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# Strong Convergence Theorems for the Split Variational Inclusion Problem and Common Fixed Point Problem for a Finite Family of Quasi-nonexpansive Mappings in Hilbert Spaces

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**Abstract :** In this work, we introduce and study an algorithm for solving the common fixed point problem of a finite family of quasi-nonexpansive mappings and the split variational inclusion problem in Hilbert spaces. We establish a strong convergence result under some suitable conditions. A numerical example supporting our main result is also given.

**Keywords :** split feasibility problem; split variational inclusion problem; common fixed point problem; quasi-nonexpansive mappings; resolvent mapping; strong convergence.

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# 1 Introduction

Let $H$ be a real Hilbert space and $C_i \subseteq H, i = 1, 2, ..., m$ be nonempty closed convex subsets of $H$. The *convex feasibility problem* (CFP) is to find a point

$$x^* \in \bigcap_{i=1}^{m} C_i. \tag{1.1}$$

Given a finite family of nonlinear mappings $T_i : H \to H, i = 1, 2, ..., m$ with $Fix(T_i) := \{x \in H : x = T_i x\} \neq \emptyset$. The *common fixed point problem* (CFPP) is to find a point

$$x^* \in \bigcap_{i=1}^{m} Fix(T_i). \tag{1.2}$$

Since each closed convex subset may be considered as a fixed point set of a projection onto the subset, hence the CFPP (1.2) is a generalization of the CFP (1.1).

Let $H_1$ and $H_2$ be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Let $C_i, i = 1, 2, ..., t$ and $Q_j, j = 1, 2, ..., r$ be nonempty closed convex subsets of $H_1$ and $H_2$, respectively. The *multiple-set split feasibility problem* (MSSFP) which was introduced by Censor et al. [1] is formulated as finding a point

$$x^* \in \bigcap_{i=1}^{t} C_i \quad \text{such that} \quad Ax^* \in \bigcap_{j=1}^{r} Q_j. \tag{1.3}$$

In particular, if $t = r = 1$, then the MSSFP (1.3) is reduced to find a point

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.4}$$

where $C$ and $Q$ are nonempty closed convex subsets of $H_1$ and $H_2$, respectively. The problem (1.4) is known as the *split feasibility problem* (SFP) which was first introduced by Censor and Elfving [2] for modeling inverse problems in finite-dimensional Hilbert spaces. To solve (1.4), Byrne [3] proposed his CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(x_n - \rho_n A^*(I - P_Q)Ax_n), \quad n \in \mathbb{N}$$

where $\rho_n \in (0, \frac{2}{\|A\|^2})$, $P_C$ and $P_Q$ are the (orthogonal) projections onto $C$ and $Q$, respectively. and $A^*$ denotes the adjoint of $A$.

Let $H$ be a real Hilbert space, and $B$ be a set-valued mapping with domain $\mathcal{D}(B) := \{x \in H : B(x) \neq \emptyset\}$. Recall that $B$ is called *monotone* if $\langle u - v, x - y\rangle \geq 0$ for any $u \in Bx$ and $v \in By$; $B$ is *maximal monotone* if its graph$\{(x, y) : x \in \mathcal{D}(B), y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. Further, for each $\beta > 0$, let $B$ is a set-valued maximal monotone mapping. Define $J_\beta^B(x) := (I + \beta B)^{-1}(x)$ for each $x \in H$. $J_\beta^B$ is called a *resolvent* of $B$ order $\beta$.

One of the most important problem for set-valued mappings is to find $\bar{x} \in H$

such that $0 \in B\bar{x}$, $\bar{x}$ is called a *zero point* of $B$. This problem contains numerous problems in optimization, economics, physics and several areas of engineering. The proximal point algorithm was first introduced by Martinet [4] which is a method for approximating a zero point of a maximal monotone mapping in a real Hilbert space and generalized by Rockafellar [5]. This iterative algorithm generates $\{x_n\}$ by

$$x_{n+1} = J_{\beta_n}^B x_n \tag{1.5}$$

where $\{\beta_n\}$ is a sequence in $(0, \infty)$, $B$ is a maximal monotone mapping in a real Hilbert space, and $J_{\beta_n}^B$ is the resolvent mapping of $B$.

In 1976, Rockafellar [5] proved that the sequence $\{x_n\}$ in (1.5) converges weakly to an element of $B^{-1}(0)$ if $B^{-1}(0)$ is nonempty and $\liminf_{n\to\infty} \beta_n > 0$.

The *split variational inclusion problem* was proposed by Moudafi [6] since 2011:

(SFVIP) Find $\bar{x} \in H_1$ such that $0 \in B_1(\bar{x})$ and $0 \in B_2(A\bar{x})$

where $H_1$ and $H_2$ be two real Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two set-valued maximal monotone mappings, $A : H_1 \to H_2$ be a bounded linear operator.

Moreover, Moudafi [6] introduced the algorithm to solve the SFVIP as following:

$$x_{n+1} := J_\lambda^{B_1}\big[x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n\big]. \tag{1.6}$$

where $\lambda$ and $\gamma$ are fixed numbers. He proved that this iteration converges weakly to a some element in the solution set of SFVIP.

