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A New Generalization of Hermite Matrix Polynomials

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Abstract : In this paper, multivariable extension of the Hermite matrix polynomials (HMP) is constructed. Then, generating matrix function and recurrence relations satisfied by these multivariable matrix polynomials are derived.

Keywords : Hermite matrix polynomials; Generating matrix function; Recurrence relation; Multilinear and multilateral generating matrix functions.
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1 Introduction

In the recent papers, matrix polynomials have significant emergence and some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials, see [1], [2],[3], [4], [5], [6], [7]. For example, Hermite matrix polynomials have been introduced and studied in [5], [6] for matrices in $\mathbb{C}^{r\times r}$ whose eigenvalues are all situated in the right open half-plane.

Throughout this paper, I and θ will denote the identity matrix and null matrix in $\mathbb{C}^{r\times r}$, respectively. For a matrix A in $\mathbb{C}^{r\times r}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. We say that a matrix A in $\mathbb{C}^{r\times r}$ is a positive stable matrix, if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$. If $f(z)$ and $g(z)$ are holomorphic functions of the

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complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus in [8], it follows that:

$$f(A)g(A)=g(A)f(A).$$

Hence if $B\in\mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B)\subset\Omega$ and $AB=BA$, then

$$f(A)g(B)=g(B)f(A). \tag{1.1}$$

If D is the complex plane cut along the negative real axis and $log(z)$ denotes the principal branch of the logarithm of z, then $z^{1/2}$ represents $\exp((1/2)log(z))$. If A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A)\subset D$, then $A^{1/2}=\sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A. The hypergeometric matrix function $F(A,B;C;z)$ has been given in the form [9]

$$F(A,B;C;z)=\sum_{n=0}^{\infty}\frac{(A)_n(B)_n}{n!}\left[(C)_n\right]^{-1}z^n,$$

for matrices A,B and C in $\mathbb{C}^{r\times r}$ such that $C+nI$ is invertible for all integer $n\geq 0$ and for $|z|<1$.

Let A be a matrix in $\mathbb{C}^{r\times r}$ where

$$Re(\mu)>0 \text{ for every eigenvalue } \mu\in\sigma(A). \tag{1.2}$$

Then Hermite matrix polynomials $H_n(x,A)$ are defined by [5]:

$$H_n(x,A)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{k!\ (n-2k)!}(x\sqrt{2A})^{n-2k}\quad,\quad n\geq 0. \tag{1.3}$$

Also, these matrix polynomials yield

$$\exp\left(xt\sqrt{2A}-t^2I\right)=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x,A)t^n\ ,\ |t|<+\infty, \tag{1.4}$$

$$\int_{-\infty}^{\infty}H_n(x,A)H_m(x,A)e^{-\frac{A}{2}x^2}dx=\begin{cases}\theta & ,\ m\neq n\\ 2^n\ n!\ \left(2\pi A^{-1}\right)^{\frac{1}{2}} & ,\ m=n\end{cases}. \tag{1.5}$$

We organize the paper as follows:

In section 2, we construct multivariable extension of Hermite matrix polynomials and show that these matrix polynomials are orthogonal with respect to weight matrix function. In section 3, generating matrix function is obtained for multivariable Hermite matrix polynomials and with the help of this generating matrix function, several recurrence formulas for multivariable Hermite matrix polynomials (MHMP) are given. Multilinear and multilateral generating matrix functions are derived for MHMP in section 4. In section 5, some applications of the result in section 4 are presented.

2 Multivariable extension of Hermite matrix polynomials

A systematic investigation of a multivariable extension of the Hermite matrix polynomials $H_n(x, A)$ is defined by

$$H_{\mathbf{n}}(\mathbf{x}, \mathbf{A}) = H_{n_1,...,n_s}(x_1, ..., x_s, A_1, ..., A_s) = H_{n_1}(x_1, A_1) ... H_{n_s}(x_s, A_s),$$

