$ and every integer $n$ we write

$$A_n(x) = \sum_k a_{nk} x_k,$$

where the sum without limits is always taken from $k = 0$ to $k = \infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of $x$ by the matrix $A$. Infinite matrix $A \in (\lambda, \mu)$ if and only if $Ax \in \mu$ whenever $x \in \lambda$.

Lemma 2.1. ([11], pp. 126) The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from $l_1$ to itself if and only if the supremum of $l_1$ norms of the columns of $A$ is bounded.
Lemma 2.2. ([4], pp. 59) $T$ has a dense range if and only if $T^*$ is one to one, where $T^*$ denotes the adjoint operator of the operator $T$.

Lemma 2.3. ([4], pp. 60) The adjoint operator $T^*$ of $T$ is onto if and only if $T$ has a bounded inverse.

3 Spectrum and point spectrum of the operator $\Delta_v$ on sequence space $l_1$

In this section we obtain spectrum and point spectrum of the operator $\Delta_v$ on sequence space $l_1$. Throughout this paper, the sequence $v = (v_k)$ satisfy conditions (1.1) and (1.2).

Theorem 3.1. The operator $\Delta_v : l_1 \to l_1$ is a bounded linear operator and
$$
\|\Delta_v\|_{(l_1, l_1)} = 2 \sup_k (v_k).
$$

Proof. Proof is simple. So we omit.

Theorem 3.2. The spectrum of $\Delta_v$ on sequence space $l_1$ is given by
$$
\sigma(\Delta_v, l_1) = \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\}.
$$

Proof. The proof of this theorem is divided into two parts. In the first part, we show that $\sigma(\Delta_v, l_1) \subseteq \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\}$ or equivalent to show that
$$
\alpha \in \mathbb{C} \text{ with } \left| 1 - \frac{\alpha}{L} \right| > 1 \text{ implies } \alpha \notin \sigma(\Delta_v, l_1), \text{ i.e., } \alpha \in \rho(\Delta_v, l_1).
$$
In the second part, we establish the reverse inequality, i.e.,
$$
\left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\} \subseteq \sigma(\Delta_v, l_1).
$$
Let $\alpha \in \mathbb{C}$ with $\left| 1 - \frac{\alpha}{L} \right| > 1$. Clearly, $\alpha = L$ as well as $\alpha = v_k$ for any $k$ do not satisfied. So $\alpha \neq L$ and $\alpha \neq v_k$ for each $k \in \mathbb{N}_0$. Consequently, $(\Delta_v - \alpha I) = (a_{nk})$ as a triangle and hence has an inverse $(\Delta_v - \alpha I)^{-1} = (b_{nk})$, where
$$
(b_{nk}) = \begin{pmatrix}
\frac{1}{(v_0 - \alpha)} & 0 & 0 & \ldots \\
\frac{(v_0 - \alpha)(v_1 - \alpha)}{v_0v_1} & \frac{1}{(v_1 - \alpha)} & 0 & \ldots \\
\frac{(v_0 - \alpha)(v_1 - \alpha)(v_2 - \alpha)}{v_0v_1v_2} & \frac{(v_1 - \alpha)(v_2 - \alpha)}{v_1} & \frac{1}{(v_2 - \alpha)} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$
By Lemma 2.1, the operator \((\Delta_v - \alpha I)^{-1}\) is in \(l_1\) if \(\sup_{n=0}^{\infty} |b_{nk}| < \infty\). In order to show \(\sup_{n=0}^{\infty} |b_{nk}| < \infty\), first we prove that the series \(\sum_{n=0}^{\infty} |b_{nk}|\) is convergent for each \(k \in \mathbb{N}_0\).

Let \(S_k = \sum_{n=0}^{\infty} |b_{nk}|\). Then the series

\[
S_0 = \sum_{n=0}^{\infty} |b_{no}| = \left| \frac{1}{v_0 - \alpha} \right| + \sum_{n=1}^{\infty} \left| \frac{v_0 v_1 \cdots v_{n-1}}{(v_0 - \alpha)(v_1 - \alpha) \cdots (v_n - \alpha)} \right|, \tag{3.1}
\]

is convergent because

\[
\lim_{n \to \infty} \frac{|b_{n+1,0}|}{|b_{n0}|} = \lim_{n \to \infty} \frac{|v_n|}{|v_{n+1} - \alpha|} = \frac{1}{|1 - \frac{\alpha}{L}|} < 1.
\]

Similarly, we can show that the series \(S_k = \sum_{n=0}^{\infty} |b_{nk}|\) is convergent for any \(k = 1, 2, 3, \ldots\).