In 2013 Chuang [7] gave a strong convergence theorems for problem SFVIP under some conditions, like the Halpern-Mann type iteration method. The following is an iteration process given by Chuang[7]:

$$x_{n+1} := a_n u + b_n x_n + c_n J_{\beta_n}^{B_1}\big[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\big] + d_n v_n \tag{1.7}$$

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences of real numbers in $[0, 1]$ with $a_n + b_n + c_n + d_n = 1$ and $0 < a_n < 1$ for each $n \in \mathbb{N}$, $\{v_n\}$ is a bounded sequence in $H_1$, $u$ is fixed and $\rho_n$ is chosen in the interval $(0, \frac{2}{\|A\|^2+1})$.

In this work, we introduce and study some algorithms for solving the common fixed point problem of a finite family of quasi-nonexpansive mappings and the split variational inclusion problem in Hilbert spaces. We establish a strong convergence result under some suitable conditions. A numerical example supporting our main result is also given.

## 2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. We shall assume that $H$ be a (real) Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, respectively. We denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [14], for each $x, y, u, v \in H$ and $t \in [0, 1]$, we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle;$$

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2;$$

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2.$$

Furthermore, we obtain the following Lemma.

**Lemma 2.1.** [8] *Let $H$ be a real Hilbert space. Then for each $m \in \mathbb{N}$*

$$\left\|\sum_{i=1}^{m} t_i x_i\right\|^2 = \sum_{i=1}^{m} t_i \|x_i\|^2 - \sum_{i=1, i\neq j}^{m} t_i t_j \|x_i - x_j\|^2,$$

*where $x_i \in H, t_i, t_j \in [0, 1]$ for all $i, j = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} t_i = 1$.*

**Lemma 2.2.** [9] *Let $H$ be a (real) Hilbert space, and let $x, y \in H$. Then $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.*

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Recall that the (metric) projection from $H$ onto $C$, denote by $P_C$ is defined for each $x \in H$, $P_C x$ is the unique element in C such that

$$\|x - P_C x\| = \inf\{\|x-y\| : y \in C\}.$$

**Lemma 2.3.** [10] *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $P_C$ be the metric projection from $H$ onto $C$. Then, for each $x \in H$ and $z \in C$, we know that $z = P_C x$ if and only if $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $T : H \to H$ be a mapping. Let $Fix(T) := \{x \in H : Tx = x\}$. Now let us recall the definitions of some mappings concerned in our study.

**Definition 2.4.** Let $H$ be a real Hilbert space. A mapping $T : H \to H$ is said to be

(*i*) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$,

(*ii*) *quasi-nonexpansive* if

$$Fix(T) \neq \emptyset \text{ and } \|Tx - q\| \leq \|x - q\| \text{ for all } x \in H \text{ and } q \in Fix(T),$$

(*iii*) *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ for all $x, y \in H$.

It is easy to see that $Fix(T)$ is a closed convex subset of $H$ if $T$ is a quasi-nonexpansive mapping.

**Lemma 2.5.** *An mapping $T : H \to H$ is called demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to $w$ and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tw = 0$.*

**Lemma 2.6.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. If $T : C \to H$ is a nonexpansive mapping, then $I - T$ is demiclosed at the origin.*

The following are important tools to study the split variational inclusion problems.

**Lemma 2.7.** [11] *Let $H$ be a real Hilbert space. Let $B : H \to 2^H$ be a set-valued maximal monotone mapping, $\beta > 0$, and let $J_\beta^B$ be a resolvent mapping of $B$ defined by $J_\beta^B(x) = (I + \beta B)^{-1}(x)$ for each $x \in H$. Thus*

- *(i)* *$J_\beta^B$ is a single-valued and firmly nonexpansive mapping for each $\beta > 0$;*
- *(ii)* *$\mathcal{D}(J_\beta^B) = H$ and $Fix(J_\beta^B) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$;*
- *(iii)* *$\left\|x - J_\beta^B x\right\| \leq \left\|x - J_\gamma^B x\right\|$ for all $0 < \beta \leq \gamma$ and for all $x \in H$;*
- *(iv)* *Suppose that $B^{-1}(0) \neq \emptyset$. Then $\left\|x - J_\beta^B x\right\|^2 + \left\|J_\beta^B x - \bar{x}\right\|^2 \leq \|x - \bar{x}\|^2$ for each $x \in H$, each $\bar{x} \in B^{-1}(0)$, and each $\beta > 0$.*
- *(v)* *Suppose that $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_\beta^B x, J_\beta^B x - w\rangle \geq 0$ for each $x \in H$, each $w \in B^{-1}(0)$, and each $\beta > 0$.*

**Lemma 2.8.** [7] *Let $H_1$ and $H_2$ be real Hilbert spaces, $A : H_1 \to H_2$ be linear operator, and $A^*$ be the adjiont of $A$, and let $\beta > 0$ be fixed, and let $\rho \in (0, \frac{2}{\|A\|^2})$. Let $B_2 : H_2 \to 2^{H_2}$ be a set-valued maximal monotone mapping, and let $J_\beta^{B_2}$ be a resolvent mapping of $B_2$. Then*

$$\begin{aligned}&\left\|\left[x - \rho A^*(I - J_\beta^{B_2})Ax\right] - \left[y - \rho A^*(I - J_\beta^{B_2})Ay\right]\right\|^2 \\ &\quad\leq \|x - y\|^2 - (2\rho - \rho^2\|A\|^2)\left\|(I - J_\beta^{B_2})Ax - (I - J_\beta^{B_2})Ay\right\|^2\end{aligned}$$