where $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{A} = (A_1, ..., A_s)$, A_i be a positive stable matrix in $\mathbb{C}^{r\times r}$ for $1 \le i \le s$ and $\mathbf{n} = (n_1, ..., n_s)$; $n_1, ..., n_s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The multivariable Hermite matrix polynomials $H_{\mathbf{n}}(\mathbf{x}, \mathbf{A})$ (MHMP) are orthogonal with respect to the weight matrix function

$$\begin{aligned} \omega(\mathbf{x}, \mathbf{A}) &= \omega(x_1, ..., x_s, A_1, ..., A_s) \\ &= \omega_1(x_1, A_1) ... \omega_s(x_s, A_s) \\ &= e^{-\frac{A_1}{2}x_1^2} ... e^{-\frac{A_s}{2}x_s^2}, \end{aligned} \tag{2.1}$$

over the domain

$$\Omega = \{(x_1, ..., x_s) : \ -\infty < x_i < \infty \ ; \ \ i = 1, 2, ..., s\}. \tag{2.2}$$

In fact, we have

$$\begin{aligned} &\int\limits_{\Omega} \omega(\mathbf{x}, \mathbf{A}) H_{\mathbf{n}}(\mathbf{x}, \mathbf{A}) H_{\mathbf{m}}(\mathbf{x}, \mathbf{A}) \, \mathbf{dx} \\ =& \int\limits_{-\infty}^{\infty} H_{n_1}(x_1, A_1) H_{m_1}(x_1, A_1) e^{-\frac{A_1}{2}x_1^2} dx_1 \times ... \\ &\times \int\limits_{-\infty}^{\infty} H_{n_s}(x_s, A_s) H_{m_s}(x_s, A_s) e^{-\frac{A_s}{2}x_s^2} dx_s \\ =& \ (2\pi)^{s/2} \prod_{i=1}^{s} 2^{n_i} \ n_i! \ (A_i^{-1})^{\frac{1}{2}} \ \delta_{m_i,n_i} \quad , (m_i, n_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad ; \quad i = 1, 2, ..., s), \end{aligned}$$

where $\mathbf{dx} = dx_1...dx_s$, A_i is a positive stable matrix in $\mathbb{C}^{r\times r}$and $A_iA_j = A_jA_i$ for $i, j = 1, ..., s$.

Thus, the following theorem has been obtained:

Theorem 2.1. *The multivariable HMP $H_{\mathbf{n}}(\mathbf{x}, \mathbf{A})$ are orthogonal with respect to the weight matrix function*

$$\omega(\mathbf{x}, \mathbf{A}) = e^{-\frac{A_1}{2}x_1^2} ... e^{-\frac{A_s}{2}x_s^2}$$

over the domain

$$\Omega = \{(x_1, ..., x_s) : \ -\infty < x_i < \infty \ ; \ \ i = 1, 2, ..., s\},$$

where A_i *is a positive stable matrix in* $\mathbb{C}^{r\times r}$ *and* $A_iA_j = A_jA_i$ *for* $i, j = 1, ..., s$.

3 Generating Matrix Function and Recurrence Relations for MHMP

In [5], it was shown that the HMP are generated by

$$\exp\left(xt\sqrt{2A} - t^2I\right) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, A)t^n \ , \ |t| < +\infty, \tag{3.1}$$

where A is a positive stable matrix in $\mathbb{C}^{r\times r}$.

In this section, we obtain generating matrix function and recurrence relations for MHMP $H_{\mathbf{n}}(\mathbf{x}, \mathbf{A})$.

Using the above expression, we can give the following theorem.

Theorem 3.1. *For the matrix polynomials MHMP* $H_{\mathbf{n}}(\mathbf{x}, \mathbf{A})$, *we have*

$$\sum_{n_1,...,n_s=0}^{\infty} H_{\mathbf{n}}(\mathbf{x}, \mathbf{A}) \frac{t_1^{n_1}}{n_1!} ... \frac{t_s^{n_s}}{n_s!} = \prod_{i=1}^{s} \exp\left(\sqrt{2A_i}x_it_i - t_i^2I\right), \tag{3.2}$$

where A_i *is a positive stable matrix in* $\mathbb{C}^{r\times r}$ *and* $|t_i| < \infty$ *for* $i = 1, ..., s$.

