Best Proximity Point Theorems for Suzuki Type Proximal Contractive Multimaps

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Abstract: The purpose of this manuscript is to establish new best proximity point results for Suzuki type proximal contractive multimaps. Our results extend some recent known results by Hussain et al. as well as other results in the literature. An illustrative example is provided to highlight our main results.

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1 Introduction and Preliminaries

The background concept on best proximity point results and related fixed point theory in (ordered) metric spaces, Banach spaces and fuzzy metric spaces
is very abundant in the literature; (see, for instance, [12], [13], [14], [15], [16], [17]) and references therein.

Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. An element $x \in A$ is said to be a fixed point of a given map $T : A \to B$ if $Tx = x$. Clearly, $T(A) \cap A \neq \emptyset$ is a necessary (but not sufficient) condition for the existence of a fixed point of $T$. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$, that is, the set of fixed point of $T$ is empty. In such a situation, one often attempts to find an element $x$ which is in some sense closed to $Tx$. Best approximation theory and best proximity point analysis have been developed in this direction. Recently, Jleli and Samet [13] introduced the notion of $\alpha$-$\psi$-proximal contractive type mappings and established some best proximity point theorems. Many authors obtained best proximity point theorems in different settings; (see [1], [2], [5], [6], [9], [10], [21], [22], for examples). Abkar and Gbeleh [3], Al-Thagafi and Shahzad [5, 6], Ali et al. [4], Xu and Fan [21] investigated best proximity points for multivalued mappings.

Later, Hussain et al. [11] introduced new Suzuki and convex type contractions and established new best proximity results for those contractions in the setting of a metric space.

Let $(X, d)$ be a metric space. For $A, B \subset X$, we use the following notations subsequently:

$$d(A, B) = \inf \{d(a, b) : a \in A, b \in B\},$$
$$D(x, B) = \inf \{d(x, b) : b \in B\},$$
$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\},$$
$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

$2^X \setminus \emptyset$ is the set of all nonempty subsets of $X$, CL$(X)$ is the set of all nonempty closed subsets of $X$, and K$(X)$ is the set of all nonempty compact subsets of $X$. For every $A, B \in \text{CL}(X)$, let

$$H(A, B) = \left\{ \begin{array}{ll}
\max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\} & \text{if the maximum exists;} \\
\infty & \text{otherwise.} 
\end{array} \right. \quad (1.1)$$

Such a map $H$ is called the generalized Hausdorff metric induced by $d$. A point $x^* \in X$ is said to be the best proximity point of a mapping $T : A \to B$ if $d(x^*, Tx^*) = d(A, B)$. When $A = B$, the best proximity point is essentially the fixed point of the mapping $T$. We review the following essential definitions.

**Definition 1.1** (see [22]). Let $(A, B)$ be a pair of nonempty subsets of a metric space $(X, d)$ with $A_0 \neq \emptyset$. Then the pair $(A, B)$ is said to have the P-property if and only if, for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$d(x_1, y_1) = d(A, B) \quad \text{and} \quad d(x_2, y_2) = d(A, B) \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2). \quad (1.2)$$
Let $\Psi$ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(a) $\psi$ is monotone nondecreasing;

(b) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

It is well-known that $\psi(t) < t$ for all $t > 0$.

Let $\Theta$ denote the set of all functions $\theta : (0, \infty) \to [1, \infty)$ with the following conditions:

(a) $\theta$ is increasing;

(b) for all sequences $\{\alpha_n\} \subset (0, \infty)$, $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} \theta(\alpha_n) = 1$;

(c) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t \to 0^+} \frac{\theta(t) - 1}{t^r} = l$.

**Definition 1.2** (see [11]). Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : A \to B$ is called $\alpha^+$-proximal admissible if there exists a mapping $\alpha : A \times A \to [0, \infty)$ such that

$$\begin{align*}
\alpha(x_1, x_2) &\geq 0 \\
\frac{d(u_1, Tx_1) + d(A, B)}{2} &= d(u_2, Tx_2) = d(A, B)
\end{align*}$$

for all $x_1, x_2, u_1, u_2 \in A$.

**Definition 1.3** (see [11]). The mapping $T : A \to B$ is called a Suzuki type $\alpha^+$-proximal contraction if there exists a mapping $\alpha : A \times A \to [0, \infty)$ such that

$$\frac{1}{2} d^*(x, Tx) \leq d(x, y) \Rightarrow \alpha(x, y) + d(Tx, Ty) \leq \psi(M(x, y))$$

for all $x, y \in A$, where $d^*(x, Tx) = d(x, Tx) - d(A, B)$, $\psi \in \Psi$, and

$$M(x, y) = \max \left\{ \frac{d(x, y)}{2}, \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B), \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B) \right\}.$$

**Definition 1.4** (see [11]). A mapping $T : A \to B$ is called a Suzuki type $\alpha^+\theta$-proximal contraction, if for all $x, y \in A$ with $\frac{1}{2} d^*(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$,

$$\Rightarrow \alpha(x, y) + \theta(d(Tx, Ty)) \leq \left[ \theta(M(x, y)) \right]^{\beta}$$

where $\alpha : A \times A \to [-\infty, \infty), 0 \leq \beta < 1$, and $\theta \in \Theta$.