*for all $x, y \in H_1$. Furthermore, $I - \rho A^*(I - J_\beta^{B_2})A$ is a nonexpansive mapping.*

**Lemma 2.9.** [12] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ which satisfies $a_{n_i} \leq a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty, a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

**Lemma 2.10.** [9] *Let $\{a_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + b_n + c_n, \quad n \geq 1$$

*where $\{\delta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a real sequence. Assume $\sum_{n=1}^{\infty} c_n < \infty$. Then the following result hold:*

(i) *If* $b_n \leq \delta_n M$ *for some* $M \geq 0$, *then* $\{a_n\}$ *is a bounded sequence.*

(ii) *If* $\sum_{n=1}^{\infty} \delta_n = \infty$ *and* $\limsup_{n\to\infty} b_n/\delta_n \leq 0$, *then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.11.** [7] *Let* $H_1$ *and* $H_2$ *be real Hilbert spaces,* $A : H_1 \to H_2$ *be linear operator, and* $A^*$ *be the adjiont of* $A$, *and let* $\beta > 0, \gamma > 0, B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *be a set-valued maximal monotone mappings. Given any* $\bar{x} \in H_1$.

(i) *If* $\bar{x}$ *is a solution of (SFVIP), then* $J_\beta^{B_1}\big[\bar{x} - \gamma A^*(I - J_\beta^{B_2})A\bar{x}\big] = \bar{x}$.

(ii) *Suppose that* $J_\beta^{B_1}\big[\bar{x} - \gamma A^*(I - J_\beta^{B_2})A\bar{x}\big] = \bar{x}$ *and the solution set of (SFVIP) is nonempty. Then* $\bar{x}$ *is a solution of (SFVIP).*

# 3 Main Results

**Theorem 3.1.** *Let* $H_1$ *and* $H_2$ *be two real Hilbert spaces,* $A : H_1 \to H_2$ *be a bounded linear operator, and let* $A^*$ *denote the adjoint of* $A$. *Let* $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *be two set-valued maximal monotone mappings. Let* $\{T_i : i = 1, 2, \ldots, N\}$ *be family of quasi-nonexpansive mappings of* $H_1$ *into itself. Let* $\{a_n\}, \{b_{n,i}\}, i = 1, 2, \ldots, N$ *and* $\{c_n\}$ *be sequences of real numbers in* $[0, 1]$ *with* $a_n + \sum_{i=1}^{N} b_{n,i} + c_n = 1$ *and* $0 < a_n < 1$ *for all* $n \in \mathbb{N}$. *Let* $\{\beta_n\}$ *be a sequence in* $(0, \infty)$. *Let* $x_1, u \in H_1$ *be fixed. Let* $\{\rho_n\} \subseteq (0, \frac{2}{\|A\|^2+1})$.

*Let* $\Omega := \big\{x \in H_1 : x \in \bigcap_{i=1}^{N} Fix(T_i), 0 \in B_1(x) \text{ and } 0 \in B_2(Ax)\big\}$ *and suppose that* $\Omega \neq \emptyset$. *Let* $\{x_n\}$ *be defined by*

$$x_{n+1} := a_n u + \sum_{i=1}^{N} b_{n,i} T_i x_n + c_n J_{\beta_n}^{B_1}\big[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\big]$$

*for each* $n \in \mathbb{N}$. *Assume that:*

(i) $\lim_{n\to\infty} a_n = 0; \sum_{n=1}^{\infty} a_n = \infty;$

(ii) $\liminf_{n\to\infty} \rho_n > 0; \liminf_{n\to\infty} c_n > 0; \liminf_{n\to\infty} \beta_n > 0; \liminf_{n\to\infty} b_{n,i} > 0 \quad \forall i = 1, 2, \ldots, N.$

(iii) $I - T_i$ *are demiclosed at origin for all* $i = 1, 2, \ldots, N$.

*Then* $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_\Omega u$.

*Proof.* Let $\bar{x} = P_\Omega u$, where $P_\Omega$ is the metric projection from $H_1$ onto $\Omega$.

Then, for each $n \in \mathbb{N}$, it follows from Lemma 2.8 that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\| &= \left\| a_n u + \sum_{i=1}^{N} b_{n,i} T_i x_n + c_n J_{\beta_n}^{B_1}\left[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\right] - \bar{x} \right\| \\
&\leq a_n \|u - \bar{x}\| + \sum_{i=1}^{N} b_{n,i} \|x_n - \bar{x}\| + c_n \left\| J_{\beta_n}^{B_1}\left[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\right] - \bar{x} \right\| \\
&\leq a_n \|u - \bar{x}\| + \sum_{i=1}^{N} b_{n,i} \|x_n - \bar{x}\| \\
&\quad + c_n \left\| J_{\beta_n}^{B_1}\left[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\right] - J_{\beta_n}^{B_1}\left[\bar{x} - \rho_n A^*(I - J_{\beta_n}^{B_2})A\bar{x}\right] \right\| \\
&\leq a_n \|u - \bar{x}\| + \sum_{i=1}^{N} b_{n,i} \|x_n - \bar{x}\| + c_n \|x_n - \bar{x}\| \\
&= a_n \|u - \bar{x}\| + \big(\sum_{i=1}^{N} b_{n,i} + c_n\big) \|x_n - \bar{x}\| \\
&= a_n \|u - \bar{x}\| + (1 - a_n) \|x_n - \bar{x}\|.
\end{aligned}$$