Now we claim that \(\sup_{k} S_k\) is finite. We have

\[
S_k = \frac{1}{|v_k - \alpha|} + \frac{|v_k|}{|v_k - \alpha|} |v_{k+1} - \alpha| + \cdots. \tag{3.2}
\]

Let \(\beta = \lim_{k \to \infty} \frac{|v_k|}{|v_{k+1} - \alpha|}\). Since modulus function is continuous, so

\[
\beta = \frac{L}{|L - \alpha|}, \tag{3.3}
\]

which shows that \(0 < \beta < 1\) and gives

\[
\lim_{k \to \infty} \frac{1}{|v_k - \alpha|} = \lim_{k \to \infty} \left( \frac{|v_{k-1}|}{|v_{k-1} - \alpha|} \frac{1}{|v_{k-1}|} \right) = \frac{\beta}{L}. \tag{3.4}
\]

Taking limit both sides of equation (3.2) and using equations (3.3) and (3.4), we get

\[
\lim_{k \to \infty} S_k = \frac{\beta}{L} \left( \frac{1}{1 - \beta} \right) < \infty.
\]

Since \(S_k\) is a sequence of positive real numbers and \(\lim_{k \to \infty} S_k < \infty\), so \(\sup_{k} S_k < \infty\). Thus,

\[
(\Delta_v - \alpha I)^{-1} \in B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left| \frac{1 - \alpha}{L} \right| > 1. \tag{3.5}
\]
Next, we show that domain of the operator \((\Delta_{\nu} - \alpha I)^{-1}\) is dense in \(l_1\) equivalent to say that range of the operator \((\Delta_{\nu} - \alpha I)\) is dense in \(l_1\), which follows immediately as the operator \((\Delta_{\nu} - \alpha I)\) is onto. Hence we have
\[
\sigma(\Delta_{\nu}, l_1) \subseteq \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\}. \tag{3.6}
\]
Conversely, it is required to show
\[
\left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\} \subseteq \sigma(\Delta_{\nu}, l_1). \tag{3.7}
\]
First we prove inclusion (3.7) under the assumption that \(\alpha \neq L\) as well as \(\alpha \neq v_k\) for each \(k \in \mathbb{N}_0\), i.e., one of the conditions of Definition 2.1 fails. Let \(\alpha \in \mathbb{C}\) with \(\left| 1 - \frac{\alpha}{L} \right| < 1\). Clearly, \((\Delta_{\nu} - \alpha I)\) is a triangle and hence \((\Delta_{\nu} - \alpha I)^{-1}\) exists. So condition (R1) is satisfied but condition (R2) fails as can be seen below:
Suppose \(\alpha \in \mathbb{C}\) with \(\left| 1 - \frac{\alpha}{L} \right| < 1\). Then by equation (3.1), the series \(S_0\) is divergent because
\[
\lim_{n \to \infty} \left| b_{n+1,0} \right| = \lim_{n \to \infty} \left| \frac{v_n}{v_{n+1} - \alpha} \right| = \frac{1}{\left| 1 - \frac{\alpha}{L} \right|} > 1.
\]
So sup \(S_k\) is unbounded. Hence
\[
(\Delta_{\nu} - \alpha I)^{-1} \notin B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left| 1 - \frac{\alpha}{L} \right| < 1. \tag{3.8}
\]
Next, we consider \(\alpha \in \mathbb{C}\) with \(\left| 1 - \frac{\alpha}{L} \right| = 1\), i.e., \(|L - \alpha| = L\) which implies \(|v_n - \alpha| \leq |v_n|\) for each \(n\), therefore \(\frac{1}{|v_n|} \leq \frac{1}{|v_n - \alpha|}\) for each \(n\). Using this inequality and equation (3.1), the series \(S_0 \geq \sum_{n=0}^{\infty} \frac{1}{v_n}\) is divergent due to the fact that \(v_n > 0\) for all \(n\) and \(\lim_{n \to \infty} \frac{1}{v_n} = \frac{1}{L} \neq 0\). Thus, sup \(S_k\) is unbounded. Hence
\[
(\Delta_{\nu} - \alpha I)^{-1} \notin B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left| 1 - \frac{\alpha}{L} \right| = 1. \tag{3.9}
\]
Finally, we prove the inclusion (3.7) under the assumption that \(\alpha = L\) as well as \(\alpha = v_k\) for all \(k \in \mathbb{N}_0\). We have
\[
(\Delta_{\nu} - v_k I)x = \begin{pmatrix}
(v_0 - v_k)x_0 \\
-v_0x_0 + (v_1 - v_k)x_1 \\
\vdots \\
-v_{k-1}x_{k-1} \\
-v_kx_k + (v_{k+1} - v_k)x_{k+1} \\
\vdots 
\end{pmatrix}.
\]
Case(i): If \((v_k)\) is a constant sequence, say \(v_k = L\) for all \(k \in \mathbb{N}_0\), then
\[
(\Delta_v - v_k I) x = 0 \quad \Rightarrow \quad x_0 = 0, \ x_1 = 0, \ x_2 = 0, \ldots
\]
This shows that the operator \((\Delta_v - \alpha I)\) is one to one, but \(R(\Delta_v - \alpha I)\) is not dense in \(l_1\). So condition (R3) fails. Hence \(L \in \sigma(\Delta_v, l_1)\).