In order to obtain some recurrence relations, we need the following lemma.

Lemma 3.2. *Let a generating matrix function for* $f_{n_1,...,n_s}(\mathbf{x}, \mathbf{A})$ *be*

$$\Psi(\mathbf{x}, \mathbf{t}, \mathbf{A}) = \sum_{n_1,...,n_s=0}^{\infty} f_{n_1,...,n_s}(\mathbf{x}, \mathbf{A}) \ t_1^{n_1} ... t_s^{n_s}, \tag{3.3}$$

where $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{A} = (A_1, ..., A_s)$, $\mathbf{t} = (t_1, ..., t_s)$ *and* $f_{n_1,...,n_s}(\mathbf{x}, \mathbf{A})$ *is a matrix polynomial of degree* n_i *with respect to* x_i*(of total degree* $n = n_1 + ... + n_s$*), provided that*

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{t}, \mathbf{A}) &= \Psi_1(x_1, t_1, A_1) ... \Psi_s(x_s, t_s, A_s), \\ \Psi_i(x_i, t_i, A_i) &= \sum_{n_i=0}^{\infty} \gamma_{n_i} \ (\sqrt{2A_i}x_it_i - t_i^2I)^{n_i} \quad , \quad \gamma_0 \neq 0 \quad ; \quad i = 1, 2, ..., s. \end{aligned}$$

Then we have

$$x_i\sqrt{2A_i}\frac{\partial}{\partial x_i}f_{n_1,...,n_s}(\mathbf{x},\mathbf{A})$$

$$= 2\frac{\partial}{\partial x_i}f_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_s}(\mathbf{x},\mathbf{A}) + n_i\sqrt{2A_i}f_{n_1,...,n_s}(\mathbf{x},\mathbf{A}),\ n_i \geq 1, \quad (3.4)$$

where $A_iA_j = A_jA_i$ for $i,j = 1,...,s$.

Proof. Differentiating (3.3) with respect to x_i and t_i, respectively, and making necessary arrangements, we obtain the desired relation. □

As a result of Lemma 3.2, considering (3.2), we can write that

$$\begin{aligned} f_{n_1,...,n_s}(\mathbf{x},\mathbf{A}) &= \frac{H_{\mathbf{n}}(\mathbf{x},\mathbf{A})}{n_1!...n_s!}, \\ \gamma_{n_i} &= \frac{1}{n_i!}\ ,\ i = 1,2,...,s. \end{aligned}$$

Under the light of the Lemma 3.2 and also considering (3.4), one can easily obtain the next theorem.

Theorem 3.3. *For the matrix polynomials* $H_{\mathbf{n}}(\mathbf{x},\mathbf{A})$, *we have*

$$\begin{aligned} & x_i\sqrt{2A_i}\frac{\partial}{\partial x_i}H_{n_1,...,n_s}(\mathbf{x},\mathbf{A}) - 2n_i\frac{\partial}{\partial x_i}H_{n_1,...,n_{i-1},n_i-1,n_{i+1},...,n_s}(\mathbf{x},\mathbf{A}) \\ = \ & n_i\sqrt{2A_i}H_{n_1,...,n_s}(\mathbf{x},\mathbf{A}),\ n_i \geq 1, \end{aligned}$$

where A_i *is a positive stable matrix in* $\mathbb{C}^{r\times r}$ *and* $A_iA_j = A_jA_i$ *for* $i,j = 1,...,s$.

Similar to Lemma 3.2, we also get the following lemma.