This implies by Lemma 2.10 that $\{x_n\}$ is a bounded sequence. For convenience, we set $y_n = J_{\beta_n}^{B_1}\left[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\right]$. By Lemma 2.7($ii$) and 2.8, we have

$$\begin{aligned}
\|y_n - \bar{x}\|^2 &= \left\| J_{\beta_n}^{B_1}\left[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\right] - J_{\beta_n}^{B_1}\left[\bar{x} - \rho_n A^*(I - J_{\beta_n}^{B_2})A\bar{x}\right] \right\|^2 \\
&\leq \left\| \left[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\right] - \left[\bar{x} - \rho_n A^*(I - J_{\beta_n}^{B_2})A\bar{x}\right] \right\|^2 \\
&\leq \|x_n - \bar{x}\|^2 - (2\rho_n - \rho_n^2 \|A\|^2) \left\| (I - J_{\beta_n}^{B_2})Ax_n - (I - J_{\beta_n}^{B_2})A\bar{x} \right\|^2 \\
&= \|x_n - \bar{x}\|^2 - (2\rho_n - \rho_n^2 \|A\|^2) \left\| (I - J_{\beta_n}^{B_2})Ax_n \right\|^2. \qquad (3.1)
\end{aligned}$$

Hence, it follows form Lemma 2.2 that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \left\| a_n u + \sum_{i=1}^{N} b_{n,i} T_i x_n + c_n y_n - \bar{x} \right\|^2 \\
&\leq \left\| \sum_{i=1}^{N} b_{n,i}(T_i x_n - \bar{x}) + c_n(y_n - \bar{x}) \right\|^2 + 2a_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle \\
&= (1 - a_n)^2 \left\| \sum_{i=1}^{N} b'_{n,i}(T_i x_n - \bar{x}) + c'_n(y_n - \bar{x}) \right\|^2 \\
&\quad + 2a_n \langle u - \bar{x}, x_{n+1} - \bar{x} \rangle \qquad (3.2)
\end{aligned}$$

where $b'_{n,i} = \dfrac{b_{n,i}}{1 - a_n} = \dfrac{b_{n,i}}{\sum_{i=1}^{N} b_{n,i} + c_n}, c'_n = \dfrac{c_n}{\sum_{i=1}^{N} b_{n,i} + c_n}$.

By (3.1),(3.2) and Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq \left\|\sum_{i=1}^{N} b_{n,i}(T_i x_n-\bar{x})+c_n(y_n-\bar{x})\right\|^2+2a_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\leq \sum_{i=1}^{N} b_{n,i}\|x_n-\bar{x}\|^2+c_n\|y_n-\bar{x}\|^2 \\
&\quad -\sum_{i=1}^{N} b_{n,i}c_n\|T_i x_n-y_n\|^2+2a_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\leq \sum_{i=1}^{N} b_{n,i}\|x_n-\bar{x}\|^2+c_n(\|x_n-\bar{x}\|^2-(2\rho_n-\rho_n^2\|A\|^2)\left\|(I-J_{\beta_n}^{B_2})Ax_n\right\|^2) \\
&\quad -\sum_{i=1}^{N} b_{n,i}c_n\|T_i x_n-y_n\|^2+2a_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&= \left(c_n+\sum_{i=1}^{N} b_{n,i}\right)\|x_n-\bar{x}\|^2-c_n(2\rho_n-\rho_n^2\|A\|^2)\left\|(I-J_{\beta_n}^{B_2})Ax_n\right\|^2 \\
&\quad -\sum_{i=1}^{N} b_{n,i}c_n\|T_i x_n-y_n\|^2+2a_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle. \qquad (3.3)
\end{aligned}$$

Since $\liminf_{n\to\infty}\beta_n>0$, we may assume that $\beta_n>\beta>0$ for each $n\in\mathbb{N}$.
Next, we consider 2 cases

**Case I**
There exists a natural number $n_0$ such taht $\|x_{n+1}-\bar{x}\|\leq\|x_n-\bar{x}\|$ for each $n\geq n_0$. Because $\{x_n\}$ is a bounded sequence, we have $\lim_{n\to\infty}\|x_n-\bar{x}\|$ exists.
From (3.3),

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq \left(c_n+\sum_{i=1}^{N} b_{n,i}\right)\|x_n-\bar{x}\|^2-c_n(2\rho_n-\rho_n^2\|A\|^2)\left\|(I-J_{\beta_n}^{B_2})Ax_n\right\|^2 \\
&\quad +2a_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\leq \|x_n-\bar{x}\|^2-c_n(2\rho_n-\rho_n^2\|A\|^2)\left\|(I-J_{\beta_n}^{B_2})Ax_n\right\|^2 \\
&\quad +2a_n\langle u-\bar{x}, x_{n+1}-\bar{x}\rangle.
\end{aligned}$$