Case(ii): If \((v_k)\) is strictly decreasing sequence, then for fixed \(k\),
\[
(\Delta_v - v_k I) x = 0
\]
\[
\Rightarrow \quad x_0 = 0, \ x_1 = 0, \ldots, \ x_{k-1} = 0, \ x_{n+1} = \left(\frac{v_n}{v_{n+1} - v_k}\right) x_n \quad \text{for all} \quad n \geq k.
\]
This shows that \((\Delta_v - v_k I)\) is not injective. So condition (R1) fails. Hence \(v_k \in \sigma(\Delta_v, l_1)\) for all \(k \in \mathbb{N}_0\).

Again, if \(\alpha = L\), then \(|v_n - \alpha| < |v_n|\) for each \(n\), i.e., \(\frac{1}{|v_n|} < \frac{1}{|v_n - \alpha|}\) for each \(n\). Using this inequality and equation (3.1), the series \(S_0 > \sum_{n=0}^{\infty} \frac{1}{v_n}\) is divergent due to fact that \(v_n > 0\) for all \(n\) and \(\lim_{n \to \infty} \frac{1}{v_n} = \frac{1}{L} \neq 0\). Thus, sup \(S_k\) is unbounded. So condition (R2) fails. Hence
\[
(\Delta_v - \alpha I)^{-1} \notin B(l_1) \quad \text{for} \quad \alpha = L.
\]
So \(L \in \sigma(\Delta_v, l_1)\). Thus, in this case also \(v_k \in \sigma(\Delta_v, l_1)\) for all \(k \in \mathbb{N}_0\) and \(L \in \sigma(\Delta_v, l_1)\). Hence we have
\[
\left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\} \subseteq \sigma(\Delta_v, l_1).
\]
From inclusions (3.6) and (3.11), we get
\[
\sigma(\Delta_v, l_1) = \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}.
\]

**Theorem 3.3.** Point spectrum of the operator \(\Delta_v\) over \(l_1\) is given by
\[
\sigma_p(\Delta_v, l_1) = \begin{cases} 
\emptyset, & \text{if } (v_k) \text{ is a constant sequence.} \\
\{v_0, v_1, v_2, \ldots\}, & \text{if } (v_k) \text{ is a strictly decreasing sequence.}
\end{cases}
\]

**Proof.** The proof of this theorem is divided into two cases.

Case(i): Suppose \((v_k)\) is a constant sequence, say \(v_k = L\) for all \(k \in \mathbb{N}_0\). Consider \(\Delta_v x = \alpha x\) for \(x \neq 0 = (0, 0, \ldots)\) in \(l_1\), which gives
\[
\begin{align*}
v_0x_0 &= \alpha x_0 \\
v_1x_1 &= \alpha x_1 \\
v_2x_2 &= \alpha x_2 \\
&\quad \vdots \\
v_{k-1}x_{k-1} + v_kx_k &= \alpha x_k \\
&\quad \vdots
\end{align*}
\]
(3.12)
Let $x_t$ be the first non-zero entry of the sequence $x = (x_n)$, so we get $-Lx_{t-1} + Lx_t = \alpha x_t$, which implies $\alpha = L$ and from the equation

$$-Lx_t + Lx_{t+1} = \alpha x_{t+1},$$

we get $x_t = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p(\Delta_v, l_1) = \emptyset.$$

Case (ii): Suppose $(v_k)$ is a strictly decreasing sequence. Consider $\Delta_v x = \alpha x$ for $x \neq 0 = (0, 0, \cdots)$ in $l_1$, which gives system of equations (3.12).