The main results of Hussain et al. in [11] are the following.
Theorem 1.5 (see [11]). Let \( A \) and \( B \) be two nonempty closed subsets of a complete metric space \((X, d)\) such that \( A_0 \neq \emptyset \). Let \( T : A \to B \) satisfy (1.3) together with the following assertions:

(i) \( T(A_0) \subseteq B_0 \) and \((A, B)\) satisfies the \( P \)-property;

(ii) \( T \) is an \( \alpha^+ \)-proximal admissible map;

(iii) there exist elements \( x_0, x_1 \) in \( A_0 \) such that
\[
d(x_1, Tx_0) = d(A, B), \quad \alpha(x_0, x_1) \geq 0
\]

(iv) \( T \) is continuous, or

(v) \( A \) is \( \alpha \)-regular, that is, if \( \{x_n\} \) is a sequence in \( A \) such that \( \alpha(x_n, x_{n+1}) \geq 0 \) and \( x_n \to x \in A \) as \( n \to \infty \), then \( \alpha(x_n, x) \geq 0 \) for all \( n \in N \).

Then there exists an element \( x^* \in A_0 \) such that \( d(x^*, Tx^*) = d(A, B) \).

Theorem 1.6 (see [11]). Let \( X, A, A_0, \) and \( B \) be as Theorem 1.5. Assume that \( T : A \to B \) satisfies (1.3) and the assertions (i)-(v) in Theorem 1.5 and
\[
\alpha(p, q) + d(Tp, Tq) \leq \psi(M(p, q))
\]
holds for all \( p, q \in A \). Then there exists an element \( x^* \in A_0 \) such that \( d(x^*, Tx^*) = d(A, B) \).

Theorem 1.7 (see [11]). Let \( X, A, A_0, \) and \( B \) be as Theorem 1.5. Assume that \( T : A \to B \) satisfies (1.3) and the assertions (i)-(v) in Theorem 1.5. Then there exists an element \( x^* \in A_0 \) such that \( d(x^*, Tx^*) = d(A, B) \).

Definition 1.8 (see [3]). An element \( x^* \in A \) is said to be the best proximity point of a multivalued nonself mapping \( T \), if \( D(x^*, Tx^*) = d(A, B) \).

Inspired and motivated by the results of Hussain et al. in [11] and by those of Ali et al. in [7], we establish the best proximity point results for Suzuki type proximal contractive multimeas. Our results extend the recent results of Hussain et al. [11] to the best proximity point results for nonself multivalued mappings. We also give an illustrative example to support our main results.

2 Main Results

We begin this section by introducing the following definitions.

Definition 2.1. Let \( A \) and \( B \) be two nonempty subsets of a metric space \((X, d)\). A mapping \( T : A \to 2^B \setminus \emptyset \) is called an \( \alpha^+ \)-proximal admissible multimap, if there exists a mapping \( \alpha : A \times A \to [-\infty, \infty) \) such that
\[
\left\{ \begin{array}{l}
\alpha(x_1, x_2) \geq 0 \\
D(u_1, Tx_1) = d(A, B) \\
D(u_2, Tx_2) = d(A, B)
\end{array} \right. \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 0
\]
for all \( x_1, x_2, u_1, u_2 \in A \).
Definition 2.2. Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$. A mapping $T : A \to \text{CL}(B)$ is called a Suzuki type $\alpha^+\psi$-proximal contractive multimap, if there exists a mapping $\alpha : A \times A \to [-\infty, \infty)$ such that

$$\frac{1}{2}D^*(x,Tx) \leq d(x,y) \Rightarrow \alpha(x,y) + H(Tx,Ty) \leq \psi(M(x,y))$$

for all $x,y \in A$, where $D^*(x,Tx) = D(x,Tx) - d(A,B)$, $\psi \in \Psi$, and

$$M(x,y) = \max \left\{ d(x,y), \frac{D(x,Tx) + D(y,Ty)}{2} - d(A,B), \frac{D(x,Ty) + D(y,Tx)}{2} - d(A,B) \right\}.$$ 

Definition 2.3. A mapping $T : A \to \text{CL}(B)$ is called a Suzuki type $\alpha^+\theta$-proximal contractive multimap, if for all $x,y \in A$ with $\frac{1}{2}D^*(x,Tx) \leq d(x,y)$ and $H(Tx,Ty) > 0$, $\Rightarrow \alpha(x,y) + \theta(H(Tx,Ty)) \leq \left[ \theta(M(x,y)) \right]^\beta$ (2.3)

where $\alpha : A \times A \to [-\infty, \infty)$, $0 \leq \beta < 1$, $\theta \in \Theta$, and

$$M(x,y) = \max \left\{ d(x,y), \frac{D(x,Tx) + D(y,Ty)}{2} - d(A,B), \frac{D(x,Ty) + D(y,Tx)}{2} - d(A,B) \right\}.$$ 

The following are our main results.