This implies that $\lim_{n\to\infty}c_n(2\rho_n-\rho_n^2\|A\|^2)\left\|(I-J_{\beta_n}^{B_2})Ax_n\right\|^2=0$.
Since $c_n(2\rho_n-\rho_n^2\|A\|^2)\geq\frac{c_n\rho_n}{\|A^2\|+1}$ for all $n\in\mathbb{N}$ and $\liminf_{n\to\infty}c_n\rho_n>0$, it follows that

$$\lim_{n\to\infty}\left\|Ax_n-J_{\beta_n}^{B_2}Ax_n\right\|=0. \qquad (3.4)$$

By Lemma 2.7($iii$), $\left\|Ax_n - J_{\beta}^{B_2} Ax_n\right\| \leq \left\|Ax_n - J_{\beta_n}^{B_2} Ax_n\right\|$
Hence,

$$\lim_{n\to\infty} \left\|Ax_n - J_{\beta}^{B_2} Ax_n\right\| = 0. \tag{3.5}$$

From (3.3), we have

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq \left(c_n + \sum_{i=1}^{N} b_{n,i}\right) \|x_n - \bar{x}\|^2 - \sum_{i=1}^{N} b_{n,i} c_n \|T_i x_n - y_n\|^2 \\ &\quad + 2a_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\leq \|x_n - \bar{x}\|^2 - \sum_{i=1}^{N} b_{n,i} c_n \|T_i x_n - y_n\|^2 + 2a_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle.\end{aligned}$$

Thus, for each $i = 1, 2, \ldots, N$,

$$\|x_{n+1} - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 - b_{n,i} c_n \|T_i x_n - y_n\|^2 + 2a_n \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle.$$

This implies $\lim_{n\to\infty} b_{n,i} c_n \|T_i x_n - y_n\| = 0 \quad \forall i = 1, 2, \ldots, N.$
Since $\liminf_{n\to\infty} c_n > 0$ and $\liminf_{n\to\infty} b_{n,i} > 0 \quad \forall i = 1, 2, \ldots, N$, it follows that

$$\lim_{n\to\infty} \|T_i x_n - y_n\| = 0 \quad \forall i = 1, 2, \ldots, N. \tag{3.6}$$

Further, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$ for some $z \in H_1$ and

$$\limsup_{n\to\infty} \langle u - \bar{x}, x_{n+1} - \bar{x}\rangle = \lim_{k\to\infty} \langle u - \bar{x}, x_{n_k} - \bar{x}\rangle = \langle u - \bar{x}, z - \bar{x}\rangle. \tag{3.7}$$

Clearly, $Ax_{n_k} \rightharpoonup Az$. From (3.5) and nonexpansiveness of $J_{\beta}^{B_2}$, we have, by Lemma 2.6, $J_{\beta}^{B_2} Az = Az$. That is $Az \in Fix(J_{\beta}^{B_2})$. By Lemma 2.7($ii$), $Az \in B_2^{-1}(0)$.
Since $J_{\beta_n}^{B_1}$ and $J_{\beta_n}^{B_2}$ are nonexpansive for each $n$, we have

$$\begin{aligned}\left\|y_n - J_{\beta_n}^{B_1} x_n\right\| &= \left\|J_{\beta_n}^{B_1}\left[x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n\right] - J_{\beta_n}^{B_1} x_n\right\| \\ &\leq \left\|x_n - \rho_n A^*(I - J_{\beta_n}^{B_2})Ax_n - x_n\right\| \\ &= \rho_n \left\|A^*(I - J_{\beta_n}^{B_2})Ax_n\right\| \\ &\leq \frac{2\|A\|}{\|A\|^2 + 1} \cdot \left\|Ax_n - J_{\beta_n}^{B_2} Ax_n\right\|.\end{aligned} \tag{3.8}$$

By (3.4),

$$\lim_{n\to\infty} \left\|y_n - J_{\beta_n}^{B_1} x_n\right\| = 0. \tag{3.9}$$

From (3.1),

$$\|y_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2. \tag{3.10}$$

Moreover,

$$\begin{aligned}
\|y_n-\bar{x}\|^2 &= \left\|J_{\beta_n}^{B_1}\left[x_n-\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\right]-\bar{x}\right\|^2\\
&\le \langle y_n-\bar{x}, x_n-\bar{x}-\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle\\
&= \frac{1}{2}\|y_n-\bar{x}\|^2+\frac{1}{2}\left\|x_n-\bar{x}-\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\right\|^2\\
&\quad -\frac{1}{2}\left\|y_n-\bar{x}-x_n+\bar{x}+\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\right\|^2\\
&= \frac{1}{2}\|y_n-\bar{x}\|^2+\frac{1}{2}\|x_n-\bar{x}\|^2+\frac{1}{2}\left\|\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\right\|^2-\frac{1}{2}\|y_n-x_n\|^2\\
&\quad -\frac{1}{2}\left\|\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\right\|^2-\langle x_n-\bar{x},\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle\\
&\quad -\langle y_n-x_n,\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle\\
&= \frac{1}{2}\|y_n-\bar{x}\|^2+\frac{1}{2}\|x_n-\bar{x}\|^2-\langle y_n-\bar{x},\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle-\frac{1}{2}\|y_n-x_n\|^2.
\end{aligned}$$

By (3.10), we have

$$\|y_n-\bar{x}\|^2\le\|x_n-\bar{x}\|^2+\langle\bar{x}-y_n,\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle-\frac{1}{2}\|y_n-x_n\|^2. \quad (3.11)$$