Lemma 3.4. *Let a generating matrix function for* $f_{n_1,...,n_s}(\mathbf{x},\mathbf{A})$ *be*

$$e^{-t_1^2-...-t_s^2}\ \Phi(x_1t_1,...,x_st_s,\mathbf{A}) = \sum_{n_1,...,n_s=0}^{\infty} f_{n_1,...,n_s}(\mathbf{x},\mathbf{A})\ t_1^{n_1}...t_s^{n_s},$$

where $\mathbf{x} = (x_1,...,x_s)$, $\mathbf{A} = (A_1,...,A_s)$ *and* $f_{n_1,...,n_s}(\mathbf{x},\mathbf{A})$ *is a matrix polynomial of degree* n_i *with respect to* x_i(*of total degree* $n = n_1 + ... + n_s$), *provided that*

$$\begin{aligned} \Phi(x_1t_1,...,x_st_s,\mathbf{A}) &= \Phi_1(x_1t_1,A_1)...\Phi_s(x_st_s,A_s), \\ \Phi_i(x_it_i,A_i) &= \sum_{n_i=0}^{\infty}\varphi_{n_i}\ (x_it_i)^{n_i}\quad ,\quad \varphi_0 \neq \theta\ ;\ i = 1,2,...,s. \end{aligned}$$

Then we have

$$n_i f_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A}) + 2f_{n_1,\ldots,n_{i-1},n_i-2,n_{i+1},\ldots,n_s}(\mathbf{x},\mathbf{A}) = x_i \frac{\partial}{\partial x_i} f_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A}) \quad , \quad n_i \geq 2, \tag{3.5}$$

where $A_iA_j = A_jA_i$ for $i,j = 1,...,s$.

As a result of Lemma 3.4, if we choose

$$\begin{aligned} f_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A}) &= \frac{H_{\mathbf{n}}(\mathbf{x},\mathbf{A})}{n_1!\ldots n_s!}, \\ \varphi_{n_i} &= \frac{(\sqrt{2A_i})^{n_i}}{n_i!} \ , \quad i = 1,2,...,s, \end{aligned}$$

and also considering (3.5), one can easily obtain the next theorem.

Theorem 3.5. *For the polynomials MHMP $H_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A})$, we have the following recurrence relation:*

$$\begin{aligned} & x_i \frac{\partial}{\partial x_i} H_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A}) - 2n_i(n_i-1)H_{n_1,\ldots,n_{i-1},n_i-2,n_{i+1},\ldots,n_s}(\mathbf{x},\mathbf{A}) \\ = \ & n_i H_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A}) \ , \ n_i \geq 2, \end{aligned}$$

where A_i is a positive stable matrix in $\mathbb{C}^{r\times r}$ and $A_iA_j = A_jA_i$ for $i,j = 1,...,s$.

By Theorem 3.3 and Theorem 3.5, we can obtain the following corollary.

Corollary 3.6. *The MHMP $H_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A})$ hold:*

$$\frac{\partial}{\partial x_i} H_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A}) = n_i\sqrt{2A_i}H_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_s}(\mathbf{x},\mathbf{A}) \ ; \ n_i \geq 1$$

and

$$\begin{aligned} & \sqrt{2}A_i x_i H_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_s}(\mathbf{x},\mathbf{A}) \\ = \ & \sqrt{A_i}H_{n_1,\ldots,n_s}(\mathbf{x},\mathbf{A}) + \sqrt{2}\frac{\partial}{\partial x_i}H_{n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_s}(\mathbf{x},\mathbf{A}) \quad ; \ n_i \geq 1, \end{aligned}$$

where A_i is a positive stable matrix in $\mathbb{C}^{r\times r}$ and $A_iA_j = A_jA_i$ for $i,j = 1,...,s$.

4 Multilinear and Multilateral Generating Matrix Functions

In recent years, by making use of the familiar group-theoretic (Lie algebraic) method a certain mixed trilateral finite-series relationships have been proved for orthogonal polynomials (see, for instance, [10]). In this section, we derive several families of multilinear and multilateral generating matrix function for the MHMP without using Lie algebraic techniques but, with the help of the similar method as considered in [11], [12].