Theorem 2.4. Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X,d)$ such that $A_0$ is nonempty. Let $\alpha : A \times A \to [-\infty, \infty)$ and let $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \to \text{CL}(B)$ is a mapping satisfying (2.2) and the following conditions:

(i) $T(A_0) \subseteq B_0$ and $(A,B)$ satisfies the $P$-property;
(ii) $T$ is an $\alpha^+\cdot$-proximal admissible multimap;
(iii) there exist elements $x_0, x_1 \in A_0$ such that

$$D(x_1,Tx_0) = d(A,B) \text{ and } \alpha(x_0,x_1) \geq 0;$$

(iv) $T$ is continuous, or

(v) $A$ is $\alpha$-regular, that is, if $\{x_n\}$ is a sequence in $A$ such that $\alpha(x_n,x_{n+1}) \geq 0$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n,x) \geq 0$ for all $n$.

Then there exists an element $x^* \in A_0$ such that

$$D(x^*,Tx^*) = d(A,B).$$
Proof. Since \( T(A_0) \subseteq B_0 \), there exists \( x_2 \in A_0 \) such that
\[
D(x_2, Tx_1) = d(A, B). \tag{2.5}
\]
As \( T \) satisfies (iii) and is \( \alpha^+ \)-proximal admissible, we obtain \( \alpha(x_1, x_2) \geq 0 \). That is
\[
D(x_2, Tx_1) = d(A, B), \quad \alpha(x_1, x_2) \geq 0. \tag{2.6}
\]
Again, since \( T(A_0) \subseteq B_0 \), there exists \( x_3 \in A_0 \) such that
\[
D(x_3, Tx_2) = d(A, B). \tag{2.7}
\]
Therefore we have
\[
D(x_2, Tx_1) = d(A, B), \quad D(x_3, Tx_2) = d(A, B), \quad \alpha(x_1, x_2) \geq 0. \tag{2.8}
\]
Again, since \( T \) is \( \alpha^+ \)-proximal admissible, we obtain \( \alpha(x_2, x_3) \geq 0 \). Hence, we have
\[
D(x_3, Tx_2) = d(A, B), \quad \alpha(x_2, x_3) \geq 0. \tag{2.9}
\]
Continuing this method, we get
\[
D(x_{n+1}, Tx_n) = d(A, B), \quad \alpha(x_n, x_{n+1}) \geq 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.10}
\]
From (2.10), definition of \( D^* \) and triangle inequality, we can write
\[
\frac{1}{2} D^*(x_{n-1}, Tx_{n-1}) = \frac{1}{2} \left( D(x_{n-1}, Tx_{n-1}) - d(A, B) \right)
\leq \frac{1}{2} \left( d(x_{n-1}, x_n) + D(x_n, Tx_{n-1}) - d(A, B) \right)
= \frac{1}{2} d(x_{n-1}, x_n)
\leq d(x_{n-1}, x_n). \tag{2.11}
\]
That is
\[
\frac{1}{2} D^*(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n). \tag{2.12}
\]
From (2.2), we get
\[
H(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n) + H(Tx_{n-1}, Tx_n) \leq \psi(M(x_{n-1}, x_n)). \tag{2.13}
\]
By using (2.10), triangle inequality and the \( P \)-property, we obtain
\[
\mathcal{M}(x_{n-1}, x_n) = \max \left\{ d(x_{n-1}, x_n), \frac{D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)}{2} - d(A, B), \right. \\
\left. \frac{D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})}{2} - d(A, B) \right\}
\]
\[
\leq \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + D(x_n, Tx_{n-1}) + d(x_n, x_{n+1}) + D(x_{n+1}, Tx_n)}{2} - d(A, B), \right. \\
\left. \frac{d(x_{n-1}, x_n) + D(x_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_n) + D(x_n, Tx_{n-1})}{2} - d(A, B) \right\}
\]
\[
= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(A, B) + d(x_n, x_{n+1}) + d(A, B)}{2} - d(A, B), \right. \\
\left. \frac{d(x_{n-1}, x_n) + d(A, B) + d(Tx_{n-1}, Tx_n) + d(A, B)}{2} - d(A, B) \right\}
\]
\[
= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(A, B) + d(x_n, x_{n+1}) + d(A, B)}{2} - d(A, B), \right. \\
\left. \frac{d(x_{n-1}, x_n) + d(A, B) + d(Tx_{n-1}, Tx_n) + d(A, B)}{2} - d(A, B) \right\}
\]
\[
= \max \left\{ d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\}
\]
\[
\leq \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\}. \tag{2.14}
\]