If $\alpha = v_0$, then

$$x_k = \left( \frac{v_{k-1}}{v_k - v_0} \right) x_{k-1} \quad \text{for all } k \geq 1$$

$$= \left[ \frac{v_{k-1}v_{k-2} \cdots v_0}{(v_k - v_0)(v_{k-1} - v_0) \cdots (v_1 - v_0)} \right] x_0 \quad \text{for all } k \geq 1.$$

If we take $x_0 \neq 0$, then get non-zero solution of $(\Delta_v - v_0 I) x = 0$.

Similarly, if $\alpha = v_k$ for all $k \geq 1$, then $x_{k-1} = 0, x_{k-2} = 0, \cdots, x_0 = 0$ and

$$x_{n+1} = \left( \frac{v_n}{v_{n+1} - v_k} \right) x_n \quad \text{for all } n \geq k$$

$$= \left[ \frac{v_n v_{n-1} \cdots v_k}{(v_{n+1} - v_k)(v_n - v_k) \cdots (v_{k+1} - v_k)} \right] x_k \quad \text{for all } n \geq k.$$

If we take $x_k \neq 0$, then get non-zero solution of $(\Delta_v - v_k I) x = 0$. Hence

$$\sigma_p(\Delta_v, l_1) = \{v_0, v_1, v_2, \cdots\}.$$ 

4 Residual and continuous spectrum of the operator $\Delta_v$ on sequence space $l_1$

We need result of point spectrum of the operator $\Delta_v^\times$ on $l_1^*$ for obtaining residual and continuous spectrum. So first we determine point spectrum of the dual operator $\Delta_v^\times$ of $\Delta_v$ on space $l_1^*$.

Let $T : l_1 \rightarrow l_1$ be a bounded linear operator having matrix representation $A$ and the dual space of $l_1$ denoted by $l_1^*$. Then the adjoint operator $T^\times : l_1^* \rightarrow l_1^*$ is defined by the transpose of the matrix $A$.

**Theorem 4.1.** Point spectrum of the operator $\Delta_v^\times$ over $l_1^*$ is

$$\sigma_p(\Delta_v^\times, l_1^*) = \{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \}.$$
Proof. Suppose \( \Delta_v^x f = \alpha f \) for \( 0 \neq f \in l_1^* \cong l_\infty \), where

\[
\Delta_v^x = \begin{pmatrix}
v_0 & -v_0 & 0 & \cdots \\
v_1 & -v_1 & 0 & \cdots \\
0 & v_2 & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\quad \text{and} \quad f = \begin{pmatrix}
f_0 \\
f_1 \\
f_2 \\
\vdots
\end{pmatrix}.
\]

This gives

\[
f_k = \left[ \frac{(v_{k-1} - \alpha)(v_{k-2} - \alpha)\cdots(v_0 - \alpha)}{v_{k-1}v_{k-2}\cdots v_0} \right] f_0 \quad \text{for all } k \geq 1.
\]

Hence

\[
|f_k| = \left| \frac{(v_{k-1} - \alpha)(v_{k-2} - \alpha)\cdots(v_0 - \alpha)}{v_{k-1}v_{k-2}\cdots v_0} \right| |f_0| \quad \text{for all } k \geq 1.
\]

(4.1)

But

\[
|v_{k-1} - \alpha| \leq (v_{k-1} - L) + |L - \alpha|
\]

\[
\Rightarrow \frac{|v_{k-1} - \alpha|}{v_{k-1}} \leq 1 \quad \text{for all } k \geq 1 \quad \text{provided} \quad \left| 1 - \frac{\alpha}{L} \right| \leq 1.
\]

Using equation (4.1), we get

\[
|f_k| \leq |f_0| \quad \text{for all } k \geq 1. \quad \text{So } \sup_k |f_k| < \infty.
\]

Hence

\[
\left| 1 - \frac{\alpha}{L} \right| \leq 1 \quad \Rightarrow \quad \sup_k |f_k| < \infty.
\]

Converse follows from the fact that

\[
\sup_k |f_k| < \infty \quad \Rightarrow \quad \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| \leq 1 \quad \text{for all } k \geq m,
\]

where \( m \) is a positive integer.