By (3.3) and (3.11),

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\le \sum_{i=1}^N b_{n,i}\|x_n-\bar{x}\|^2+c_n\|y_n-\bar{x}\|^2+2a_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\le c_n(\|x_n-\bar{x}\|^2+\langle\bar{x}-y_n,\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle-\frac{1}{2}\|y_n-x_n\|^2)^2\\
&\quad +\sum_{i=1}^N b_{n,i}\|x_n-\bar{x}\|^2+2a_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\
&= \left(c_n+\sum_{i=1}^N b_{n,i}\right)\|x_n-\bar{x}\|^2+c_n\langle\bar{x}-y_n,\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle\\
&\quad -\frac{c_n}{2}\|y_n-x_n\|^2+2a_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\le \|x_n-\bar{x}\|^2+c_n\langle\bar{x}-y_n,\rho_nA^*(I-J_{\beta_n}^{B_2})Ax_n\rangle\\
&\quad -\frac{c_n}{2}\|y_n-x_n\|^2+2a_n\langle u-\bar{x},x_{n+1}-\bar{x}\rangle. \qquad (3.12)
\end{aligned}$$

This together with the condition $\liminf_{n\to\infty} c_n>0$, we get

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \quad (3.13)$$

From (3.6) and (3.13), we obtain

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=0 \qquad \forall i=1,2,\ldots,N. \quad (3.14)$$

Again, by (3.9) and (3.13), we get

$$\lim_{n\to\infty}\left\|x_n - J_{\beta_n}^{B_1}x_n\right\| = 0 \tag{3.15}$$

By (3.15) and Lemma 2.7($iii$),

$$\lim_{n\to\infty}\left\|x_n - J_{\beta}^{B_1}x_n\right\| = 0 \tag{3.16}$$

By Lemma 2.6, we obtain $J_\beta^{B_1}(z) = z$. That is $z \in Fix(J_\beta^{B_1})$. By Lemma 2.7($ii$), $z \in B_1^{-1}(0)$. So, $z$ is a solution of (SFVIP). Since $I - T_i$ are demiclosed at origin for all $i = 1, 2, \ldots, N$, we also get $z \in \bigcap_{i=1}^N FixT_i$. Thus $z \in \Omega$. From $\bar{x} = P_\Omega u$, we obtain that $\langle u - \bar{x}, \bar{x} - z\rangle \geq 0$ by Lemma 2.3. Hence

$$\limsup_{n\to\infty}\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle = \langle u - \bar{x}, z - \bar{x}\rangle \leq 0. \tag{3.17}$$

From (3.3), we get

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq \left(c_n + \sum_{i=1}^N b_{n,i}\right)\|x_n - \bar{x}\|^2 + 2a_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &= (1 - a_n)\|x_n - \bar{x}\|^2 + 2a_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle.\end{aligned}$$

By Lemma 2.10, we have $\lim\limits_{n\to\infty}\|x_n - \bar{x}\|$ . Therefore $\lim\limits_{n\to\infty} x_n = \bar{x}$.

**Case II**

Suppose that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that for each $j \in \mathbb{N}$ $\left\|x_{n_j} - \bar{x}\right\| \leq \left\|x_{n_j+1} - \bar{x}\right\|$. By Lemma 2.9, there exists a nondecreasing sequence $\{m_k\}$ in $\mathbb{N}$ such that $m_k \to \infty$,

$$\|x_{m_k} - \bar{x}\| \leq \|x_{m_k+1} - \bar{x}\| \quad \text{and} \quad \|x_k - \bar{x}\| \leq \|x_{m_k+1} - \bar{x}\| \qquad \forall k \in \mathbb{N}. \tag{3.18}$$

By (3.3) and (3.18), we have

$$\begin{aligned}\|x_{m_k} - \bar{x}\|^2 &\leq \|x_{m_k+1} - \bar{x}\|^2 \\ &\leq \left(c_{m_k} + \sum_{i=1}^N b_{m_k,i}\right)\|x_{m_k} - \bar{x}\|^2 + 2a_{m_k}\langle u - \bar{x}, x_{m_k+1} - \bar{x}\rangle \\ &\quad - c_{m_k}(2\rho_{m_k} - \rho_{m_k}^2\|A\|^2)\left\|(I - J_{\beta_{m_k}}^{B_2})Ax_{m_k}\right\|^2 \\ &\quad - \sum_{i=1}^N b_{m_k,i}c_{m_k}\|T_ix_{m_k} - y_{m_k}\|^2 \\ &\leq \|x_{m_k} - \bar{x}\|^2 - c_{m_k}(2\rho_{m_k} - \rho_{m_k}^2\|A\|^2)\left\|(I - J_{\beta_{m_k}}^{B_2})Ax_{m_k}\right\|^2 \\ &\quad - \sum_{i=1}^N b_{m_k,i}c_{m_k}\|T_ix_{m_k} - y_{m_k}\|^2 + 2a_{m_k}\langle u - \bar{x}, x_{m_k+1} - \bar{x}\rangle \end{aligned} \tag{3.19}$$

It follows that $\lim_{k\to\infty} c_{m_k}(2\rho_{m_k} - \rho_{m_k}^2 \|A\|^2) \left\|(I - J_{\beta_{m_k}}^{B_2})Ax_{m_k}\right\|^2 = 0$.
Following a similar argument as the proof of case I, we have

$$\lim_{k\to\infty} \left\|Ax_{m_k} - J_\beta^{B_2} Ax_{m_k}\right\| = 0. \tag{3.20}$$

and

$$\lim_{k\to\infty} \|T_i x_{m_k} - y_{m_k}\| = 0 \quad \forall i = 1, 2, \ldots, N. \tag{3.21}$$