Since \((A, B)\) satisfies the \( P \)-property, we obtain
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\
\leq H(Tx_{n-1}, Tx_n) \\
\leq \alpha(x_{n-1}, x_n) + H(Tx_{n-1}, Tx_n) \tag{2.15}
\]
\[
\leq \psi\left( \mathcal{M}(x_{n-1}, x_n) \right) \text{ for all } n \in \mathbb{N}.
\]

From (2.14) and (2.15), we have
\[
d(x_n, x_{n+1}) \leq \psi\left( \mathcal{M}(x_{n-1}, x_n) \right) \\
\leq \psi\left( \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\} \right) \text{ for all } n \in \mathbb{N}. \tag{2.16}
\]

If \(x_{n_0} = x_{n_0 + 1}\), for some \(n_0 \in \mathbb{N}\), from (2.10), we obtain
\[
D(x_{n_0}, Tx_{n_0}) = D(x_{n_0 + 1}, Tx_{n_0}) = d(A, B),
\]
This means \(x_{n_0}\) is a best proximity point of \(T\). Therefore, we suppose that
\[
d(x_{n+1}, x_n) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.17}
\]
If $\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then (2.10) implies
\[
d(x_n, x_{n+1}) \leq \psi \left( d(x_n, x_{n+1}) \right) < d(x_n, x_{n+1}),
\]
which is a contradiction. Therefore,
\[
d(x_n, x_{n+1}) \leq \psi(\mathcal{M}(x_{n-1}, x_n)) \leq \psi(d(x_{n-1}, x_n)) \text{ for all } n \in \mathbb{N}.
\]
By the monotonicity of $\psi$ and by induction, it follows from (2.19) that
\[
d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)) \text{ for all } n \in \mathbb{N} \cup \{0\}.
\]
Suppose $\epsilon$ is any positive real number. There exists $N \in \mathbb{N}$ such that
\[
\sum_{n \geq N} \psi^n \left( d(x_0, x_1) \right) < \epsilon \text{ for all } n \in \mathbb{N}.
\]
If $m, n \in \mathbb{N}$ with $m > n \geq N$. By the triangle inequality, we have
\[
d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{n \geq N} \psi^n \left( d(x_0, x_1) \right) < \epsilon.
\]
Consequently, $\lim_{m, n \to \infty} d(x_n, x_m) = 0$, which implies $\{x_n\}$ is a Cauchy sequence.
Since $X$ is complete, $x_n \to x^* \in X$. If (iv) holds, Then $Tx_n \to Tx^*$ as $n \to \infty$ and
\[
d(A, B) = \lim_{n \to \infty} D(x_{n+1}, Tx_n) = D(x^*, Tx^*).
\]
Hence $x^*$ is the best proximity point of $T$.

Next, assume that (v) holds. Then $\alpha(x_n, x^*) \geq 0$. If the following inequalities hold:
\[
\frac{1}{2} D^*(x_n, Tx_n) > d(x_n, x^*) \quad \text{and} \quad \frac{1}{2} D^*(x_{n+1}, Tx_{n+1}) > d(x_{n+1}, x^*),
\]
for some $n \in \mathbb{N}$, then by the triangle inequality, (2.10) and definition of $D^*$, we obtain
\[
d(x_n, x_{n+1}) \leq d(x_n, x^*) + d(x^*, x_{n+1})
\]
\[
< \frac{1}{2} \left[ D^*(x_n, Tx_n) + D^*(x_{n+1}, Tx_{n+1}) \right]
\]
\[
= \frac{1}{2} \left[ D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1}) - 2d(A, B) \right]
\]
\[
\leq \frac{1}{2} \left[ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \right]
\]
\[
\leq d(x_n, x_{n+1}).
\]
which is a contradiction. Consequently, for any \( n \in \mathbb{N} \), either
\[
\frac{1}{2} D^*(x_n, Tx_n) \leq d(x_n, x^*) \quad \text{or} \quad \frac{1}{2} D^*(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, x^*)
\]
holds.

Then, we may choose a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that
\[
\frac{1}{2} D^*(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, x^*) \quad \text{and} \quad \alpha(x_{n_k}, x_{n_k+1}) \geq 0,
\]
for all \( k \in \mathbb{N} \). By (2.2), we have
\[
H(Tx_{n_k}, Tx^*) \leq \alpha(x_{n_k}, x^*) + H(Tx_{n_k}, Tx^*) \leq \psi(\mathcal{M}(x_{n_k}, x^*)).
\] (2.21)

Observe that
\[
\mathcal{M}(x_{n_k}, x^*) = \max \left\{ d(x_{n_k}, x^*), \frac{D(x_{n_k}, Tx_{n_k}) + D(x^*, Tx^*)}{2} - d(A, B), \frac{D(x_{n_k}, Tx^*) + D(x^*, Tx_{n_k})}{2} - d(A, B) \right\}
\]
\[
\leq \max \left\{ d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + D(x_{n_k+1}, Tx_{n_k}) + D(x^*, Tx^*)}{2} - d(A, B), \frac{d(x_{n_k}, x^*) + D(x^*, Tx^*) + d(x^*, x_{n_k+1}) + D(x_{n_k+1}, Tx_{n_k})}{2} - d(A, B) \right\}
\]
\[
= \max \left\{ d(x_{n_k}, x^*), \frac{d(x_{n_k}, x_{n_k+1}) + d(A, B) + D(x^*, Tx^*)}{2} - d(A, B), \frac{d(x_{n_k}, x_{n_k+1}) + d(A, B) + D(x^*, Tx^*)}{2} - d(A, B) \right\}.
\]