\[
\Rightarrow \quad \lim_{k \to \infty} \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| \leq 1
\]

\[
\Rightarrow \quad \left| 1 - \frac{\alpha}{L} \right| \leq 1.
\]

Hence

\[
\sup_k |f_k| < \infty \quad \Rightarrow \quad \left| 1 - \frac{\alpha}{L} \right| \leq 1.
\]

This means that \( f \in l_1^* \) if and only if \( f_0 \neq 0 \) and \( \left| 1 - \frac{\alpha}{L} \right| \leq 1 \). Hence

\[
\sigma_p(\Delta_v^x, l_1^*) = \left\{ \alpha \in \mathbb{C} : \left| 1 - \frac{\alpha}{L} \right| \leq 1 \right\}.
\]
Theorem 4.2. Residual spectrum \( \sigma_r(\Delta_v, l_1) \) of the operator \( \Delta_v \) over \( l_1 \) is

\[
\sigma_r(\Delta_v, l_1) = \begin{cases} 
\{ \alpha \in \mathbb{C} : |1 - \frac{\alpha}{L}| \leq 1 \} , & \text{if } (v_k) \text{ is a constant sequence.} \\
\{ \alpha \in \mathbb{C} : |1 - \frac{\alpha}{L}| \leq 1 \} \setminus \{v_0, v_1, v_2, \ldots \} , & \text{if } (v_k) \text{ is a strictly decreasing sequence.}
\end{cases}
\]

Proof. The proof of this theorem is divided into two cases.

Case(i): Let \( (v_k) \) be a constant sequence. For \( \alpha \in \mathbb{C} \) with \( |1 - \frac{\alpha}{L}| \leq 1 \), the operator \( (\Delta_v - \alpha I) \) is a triangle except \( \alpha = L \) and consequently, the operator \( (\Delta_v - \alpha I) \) has an inverse. Further by Theorem 3.3, the operator \( (\Delta_v - \alpha I) \) is one to one for \( \alpha = L \) and hence has an inverse.

But by Theorem 4.1, the operator \( (\Delta_v - \alpha I)^\times = \Delta_v^\times - \alpha I \) is not one to one for \( \alpha \in \mathbb{C} \) with \( |1 - \frac{\alpha}{L}| \leq 1 \). Hence by Lemma 2.2, the range of the operator \( (\Delta_v - \alpha I) \) is not dense in \( l_1 \). Thus,

\[
\sigma_r(\Delta_v, l_1) = \{ \alpha \in \mathbb{C} : |1 - \frac{\alpha}{L}| \leq 1 \}.
\]

Case(ii): Let \( (v_k) \) be a strictly decreasing sequence with \( \lim_{k \to \infty} v_k = L \). For \( \alpha \in \mathbb{C} \) such that \( |1 - \frac{\alpha}{L}| \leq 1 \), the operator \( (\Delta_v - \alpha I) \) is a triangle except \( \alpha = v_k \) for all \( k \in \mathbb{N}_0 \) and consequently, the operator \( (\Delta_v - \alpha I) \) has an inverse. Further by Theorem 3.3, the operator \( (\Delta_v - v_k I) \) is not one to one and hence \( (\Delta_v - v_k I)^{-1} \) does not exist for all \( k \in \mathbb{N}_0 \).

On the basis of argument as given in case(i), it is easy to verify that the range of the operator \( (\Delta_v - \alpha I) \) is not dense in \( l_1 \). Thus,

\[
\sigma_r(\Delta_v, l_1) = \{ \alpha \in \mathbb{C} : |1 - \frac{\alpha}{L}| \leq 1 \} \setminus \{v_0, v_1, v_2, \ldots \}.
\]

\[\square\]

Theorem 4.3. Continuous spectrum \( \sigma_c(\Delta_v, l_1) \) of the operator \( \Delta_v \) over \( l_1 \) is \( \sigma_c(\Delta_v, l_1) = \emptyset \).