Theorem 4.1. *Corresponding to a non-vanishing function* $\{\Omega_\mu(y_1,...,y_r)\}_{\mu\in\mathbb{C}}$ *of non-vanishing functions of* r *complex variables* ($r\in\mathbb{N}$) *and of complex order* μ, *let*

$$\Lambda_{\mu,\nu}(y_1,...,y_r;z):=\sum_{k=0}^{\infty}a_k\ \Omega_{\mu+\nu k}(y_1,...,y_r)\ z^k, \tag{4.1}$$

$(a_k\neq 0\,,\ \ \mu,\nu\in\mathbb{N}_0)$ *and*

$$\Theta_{\mathbf{n},p,\mu,\nu}(\mathbf{x};y_1,...,y_r;\zeta):=\sum_{k=0}^{[n_1/p]}a_k\ \frac{H_{n_1-pk,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{(n_1-pk)!\ n_2!...n_s!}\ \Omega_{\mu+\nu k}(y_1,...,y_r)\ \zeta^k, \tag{4.2}$$

where A_i *is a positive stable matrix in* $\mathbb{C}^{r\times r}$ *for* $i=1,...,s$, $\mathbf{n}=(n_1,...,n_s)$, $n_1,...,n_s,p\in\mathbb{N}$, $\mathbf{x}=(x_1,...,x_s)$, $\mathbf{A}=(A_1,...,A_s)$ *and (as usual)* $[\lambda]$ *represents integer part in* $\lambda\in\mathbb{R}$. *Then we have*

$$\begin{aligned}&\sum_{n_1,...,n_s=0}^{\infty}\Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x};y_1,...,y_r;\frac{\eta}{t_1^p}\right)\ t_1^{n_1}....t_s^{n_s}\\ =\ &\left\{\prod_{i=1}^{s}\exp\left(\sqrt{2A_i}x_it_i-t_i^2I\right)\right\}\Lambda_{\mu,\nu}(y_1,...,y_r;\eta),\ \ |t_i|<\infty\ \textit{for}\ \ i=1,...,s,\end{aligned} \tag{4.3}$$

provided that each member of (4.3) *exists.*

Proof. For convenience, let S denote the first member of the assertion (4.3) of Theorem 4.1. Then, plugging the polynomials

$$\Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x};y_1,...,y_r;\frac{\eta}{t_1^p}\right)$$

from the definition (4.2) into the left-hand side of (4.3), we obtain

$$S=\sum_{n_1,...,n_s=0}^{\infty}\sum_{k=0}^{[n_1/p]}a_k\ \frac{H_{n_1-pk,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{(n_1-pk)!\ n_2!...n_s!}\ \Omega_{\mu+\nu k}(y_1,...,y_r)\ \eta^k\ t_1^{n_1-pk}\ \ t_2^{n_2}...t_s^{n_s}. \tag{4.4}$$

Upon changing the order of summation in (4.4), if we replace n_1 by n_1+pk, we can write

$$\begin{aligned}S\ &=\ \sum_{n_1,...,n_s=0}^{\infty}\sum_{k=0}^{\infty}a_k\frac{H_{n_1,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{n_1!\ n_2!...n_s!}\ \Omega_{\mu+\nu k}(y_1,...,y_r)\ \eta^k\ t_1^{n_1}...t_s^{n_s}\\ &=\ \left(\sum_{n_1,...,n_s=0}^{\infty}\frac{H_{n_1,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{n_1!\ n_2!...n_s!}\ t_1^{n_1}...t_s^{n_s}\right)\left(\sum_{k=0}^{\infty}a_k\ \Omega_{\mu+\nu k}(y_1,...,y_r)\ \eta^k\right)\\ &=\ \left\{\prod_{i=1}^{s}\exp\left(\sqrt{2A_i}x_it_i-t_i^2I\right)\right\}\Lambda_{\mu,\nu}(y_1,...,y_r;\eta),\end{aligned}$$

which completes the proof of Theorem 4.1. $\square$

Remark 4.2. *If we set $r = s$ and $\Omega_{\mu+\nu k}(y_1, ..., y_s) = H_{\mu+\nu k}(y_1, ..., y_s, \mathbf{A})$, $(\mu, \nu \in \mathbb{N}_0)$, we obtain result which provides a class of bilinear generating matrix functions for the MHMP.*