Taking the limit as \( k \to \infty \), we obtain
\[
\lim_{k \to \infty} \mathcal{M}(x_{n_k}, x^*) \leq \frac{D(x^*, Tx^*) - d(A, B)}{2}. \quad (2.22)
\]

Further,
\[
D(x^*, Tx^*) \leq d(x^*, x_{n_k+1}) + D(x_{n_k+1}, Tx_{n_k}) + d(Tx_{n_k}, Tx^*)
\]
\[
= d(x^*, x_{n_k+1}) + d(A, B) + d(Tx_{n_k}, Tx^*),
\]
which gives
\[
D(x^*, Tx^*) - d(x^*, x_{n_k+1}) - d(A, B) \leq d(Tx_{n_k}, Tx^*). \quad (2.23)
\]

Taking the limit as \( k \to \infty \) in (2.23), we obtain
\[
D(x^*, Tx^*) - d(A, B) \leq \lim_{k \to \infty} d(Tx_{n_k}, Tx^*). \quad (2.24)
\]
Therefore, from (2.21), (2.22) and (2.24), we have
\[
D(x^*, Tx^*) - d(A, B) \leq \lim_{k \to \infty} d(Tx_{n_k}, Tx^*) \\
\leq \lim_{k \to \infty} H(Tx, Tx^*) \\
\leq \psi \left( \lim_{k \to \infty} M(x_{n_k}, x^*) \right) \\
\leq \psi \left( \frac{D(x^*, Tx^*) - d(A, B)}{2} \right).
\]

Now, if \(D(x^*, Tx^*) - d(A, B) > 0\), we have that
\[
D(x^*, Tx^*) - d(A, B) < \frac{D(x^*, Tx^*) - d(A, B)}{2}.
\]
This is a contradiction. Hence, \(D(x^*, Tx^*) = d(A, B)\). Therefore \(x^*\) is the best proximity point of \(T\).

Theorem 2.5. Let \(A\) and \(B\) be two nonempty closed subsets of a complete metric space \((X, d)\) such that \(A_0\) is nonempty. Let \(\alpha : A \times A \to [-\infty, \infty)\) and let \(T : A \to K(B)\) be a mapping satisfying (2.22) and the following conditions:

(i) \(T(A_0) \subseteq B_0\) for each \(x \in A_0\) and \((A, B)\) satisfies the P-property;

(ii) \(T\) is an \(\alpha^+\)-proximal admissible multimap;

(iii) there exist elements \(x_0, x_1\) in \(A_0\) such that
\[
D(x_1, Tx_0) = d(A, B), \quad \alpha(x_0, x_1) \geq 0;
\]

(iv) \(T\) is continuous, or

(v) \(A\) is \(\alpha\)-regular, that is, if \(\{x_n\}\) is a sequence in \(A\) such that \(\alpha(x_n, x_{n+1}) \geq 0\) and \(x_n \to x \in A\) as \(n \to \infty\), then \(\alpha(x_n, x) \geq 0\) for all \(n\).

Then there exists an element \(x^* \in A_0\) such that
\[
D(x^*, Tx^*) = d(A, B).
\]

The following result can be deduced easily from Theorem 2.4.

Theorem 2.6. Let \(X, A, A_0\) and \(B\) be the same as in Theorem 2.5. Assume that \(T : A \to CL(B)\) satisfies the conditions (i)-(v) in Theorem 2.5 and
\[
\alpha(p, q) + H(Tp, Tq) \leq \psi(M(p, q))
\]
holds for all \(p, q \in A\). Then there exists an element \(x^* \in A_0\) such that
\[
D(x^*, Tx^*) = d(A, B).
\]
Therefore we also obtain the following theorem.

**Theorem 2.7.** Let \( X, A, A_0 \) and \( B \) be the same as in Theorem 2.4. Assume that \( T : A \to K(B) \) satisfies the conditions (i)-(v) in Theorem 2.4 and

\[
\alpha(p, q) + H(Tp, Tq) \leq \psi(M(p, q))
\]

holds for all \( p, q \in A \). Then there exists an element \( x^* \in A_0 \) such that

\[
D(x^*, Tx^*) = \text{dist}(A, B).
\]

**Theorem 2.8.** Let \( X, A, A_0, \) and \( B \) be as in Theorem 2.4. Assume that \( T : A \to CL(B) \) satisfies (2.3) and the following assertions:

(i) \( T(A_0) \subseteq B_0 \) and \( (A, B) \) satisfies the \( P \)-property;

(ii) \( T \) is an \( \alpha^+ \)-proximal admissible multimap;

(iii) there exist elements \( x_0, x_1 \in A_0 \) such that

\[
D(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 0;
\]

(iv) \( T \) is continuous, or

(v) \( A \) is \( \alpha \)-regular, that is, if \( \{x_n\} \) is a sequence in \( A \) such that \( \alpha(x_n, x_{n+1}) \geq 0 \) and \( x_n \to x \in A \) as \( n \to \infty \), then \( \alpha(x_n, x) \geq 0 \) for all \( n \).