Further, there exists a subsequence $\{x_{m_{k_l}}\}$ of $\{x_{m_k}\}$ such that $x_{m_{k_l}} \rightharpoonup z$ for some $z \in H_1$ and

$$\limsup_{k\to\infty} \langle u - \bar{x}, x_{m_k+1} - \bar{x}\rangle = \lim_{l\to\infty} \langle u - \bar{x}, x_{m_{k_l}} - \bar{x}\rangle. \tag{3.22}$$

Clearly, $Ax_{m_{k_l}} \rightharpoonup Az$. From (3.5) and nonexpansiveness of $J_\beta^{B_2}$, by Lemma 2.6, we have $J_\beta^{B_2} Az = Az$. That is $Az \in Fix(J_\beta^{B_2})$. By Lemma 2.7($ii$), $Az \in B_2^{-1}(0)$. Moreover, by (3.9),

$$\lim_{k\to\infty} \left\|y_{m_k} - J_{\beta_{m_k}}^{B_1} x_{m_k}\right\| = 0 \tag{3.23}$$

From (3.12) and (3.18), we have

$$\begin{aligned}
\|x_{m_k+1} - \bar{x}\|^2 &\leq \|x_{m_k} - \bar{x}\|^2 + c_{m_k}\langle \bar{x} - y_{m_k}, \rho_{m_k} A^*(I - J_{\beta_{m_k}}^{B_2})Ax_{m_k}\rangle \\
&\quad - \frac{c_{m_k}}{2} \|y_{m_k} - x_{m_k}\|^2 + 2a_{m_k}\langle u - \bar{x}, x_{m_k+1} - \bar{x}\rangle \\
&\leq \|x_{m_k+1} - \bar{x}\|^2 + c_{m_k}\langle \bar{x} - y_{m_k}, \rho_{m_k} A^*(I - J_{\beta_{m_k}}^{B_2})Ax_{m_k}\rangle \\
&\quad - \frac{c_{m_k}}{2} \|y_{m_k} - x_{m_k}\|^2 + 2a_{m_k}\langle u - \bar{x}, x_{m_k+1} - \bar{x}\rangle
\end{aligned} \tag{3.24}$$

This implies

$$\lim_{k\to\infty} \|y_{m_k} - x_{m_k}\| = 0. \tag{3.25}$$

From $\left\|x_{m_k} - J_{\beta_{m_k}}^{B_1} x_{m_k}\right\| \leq \left\|x_{m_k} - y_{m_k} + y_{m_k} - J_{\beta_{m_k}}^{B_1} x_{m_k}\right\|$,
by (3.23) and (3.25), we get

$$\lim_{k\to\infty} \left\|x_{m_k} - J_{\beta_{m_k}}^{B_1} x_{m_k}\right\| = 0. \tag{3.26}$$

By Lemma 2.7($iii$),we also get

$$\lim_{k\to\infty} \left\|x_{m_k} - J_\beta^{B_1} x_{m_k}\right\| = 0. \tag{3.27}$$

By Lemma 2.6 and nonexpansiveness of $J_\beta^{B_1}$, we have $J_\beta^{B_1}(z) = z$. That is $z \in Fix(J_\beta^{B_1})$. By Lemma 2.7($ii$), $z \in B_1^{-1}(0)$. So, $z$ is a solution of (SFVIP). Since $I - T_i$ are demiclosed at origin for all $i = 1, 2, \ldots, N$, we have $z \in \bigcap_{i=1}^N FixT_i$.

Thus $z \in \Omega$. From $\bar{x} = P_\Omega u$, we obtain that $\langle u - \bar{x}, \bar{x} - z \rangle \geq 0$, by Lemma 2.3. Hence

$$\limsup_{k\to\infty} \langle u - \bar{x}, x_{m_k+1} - \bar{x} \rangle = \langle u - \bar{x}, z - \bar{x} \rangle \leq 0. \tag{3.28}$$

By (3.19),

$$\begin{aligned} \|x_{m_k} - \bar{x}\|^2 &\leq \left( c_{m_k} + \sum_{i=1}^{N} b_{m_k,i} \right) \|x_{m_k} - \bar{x}\|^2 + 2a_{m_k} \langle u - \bar{x}, x_{m_k+1} - \bar{x} \rangle \\ &\leq (1 - a_{m_k}) \|x_{m_k} - \bar{x}\|^2 + 2a_{m_k} \langle u - \bar{x}, x_{m_k+1} - \bar{x} \rangle. \end{aligned} \tag{3.29}$$

It follows that $\|x_{m_k} - \bar{x}\|^2 \leq 2\langle u - \bar{x}, x_{m_k+1} - \bar{x} \rangle$.
By (3.28) and (3.29), we get

$$\lim_{k\to\infty} \|x_{m_k} - \bar{x}\| = 0. \tag{3.30}$$

For each $k \in \mathbb{N}$, we have

$$\begin{aligned} \|x_{m_k+1} - x_{m_k}\| &= \left\| a_{m_k} u + \sum_{i=1}^{N} b_{m_k,i} T_i x_{m_k} + c_{m_k} y_{m_k} - x_{m_k} \right\| \\ &\leq a_{m_k} \|u - x_{m_k}\| + \sum_{i=1}^{N} b_{m_k,i} \|T_i x_{m_k} - x_{m_k}\| + c_{m_k} \|y_{m_k} - x_{m_k}\|. \end{aligned}$$