5 Further Consequences

By expressing the multivariable function $\Omega_{\mu+\nu k}(y_1, ..., y_r), (k \in \mathbb{N}_0, \ r \in \mathbb{N})$ in terms of a simpler function of one and more variables, we can give further applications of Theorem 4.1. For example, consider the case of $r = 1$ and $\Omega_{\mu+\nu k}(y) = L^{(B,\lambda)}_{\mu+\nu k}(y)$, $(\mu, \nu \in \mathbb{N}_0)$ in Theorem 4.1. Here, the Laguerre matrix polynomials $L^{(B,\lambda)}_{n}(y)$ are defined by [4] as:

$$L^{(B,\lambda)}_{n}(y) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k! \ (n-k)!} \ (B+I)_n \ \left[(B+I)_k\right]^{-1} \ y^k,$$

in which B is a matrix in $\mathbb{C}^{r\times r}$, $B + nI$ is invertible for every integer $n \geq 0$ and λ is a complex number with $Re(\lambda) > 0$. Notice that Laguerre matrix polynomials are generated as follows

$$\sum_{n=0}^{\infty} L^{(B,\lambda)}_{n}(y) \ \eta^n = (1-\eta)^{-(B+I)} \exp\left(\frac{-\lambda y \eta}{1-\eta}\right), \tag{5.1}$$

where $|\eta| < 1$ and $-\infty < y < \infty$. Then we obtain the following corollary which provides a class of bilateral generating matrix functions for the MHMP and the Laguerre matrix polynomials.

Corollary 5.1. *Let $\Lambda_{\mu,\nu}(y; z) := \sum\limits_{k=0}^{\infty} a_k \ L^{(B,\lambda)}_{\mu+\nu k}(y) \ z^k$ where $(a_k \neq 0, \ \mu, \nu \in \mathbb{N}_0)$ and*

$$\Theta_{\mathbf{n},p,\mu,\nu}(\mathbf{x}; y; \zeta) := \sum_{k=0}^{[n_1/p]} a_k \ \frac{H_{n_1-pk,n_2,...,n_s}(\mathbf{x}, \mathbf{A})}{(n_1-pk)! \ n_2!...n_s!} \ L^{(B,\lambda)}_{\mu+\nu k}(y) \ \zeta^k,$$

where $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{A} = (A_1, ..., A_s)$, $\mathbf{n} = (n_1, ..., n_s), n_1, ..., n_s, p \in \mathbb{N}$ and A_i, B are matrices in $\mathbb{C}^{r\times r}$ satisfying condition $Re(\lambda_i) > 0$ for all eigenvalue $\lambda_i \in \sigma(A_i)$ for $i = 1, ..., s$ and $Re(\gamma) > -1$ for all eigenvalue $\gamma \in \sigma(B)$. Then we have

$$\sum_{n_1,...,n_s=0}^{\infty} \Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x}; y; \frac{\eta}{t_1^p}\right) \ t_1^{n_1}....t_s^{n_s} = \left\{\prod_{i=1}^{s} \exp\left(\sqrt{2A_i} x_i t_i - t_i^2 I\right)\right\} \Lambda_{\mu,\nu}(y; \eta), \tag{5.2}$$

provided that each member of (5.2) exists, $|t_i| < \infty$ for $i = 1, ..., s$.

Remark 5.2. *Taking $a_k = 1$, $\mu = 0$, $\nu = 1$, using the generating matrix function*

(5.1) *for the Laguerre matrix polynomials and (3.2), we have*

$$\sum_{n_1,...,n_s=0}^{\infty}\sum_{k=0}^{[n_1/p]}\frac{H_{n_1-pk,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{(n_1-pk)!\ n_2!...n_s!}L_k^{(B,\lambda)}(y)\ \eta^k\ t_1^{n_1-pk}\ t_2^{n_2}...t_s^{n_s}$$
$$=\left\{\prod_{i=1}^{s}\exp\left(\sqrt{2A_i}x_it_i-t_i^2I\right)\right\}(1-\eta)^{-(B+I)}\exp\left(\frac{-\lambda y\eta}{1-\eta}\right),$$

where $|\eta|<1,\ -\infty<y<\infty$ *and* $|t_i|<\infty$ *for* $i=1,...,s$.