Then there exists an element \( x^* \in A_0 \) such that

\[
D(x^*, Tx^*) = d(A, B).
\]

**Proof.** As in the proof of Theorem 2.4, we can construct a sequence \( \{x_n\} \) satisfying

\[
D(x_{n+1}, Tx_n) = d(A, B),
\]

and

\[
\frac{1}{2}D^*(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n) \quad \text{and} \quad \alpha(x_{n-1}, x_n) \geq 0 \quad \text{for all} \ n \in \mathbb{N}.
\]

Now (2.3) implies

\[
\Theta(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n) + \theta(H(Tx_{n-1}, Tx_n)) \leq \left[\theta(M(x_{n-1}, x_n))\right]^2.
\]

(2.28)

From Theorem 2.4, we obtain

\[
M(x_{n-1}, x_n) \leq \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}
\]

(2.29)

and

\[
d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) \quad \text{for all} \ n \in \mathbb{N}.
\]
Therefore from \( (2.28) \) and \( (2.29) \), we get
\[
\theta(d(x_n, x_{n+1})) \leq \theta(H(Tx_{n-1}, Tx_n)) \\
\leq \left[ \theta(\mathcal{M}(x_{n-1}, x_n)) \right]^\beta \\
\leq \left[ \theta(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \right]^\beta \text{ for all } n \in \mathbb{N}.
\]

Now if \( \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1}) \), then from \( (2.30) \) we get
\[
\theta(d(x_n, x_{n+1})) \leq \left[ \theta(d(x_n, x_{n+1})) \right]^\beta < \theta(d(x_n, x_{n+1})),
\]
which is a contradiction. Therefore, we have
\[
\theta(d(x_n, x_{n+1})) \leq \left[ \theta(d(x_{n-1}, x_n)) \right]^\beta \text{ for all } n \in \mathbb{N}. \tag{2.31}
\]

Therefore from \( (2.31) \), we get
\[
1 \leq \theta(d(x_n, x_{n+1})) \leq (\theta(d(x_{n-1}, x_n)))^\beta \\
\leq ((\theta(d(x_{n-2}, x_{n-1})))^\beta)^\beta \\
\quad \vdots \\
\leq (\theta(d(x_0, x_1)))^{\beta^n} \tag{2.32}
\]

Letting \( n \to \infty \) in \( (2.32) \), we obtain
\[
\lim_{n \to \infty} \theta(d(x_n, x_{n+1})) = 1, \tag{2.33}
\]
and since \( \theta \in \Theta \), we have
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.34}
\]

Again since \( \theta \in \Theta \), there exists \( 0 < r < 1 \) and \( 0 < l \leq \infty \) with
\[
\lim_{n \to \infty} \frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} = l. \tag{2.35}
\]

Assume that \( l < \infty \). Let \( C = \frac{l}{2} \). Thus there exists \( n_0 \in \mathbb{N} \) such that
\[
\left| \frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} - l \right| \leq C \text{ for all } n \geq n_0.
\]

Hence
\[
\frac{\theta(d(x_n, x_{n+1})) - 1}{d(x_n, x_{n+1})^r} \geq l - C = C \text{ for all } n \geq n_0,
\]
and also
\[
n[d(x_n, x_{n+1})]^r \leq nK[\theta(d(x_n, x_{n+1})) - 1] \text{ for all } n \geq n_0,
\]
where $K = \frac{1}{C}$. If $l = \infty$, then there exists $n_0 \in \mathbb{N},$

$$\frac{\theta(d(x_n, x_{n+1})) - 1}{[d(x_n, x_{n+1})]^r} \geq C \quad \text{for all } n \geq n_0,$$

which implies

$$n[d(x_n, x_{n+1})]^r \leq nK[\theta(d(x_n, x_{n+1})) - 1] \quad \text{for all } n \geq n_0,$$

where $K = \frac{1}{C}$. Hence, in all cases there exist $K > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[d(x_n, x_{n+1})]^r \leq nK[\theta(d(x_n, x_{n+1})) - 1] \quad \text{for all } n \geq n_0. \quad (2.36)$$

From (2.33) and (2.36), letting $n \to \infty$, we obtain

$$\lim_{n \to \infty} n[d(x_n, x_{n+1})]^r = 0. \quad (2.37)$$

It follows from (2.37) that there is $n_1 \in \mathbb{N}$ with

$$n[d(x_n, x_{n+1})]^r \leq 1 \quad \text{for all } n > n_1.$$

This implies

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/r}} \quad \text{for all } n > n_1.$$