It follows by (3.14),(3.25) and $\lim_{n\to\infty} a_n = 0$ that

$$\lim_{k\to\infty} \|x_{m_k+1} - x_{m_k}\| = 0. \tag{3.31}$$

Therefore,

$$\|x_{m_k+1} - \bar{x}\| \leq \|x_{m_k+1} - x_{m_k}\| + \|x_{m_k} - \bar{x}\|. \tag{3.32}$$

Hence, by (3.18) and (3.32),

$$\lim_{k\to\infty} \|x_k - \bar{x}\| = 0.$$

It implies that $\lim_{k\to\infty} x_k = \bar{x}$. Moreover, by Lemma 2.6, $T_i \bar{x} = \bar{x}$ for all $i = 1, 2, \ldots, N$. Therefore,

$$\bar{x} \in \bigcap_{i=1}^{N} Fix(T_i).$$

Therefore, the proof is completed. □

# 4 Numerical example

In this section, we give a numerical example to demonstrate the convergence of our algorithm.
Let $H_1 = \mathbb{R}^2, H_2 = \mathbb{R}^3$. Let $B_1 : \mathbb{R}^2 \to \mathbb{R}^2, B_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$B_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, B_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 4 & 3 & 1 \\ 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and let $A : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. We see that both $B_1$ and $B_2$ are maximal monotone mappings and $A$ is a bounded linear operator. For each $i = 1, 2, \ldots, N$, define a mapping $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_i(x, y)^\top = \begin{cases} \frac{i}{i+1}(x \sin \frac{1}{x}, y \sin \frac{1}{y})^\top, & \text{if } x \neq 0 \text{ and } y \neq 0; \\ (0, 0)^\top, & \text{otherwise.} \end{cases}$$

Then $T_1$ and $T_2$ are quasi-nonexpansive mapping (but not nonexpansive) with a unique fixed point $(0, 0)^\top$. It's not hard to see that $I - T_1$ and $I - T_2$ are demiclosed at origin. Let $\Omega := \{x \in \mathbb{R}^2 : x \in Fix(T_1) \cap Fix(T_2), 0 \in B_1(x) \text{ and } 0 \in B_2(Ax)\}$. We see that $(0, 0)^\top \in \Omega$. Choose $a_n = \frac{1}{100n+1}, b_{n,1} = b_{n,2} = \frac{1}{4} - \frac{1}{200n}, c_n = \frac{1}{2} + \frac{1}{100n} - a_n, \rho_n = \frac{1}{\|A\|^2+1}$ and $\beta_n = \frac{n+1}{2n}$ for all $n \in \mathbb{N}$.

First, we start with the initial point $x_1 = (4, -7)^\top$ and $u = (-5, 5)^\top$. The stopping criterion for our testing method is taken as: $\|x_{n+1} - x_n\| < 10^{-4}$. Now, a convergence of our algorithm is shown in Table 1.

Table 1: Numerical experiment of the algorithm in Theorem 3.1

| $n$ | $x_n$ | $\|x_{n+1} - x_n\|$ |
|---|---|---|
| 2 | (1.6007,-3.956146) | 2.007613 |
| 3 | (0.48182,-2.289227) | 1.336871 |
| 4 | (-0.350435,-1.243007) | 0.703085 |
| 5 | (-0.18089,-0.560671) | 0.608410 |
| 6 | (-0.28325,0.039067) | 0.30006 |
| . | . | . |
| . | . | . |
| . | . | . |
| 60 | (-0.000817,0.001026) | 0.000498 |
| 61 | (-0.001045,0.001468) | 0.000864 |
| 62 | (-0.000540,0.000767) | 9.29E-05 |

From Table 1, we observe that a sequence $\{x_n\}$ strongly converges to $(0, 0)^\top$ and $(0, 0)^\top$ is a solution of SFVIP and common fixed point of $T_1$ and $T_2$.

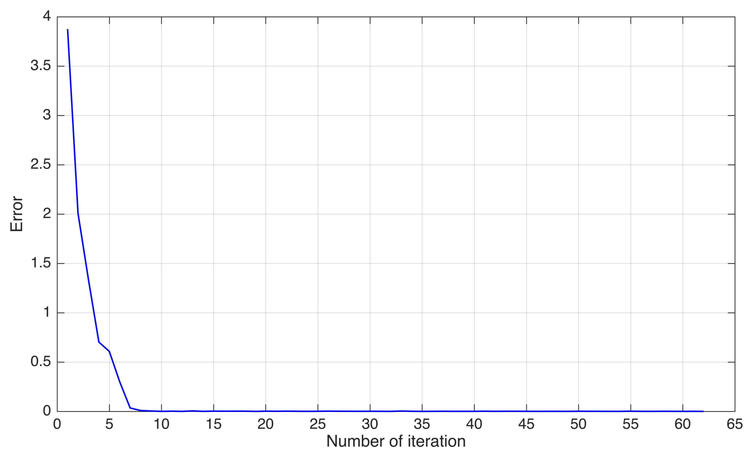


Figure 1: Figure of error $\|x_{n+1} - x_n\|$

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