Set $r=1$ and $\Omega_{\mu+\nu k}(y)=P_{\mu+\nu k}^{(B,C)}(y),\ (\mu,\nu\in\mathbb{N}_0)$, in Theorem 4.1, where the Jacobi matrix polynomials $P_n^{(B,C)}(y)$ are defined by [1] as:

$$P_n^{(B,C)}(y)$$
$$=\frac{(-1)^n}{n!}\ F\left(B+C+(n+1)I,-nI;C+I;\frac{1+y}{2}\right)\ \Gamma^{-1}(C+I)\ \Gamma(C+(n+1)I),$$

where B and C are matrices in $\mathbb{C}^{r\times r}$ whose eigenvalues, z, all satisfy the condition $Re(z)>-1$. Here Jacobi matrix polynomials are generated by

$$\sum_{n=0}^{\infty}(B+C+I)_n\ P_n^{(B,C)}(y)\ \left[(C+I)_n\right]^{-1}\eta^n$$
$$=(1+\eta)^{-(B+C+I)}\ F\left(\frac{B+C+I}{2},\frac{B+C+2I}{2};C+I;\frac{2\eta(y+1)}{(1+\eta)^2}\right),\tag{5.3}$$

where $|\eta|<1$ and $|y|<1$ which were in [11]. Then we obtain the following corollary which provides a class of bilateral generating matrix functions for MHMP and the Jacobi matrix polynomials.

Corollary 5.3. *Let* $\Lambda_{\mu,\nu}(y;z):=\sum_{k=0}^{\infty}a_k\ (B+C+I)_k\ P_{\mu+\nu k}^{(B,C)}(y)\ \left[(C+I)_k\right]^{-1}z^k$, *where* $(a_k\neq 0,\ \mu,\nu\in\mathbb{N}_0)$ *and*

$$\Theta_{\mathbf{n},p,\mu,\nu}(\mathbf{x};y;\zeta)$$
$$=\sum_{k=0}^{[n_1/p]}a_k\frac{H_{n_1-pk,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{(n_1-pk)!\ n_2!...n_s!}(B+C+I)_k\ P_{\mu+\nu k}^{(B,C)}(y)\ \left[(C+I)_k\right]^{-1}\ \zeta^k,$$

where $\mathbf{x}=(x_1,...,x_s),\ \mathbf{A}=(A_1,...,A_s),\ \mathbf{n}=(n_1,...,n_s),n_1,...,n_s,p\in\mathbb{N}$ *and* A,B *and* C *are matrices in* $\mathbb{C}^{r\times r}$ *satisfying condition* $Re(\lambda)>0$ *for all eigenvalue* $\lambda\in\sigma(A)$, $Re(\gamma)>-1$ *for all eigenvalue* $\gamma\in\sigma(B)$ *and* $Re(\xi)>-1$ *for all eigenvalue* $\xi\in\sigma(C)$. *Then we have*

$$\sum_{n_1,...,n_s=0}^{\infty}\Theta_{\mathbf{n},p,\mu,\nu}\left(\mathbf{x};y;\frac{\eta}{t_1^p}\right)\ t_1^{n_1}....t_s^{n_s}=\left\{\prod_{i=1}^{s}\exp\left(\sqrt{2A_i}x_it_i-t_i^2I\right)\right\}\Lambda_{\mu,\nu}(y;\eta),\tag{5.4}$$

provided that each member of (5.4) *exists* $|t_i|<\infty$ *for* $i=1,...,s$.

Remark 5.4. *Taking $a_k = 1$, $\mu = 0$, $\nu = 1$, using the generating matrix function* (5.3) *for the Jacobi matrix polynomials and (3.2), we have*

$$\begin{aligned}
&\sum_{n_1,...,n_s=0}^{\infty} \sum_{k=0}^{[n_1/p]} \left\{ \frac{H_{n_1-pk,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{(n_1-pk)!\ n_2!...n_s!} (B+C+I)_k\ P_k^{(B,C)}(y) \right. \\
&\left. \times \left[(C+I)_k\right]^{-1}\ \eta^k\ t_1^{n_1-pk}\ t_2^{n_2}...t_s^{n_s} \right\} \\
= &\left\{ \prod_{i=1}^{s} \exp\left(\sqrt{2A_i}x_it_i - t_i^2 I\right) \right\} \\
&\times (1+\eta)^{-(B+C+I)}\ F\left(\frac{B+C+I}{2}, \frac{B+C+2I}{2}; C+I; \frac{2\eta(y+1)}{(1+\eta)^2}\right),
\end{aligned}$$

where $|\eta| < 1$, $|y| < 1$ and $|t_i| < \infty$ for $i = 1, ..., s$.