If $m > n > n_1$, then by the triangle inequality, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Since $0 < r < 1$, $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}} < \infty$. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$, which means that $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Suppose that (iv) holds. Thus $Tx_n \to Tx^*$ as $n \to \infty$, which implies

$$d(A, B) = \lim_{n \to \infty} (D(x_{n+1}, Tx_n)) = D(x^*, Tx^*),$$

as required. Next, assume that (v) holds. As in the proof of Theorem 2.3, we can deduce that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\frac{1}{2}D^*(x_{nk}, Tx_{nk}) \leq d(x_{nk}, x^*) \quad \text{and} \quad \alpha(x_{nk}, x_{nk+1}) \geq 0,$$

for all $k \in \mathbb{N}$. By (2.3) we have

$$\theta(H(Tx_{nk}, Tx^*)) \leq \theta(M(x_{nk}, x^*))^{\beta} < \theta(M(x_{nk}, x^*)),$$
where $0 \leq \beta < 1, \theta \in \Theta$. This implies

$$H(Tx_{n_k}, Tx^*) \leq M(x_{n_k}, x^*).$$ (2.38)

As in Theorem 2.4, we obtain

$$\lim_{k \to \infty} M(x_{n_k}, x^*) \leq \frac{D(x^*, Tx^*) - d(A, B)}{2}$$ (2.39)

and

$$D(x^*, Tx^*) - d(A, B) \leq \lim_{k \to \infty} d(Tx_{n_k}, Tx^*).$$ (2.40)

If $D(x^*, Tx^*) - d(A, B) > 0$, therefore from (2.38), (2.39) and (2.40), we obtain

$$D(x^*, Tx^*) - d(A, B) \leq \frac{D(x^*, Tx^*) - d(A, B)}{2},$$

which is a contradiction. Therefore $D(x^*, Tx^*) = d(A, B)$, as required.

**Theorem 2.9.** Let $X, A, A_0$, and $B$ be as Theorem 2.4. Assume that $T : A \to K(B)$ satisfies (2.3) and the assertions (i)-(v) in Theorem 2.8. Then there exists an element $x^* \in A_0$ such that

$$D(x^*, Tx^*) = d(A, B).$$

**Theorem 2.10.** Let $X, A, A_0$ and $B$ be the same as in Theorem 2.4. Assume that $T : A \to CL(B)$ satisfies the conditions (i)-(v) in Theorem 2.8 and

$$\alpha(p, q) + \theta(H(Tp, Tq)) \leq \left[\theta(M(p, q))\right]^\beta$$

holds for all $p, q \in A$, where $\alpha : A \times A \to [-\infty, \infty), 0 \leq \beta < 1, \theta \in \Theta$. Then there exists an element $x^* \in A_0$ such that

$$D(x^*, Tx^*) = d(A, B).$$

**Theorem 2.11.** Let $X, A, A_0$ and $B$ be the same as in Theorem 2.4. Assume that $T : A \to K(B)$ satisfies the conditions (i)-(v) in Theorem 2.8 and

$$\alpha(p, q) + \theta(H(Tp, Tq)) \leq \left[\theta(M(p, q))\right]^\beta$$

holds for all $p, q \in A$, where $\alpha : A \times A \to [-\infty, \infty), 0 \leq \beta < 1, \theta \in \Theta$. Then there exists an element $x^* \in A_0$ such that

$$D(x^*, Tx^*) = d(A, B).$$

**Example 2.12.** Let $X = [0, \infty) \times [0, \infty)$ be a product space endowed with the usual metric $d$. Suppose that $A = \{(1/2, x) : 0 \leq x < \infty\}$ and $B = \{(0, x) : 0 \leq x < \infty\}$. 
Define \( T : A \to \text{CL}(B) \) by

\[
T\left(\frac{1}{2}, a\right) = \begin{cases} \{(0, \frac{a}{2}) : 0 \leq x \leq a\} & \text{if } a \leq 1 \\ \{(0, x^2) : 0 \leq x \leq a^2\} & \text{if } a > 1, \end{cases}
\] (2.41)

and define \( \alpha : A \times A \to [-\infty, \infty) \) by

\[
\alpha(x, y) = \begin{cases} 0 & \text{if } x, y \in \left(\frac{1}{2}, a\right) : 0 \leq a \leq 1 \\ -\infty & \text{otherwise.} \end{cases}
\]