Proof. It is known that \( \sigma_p(\Delta_v, l_1) \), \( \sigma_r(\Delta_v, l_1) \) and \( \sigma_c(\Delta_v, l_1) \) are pairwise disjoint sets and union of these sets is \( \sigma(\Delta_v, l_1) \). But by Theorems 3.2, 3.3 and 4.2; we get

\[
\sigma(\Delta_v, l_1) = \sigma_p(\Delta_v, l_1) \cup \sigma_r(\Delta_v, l_1).
\]

Therefore, \( \sigma_c(\Delta_v, l_1) = \emptyset \).

\[\square\]

5 Fine spectrum of the operator \( \Delta_v \) on sequence space \( l_1 \)

Theorem 5.1. If \( \alpha \) satisfies \( |1 - \frac{\alpha}{L}| > 1 \), then \( (\Delta_v - \alpha I) \in A_1 \).
Proof. It is required to show that the operator \((\Delta_v - \alpha I)\) is bijective and has a continuous inverse for \(\alpha \in \mathbb{C}\) with \(|1 - \frac{\alpha}{L}| > 1\). Since \(\alpha \neq L\) and \(\alpha 
eq v_k\) for each \(k \in \mathbb{N}_0\), therefore \((\Delta_v - \alpha I)\) is a triangle. Hence it has an inverse. The inverse of the operator \((\Delta_v - \alpha I)\) is continuous for \(\alpha \in \mathbb{C}\) with \(|1 - \frac{\alpha}{L}| > 1\) by statement (3.5). Also the equation

\[(\Delta_v - \alpha I) x = y\]  

implies \(x = (\Delta_v - \alpha I)^{-1} y\), i.e., \(x_n = ((\Delta_v - \alpha I)^{-1} y)_n, n \in \mathbb{N}_0\).

Thus for every \(y \in l_1\), we can find \(x \in l_1\) such that \((\Delta_v - \alpha I) x = y\), since \((\Delta_v - \alpha I)^{-1} \in (l_1, l_1)\). This shows that operator \((\Delta_v - \alpha I)\) is onto and hence \((\Delta_v - \alpha I) \in A_1\).

Theorem 5.2. Let \((v_k)\) be a constant sequence, say \(v_k = L\) for all \(k \in \mathbb{N}_0\). Then 
\(L \in C_1\sigma(\Delta_v, l_1)\).

Proof. We have
\[
\sigma_r(\Delta_v, l_1) = \left\{ \alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1 \right\}.
\]
Clearly, \(L \in \sigma_r(\Delta_v, l_1)\). It is sufficient to show that the operator \((\Delta_v - LI)^{-1}\) is continuous. By Lemma 2.3, it is enough to show that \((\Delta_v - LI)^{\times}\) is onto, i.e., for given \(y = (y_n) \in l_{\infty}\), we have to find \(x = (x_n) \in l_{\infty}\) such that \((\Delta_v - LI)^{\times} x = y\).

Now \((\Delta_v - LI)^{\times} x = y\), i.e.,
\[
-Lx_1 = y_0 \\
-Lx_2 = y_1 \\
\vdots \\
-Lx_i = y_{i-1} \\
\vdots
\]

Thus, \(-Lx_n = y_{n-1}\) for all \(n \geq 1\), which implies \(\sup_n |x_n| < \infty\), since \(y \in l_{\infty}\) and \(L \neq 0\). This shows that operator \((\Delta_v - LI)^{\times}\) is onto and hence \(L \in C_1\sigma(\Delta_v, l_1)\).

Theorem 5.3. Let \((v_k)\) be a constant sequence, say \(v_k = L\) for all \(k \in \mathbb{N}_0\) and \(\alpha \neq L, \alpha \in \sigma_r(\Delta_v, l_1)\). Then \(\alpha \in C_2\sigma(\Delta_v, l_1)\).

Proof. It is sufficient to show that the operator \((\Delta_v - \alpha I)^{-1}\) is discontinuous for \(\alpha \neq L\) and \(\alpha \in \sigma_r(\Delta_v, l_1)\). The operator \((\Delta_v - \alpha I)^{-1}\) is discontinuous by statements (3.8) and (3.9) for \(L \neq \alpha \in \mathbb{C}\) with \(|1 - \frac{\alpha}{L}| \leq 1\).
Theorem 5.4. Let \((v_k)\) be a strictly decreasing sequence of positive real numbers and \(\alpha \in \sigma_r(\Delta_v, l_1)\). Then \(\alpha \in C_2\sigma(\Delta_v, l_1)\).

Proof. It is sufficient to show that the operator \((\Delta_v - \alpha I)^{-1}\) is discontinuous for \(\alpha \in \sigma_r(\Delta_v, l_1)\). The operator \((\Delta_v - \alpha I)^{-1}\) is discontinuous by statements (3.8), (3.9) and (3.10) for \(v_k \neq \alpha \in \mathbb{C}\) with \(|1 - \frac{\alpha}{l_1}| \leq 1\).

References


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