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1, ..., y_r)$, $(r \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorem 4.1 can be applied in order to derive various families of multilinear and multilateral generating matrix functions for the MHMP.

We set $r = 2$ and $\Omega_{\mu+\nu k}(y, z) = H_{\mu+\nu k}(y, z, B)$, $(\mu, \nu \in \mathbb{N}_0)$, in Theorem 4.1, where the two-variable Hermite matrix polynomials $H_n(y, z, B)$ are defined by means of the generating matrix function in [6] as:

$$\exp\left(y\eta\sqrt{2B} - z\eta^2 I\right) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(y, z, B)\eta^n; \quad |\eta| < \infty, \tag{5.5}$$

where B is a positive stable matrix in $\mathbb{C}^{r\times r}$. Then we obtain the following corollary which provides a class of bilateral generating matrix functions for the two-variable Hermite matrix polynomials and the MHMP defined by (3.2).

Corollary 5.5. *Let $\Lambda_{\mu,\psi}(y, z; r) := \sum_{k=0}^{\infty} a_k\ H_{\mu+\nu k}(y, z, B)\ r^k$, where*

$$(a_k \neq 0\,,\ \ \mu, \nu \in \mathbb{N}_0)$$

and

$$\Theta_{\mathbf{n},p,\mu,\psi}(\mathbf{x}; y, z; \zeta) := \sum_{k=0}^{[n_1/p]} a_k \frac{H_{n_1-pk,n_2,...,n_s}(\mathbf{x},\mathbf{A})}{(n_1-pk)!\ n_2!...n_s!} H_{\mu+\nu k}(y, z, B)\ \zeta^k,$$

where $\mathbf{x} = (x_1, ..., x_s)$, $\mathbf{A} = (A_1, ..., A_s)$, $\mathbf{n} = (n_1, ..., n_s), n_1, ..., n_s, p \in \mathbb{N}$ and A_i, B are positive stable matrices in $\mathbb{C}^{r\times r}$ for $i = 1, ..., s$. Then we have

$$\sum_{n_1,...,n_s=0}^{\infty} \Theta_{\mathbf{n},p,\mu,\psi}\left(\mathbf{x}; y, z; \frac{\eta}{t_1^p}\right)\ t_1^{n_1}....t_s^{n_s} = \left\{ \prod_{i=1}^{s} \exp\left(\sqrt{2A_i}x_it_i - t_i^2 I\right) \right\} \Lambda_{\mu,\psi}(y, z; \eta), \tag{5.6}$$

provided that each member of (5.6) exists, $|t_i| < \infty$ for $i = 1, ..., s$.

Remark 5.6. *Taking* $a_k = \frac{1}{k!}$, $\mu = 0$, $\nu = 1$, *using the generating matrix function (5.5) for the two-variable Hermite matrix polynomials and (3.2), we have*

$$\begin{aligned}&\sum_{n_1,\ldots,n_s=0}^{\infty} \sum_{k=0}^{[n_1/p]} \frac{H_{n_1-pk,n_2,\ldots,n_s}(\mathbf{x},\mathbf{A})}{(n_1-pk)!\; n_2!\ldots n_s!} \frac{H_k(y,z,B)}{k!} \eta^k\; t_1^{n_1-pk}\; t_2^{n_2}\ldots t_s^{n_s} \\ = &\left\{\prod_{i=1}^{s} \exp\left(\sqrt{2A_i}x_i t_i - t_i^2 I\right)\right\} \exp\left(y\eta\sqrt{2B} - z\eta^2 I\right),\end{aligned}$$

where $|\eta| < \infty$ *and* $|t_i| < \infty$ *for* $i = 1, \ldots, s$.

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