Let \( \psi(t) = \frac{t}{2} \) for all \( t \geq 0 \). Note that \( A_0 = A, B_0 = B, \) and \( Tx \subseteq B_0 \) for each \( x \in A_0 \). Also, the pair \( (A, B) \) satisfies the \( P \)-property. For \( x_0, x_1 \in \{(\frac{1}{2}, x) : 0 \leq x \leq 1\} \); then \( Tx_0, Tx_1 \subseteq \{(0, \frac{x}{2}) : 0 \leq x \leq 1\} \). Consider \( u_1 \in Tx_0, u_2 \in Tx_1 \) and \( w_1, w_2 \in A \) such that \( d(w_1, u_1) = d(A, B) \) and \( d(w_2, u_2) = d(A, B) \). Then we have \( w_1, w_2 \in \{(\frac{1}{2}, x) : 0 \leq x \leq \frac{1}{2}\} \), so \( \alpha(w_1, w_2) = 0 \). Therefore, \( T \) is an \( \alpha^+ \)-proximal admissible map. For \( x_0 = (\frac{1}{2}, 1) \in A_0 \) and \( u_1 = (0, \frac{1}{2}) \in Tx_0 \in B_0 \), we have \( x_1 = (\frac{1}{2}, \frac{1}{2}) \in A_0 \) such that

\[
d(x_1, u_1) = \text{dist}(A, B), \quad \alpha(x_0, x_1) = \alpha\left((\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2})\right) = 0.
\]

Note that \( d(A, B) = \frac{1}{2}, A_0 = \{(\frac{1}{2}, x) : 0 \leq x < \infty\} \) and \( B_0 = \{(0, x) : 0 \leq x < \infty\} \).

Let \( d(x_1, y_1) = d(A, B) = \frac{1}{2} \) and \( d(x_2, y_2) = d(A, B) = \frac{1}{2} \), where \( x_1 = (\frac{1}{2}, u_1), x_2 = (\frac{1}{2}, u_2) \in A_0 \) and \( y_1 = (0, v_1), y_2 = (0, v_2) \in B_0 \). Then

\[
\frac{1}{2} + |u_1 - v_1| = \frac{1}{2}
\]

and

\[
\frac{1}{2} + |u_2 - v_2| = \frac{1}{2}
\]

so \( |u_1 - v_1| = 0 \) and \( |u_2 - v_2| = 0 \). So, we have \( v_1 = u_1 \) and \( v_2 = u_2 \). This shows that \( d(x_1, x_2) = d(y_1, y_2) \). So \( (A, B) \) satisfies the \( P \)-property.

Notice that \( T(A_0) \subseteq B_0 \). Assume \( \frac{1}{2}D^*(p, Tp) \leq d(p, q) \) and \( \alpha(p, q) \geq 0 \), for \( p, q \in A \). Then

\[
\begin{align*}
p &= \left(\frac{1}{2}, 1\right), q &= \left(\frac{1}{2}, \frac{1}{2}\right) & \text{or} \\
p &= \left(\frac{1}{2}, \frac{1}{2}\right), q &= \left(\frac{1}{2}, 1\right) & \text{or} \\
q &= \left(\frac{1}{2}, 0\right), p &= \left(\frac{1}{2}, \frac{1}{2}\right) & \text{or} \\
q &= \left(\frac{1}{2}, 1\right), p &= \left(\frac{1}{2}, 0\right). 
\end{align*}
\]
Since $d(Tp, Tq) = d(Tq, Tp)$ and $\mathcal{M}(p, q) = \mathcal{M}(q, p)$ for all $p, q \in A$, we can suppose that
\[
(p, q) = ((\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2})) \text{ or } (p, q) = ((\frac{1}{2}, 1), (\frac{1}{2}, 0)).
\]

Now we consider the following cases:

(i) if $(p, q) = ((\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2})) \in A_0$, then
\[
H(T((\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2})) \leq \frac{1}{4} = \psi(d((\frac{1}{2}, 1), (\frac{1}{2}, \frac{1}{2})) \leq \psi(\mathcal{M}(p, q)).
\]

(ii) if $(p, q) = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0)) \in A_0$, then
\[
H(T((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0))) \leq \frac{1}{4} = \psi(d((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0)) \leq \psi(\mathcal{M}(p, q)).
\]

Consequently, we have
\[
\frac{1}{2}D^*(p, Tp) \leq d(p, q) \Rightarrow \alpha(p, q) + H(Tp, Tq) \leq \psi(\mathcal{M}(p, q))
\]

If $x, y \in \{(\frac{1}{2}, a) : 0 \leq a \leq 1\}$, then we have
\[
\alpha(x, y) + H(Tx, Ty) = 0 + \frac{|x - y|}{2} = \frac{1}{2}d(x, y) = \psi(d(x, y)) \leq \psi(\mathcal{M}(x, y)).
\]

Therefore
\[
\alpha(x, y) + H(Tx, Ty) \leq \psi(\mathcal{M}(x, y)).
\]

Hence, $T$ is an $\alpha^+\psi$-proximal contractive multimap. Moreover, if $\{x_n\}$ is a sequence in $A$ such that $\alpha(x_n, x_{n+1}) \geq 0$ for all $n$ and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 0$ for all $k$. Therefore, all the conditions of Theorem 2.3 hold true and $T$ has the best proximity point. Here $p = (\frac{1}{2}, 0)$ is the best proximity point of $T$.

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References


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