The Hybrid Method for Generalized Mixed Equilibrium Problems for an Infinite Family of Asymptotically Nonexpansive Mappings

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Abstract: In this paper, we introduce a hybrid method for finding a common element of the set of common fixed points for an infinite family of asymptotically nonexpansive mappings and the set of solutions of a generalized mixed equilibrium problem in Hilbert spaces. The results obtained in this paper improve and extend the recently corresponding results.

Keywords: Generalized mixed equilibrium problem; Asymptotically nonexpansive mapping.

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1 Introduction

Let $C$ be a closed convex subset of a real Hilbert space $H$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, $A: C \to H$ a mapping and $\varphi: C \to \mathbb{R}$ a real-valued function. The generalized mixed equilibrium problem is for finding $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \quad (1.1)$$

The set of solutions of (1.1) is denoted by $\text{GMEP}(F, \varphi, A)$, that is,

$$\text{GMEP}(F, \varphi, A) = \{ x \in C : F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C \}. \quad (1.2)$$
If $F \equiv 0$, the problem (1.1) is reduced into the mixed variational inequality of Browder type [1], for finding $x \in C$ such that

$$\langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\text{MVI}(C, \varphi, A)$.

If $A \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced into the equilibrium problem [2] for finding $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $\text{EP}(F)$. This problem contains fixed point problems and includes as special cases numerous problems in physics, optimization, and economics. Some methods have been proposed to solve the equilibrium problem; see [3–5].

If $F \equiv 0$ and $\varphi \equiv 0$, the problem (1.1) is reduced into the Harmann-Stampacchia variational inequality [6] for finding $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $\text{VI}(C, A)$. The variational inequality has been extensively studied in the literature [7].

If $F \equiv 0$ and $A \equiv 0$, the problem (1.1) is reduced into the minimize problem for finding $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by $\text{Arg min}(\varphi)$.

Recall that a mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in C. \tag{1.7}$$

A mapping $A$ of $C$ into $H$ is called $\alpha$-inverse strongly monotone, see [8–10], if there exists a positive real number $\alpha$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \| Ax - Ay \|^2, \forall x, y \in C. \tag{1.8}$$

It is obvious that any $\alpha$-inverse strongly monotone mapping $A$ is monotone and Lipschitz continuous.

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, $S : C \to C$ be a mapping. We denote $F(S)$ to be the set of fixed points of $S$, i.e. $F(S) = \{ x \in C : x = Sx \}$. A mapping $S$ is said to be

(i) nonexpansive, if $\| Sx - Sy \| \leq \| x - y \| \ \forall x, y \in C$;

(ii) asymptotically nonexpansive, if there exist a sequence $k_n \geq 1$ such that

$$\lim_{n \to \infty} k_n = 1 \text{ and } \| S^n x - S^n y \| \leq k_n \| x - y \|, \forall x, y \in C, n \geq 1; \tag{1.9}$$
uniformly $L$-Lipschitzian, if there exist a constant $L > 0$ such that
\[ \|S^n x - S^n y\| \leq L\|x - y\|, \ \forall x, y \in C, \ n \geq 1; \] (1.10)

In 2003, Nakajo and Takahashi [11] proposed the following modification of the Mann iteration method for a nonexpansive mapping $T$ in a Hilbert space $H$:

\[ x_0 \in C \text{ chosen arbitrarily}, \]
\[ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \]
\[ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \]
\[ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n \rangle \geq 0\}, \]
\[ x_{n+1} = P_{C_n \cap Q_n}x_0. \] (1.11)

where $P_C$ is denoted the metric projection from $H$ onto a closed and convex subset $C$ of $H$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ is defined by (1.11) converges strongly to $P_{F(T)}x_0$.

Inchan and Plubtieng [12] introduced the modified Ishikawa iteration process by shrinking hybrid method [13] for two asymptotically nonexpansive mappings $S$ and $T$, with a closed convex bounded subset $C$ of a Hilbert space $H$. For $C_1 = C$ and $x_1 = P_Cx_0$, $\{x_n\}$ is defined as follows:
\[ y_n = \alpha_n x_n + (1 - \alpha_n)T^n z_n, \]
\[ z_n = \beta_n x_n + (1 - \beta_n)S^n x_n, \]
\[ C_{n+1} = \{v \in C_n : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \]
\[ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N} \] (1.12)

where $\theta_n = (1 - \alpha_n)(t^2 - 1) + \beta_n s^2 - 1)\|y_n - v\|^2 \leq (diam C)^2 \rightarrow 0$, as $n \rightarrow \infty$ and $0 \leq \alpha_n \leq \alpha < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$. They proved that the sequence $\{x_n\}$ is generated by (1.12) converges strongly to a common fixed point of two asymptotically nonexpansive mappings $S$ and $T$.

The purpose of this paper is to introduce the Mann iteration process for finding a common element of the set of common fixed points of an infinite family of asymptotically nonexpansive mappings and the set of solutions of a generalized mixed equilibrium problem under some control conditions. We prove that the strong convergence theorem which extends and improves the result of many others [11, 12].

2 Preliminaries

In this section, we present some useful lemmas which will be used in our main result and we will use the notation:

- $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence.
- $\omega_{\omega}(x_n) = \{x : x_{n_i} \rightharpoonup x\}$ denotes the weak $\omega$-limit set of $\{x_n\}$. 

Let $H$ be a real Hilbert space. Then
\[ \|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2 \langle x - y, y \rangle \quad \forall \, x, y \in H. \quad (2.1) \]

For each $x, y \in H$ and $\lambda \in \mathbb{R}$, we known that
\[ \|\lambda x - (1 - \lambda)y\|^2 = \lambda \|x\|^2 - (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|. \quad (2.2) \]

Let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$, then
\[ \|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \| (I - P_C) x - (I - P_C) y \|^2, \quad \forall \, x, y \in H, \quad (2.3) \]

where $I$ is the identity mapping.

**Lemma 2.1** (Opial’s condition [14]). For any sequence $\{x_n\}$ in a Hilbert space $H$ with $x_n \rightharpoonup x$, the inequality
\[ \liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad (2.4) \]

holds for every $y \in H$ with $y \neq x$.

**Lemma 2.2** (The Kadec-Klee property [15]). For any sequence $\{x_n\}$ in a Hilbert space $H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

**Lemma 2.3** (Demiclosedness Principle [16]). Suppose $X$ is a Banach space satisfying the locally uniform Opial’s condition, $C$ is a nonempty weakly compact convex subset of $X$, and $T : C \to C$ is an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, i.e. if $\{x_n\}$ is a sequence in $C$ which converge weakly to $x$ and if the sequence $\{x_n - Tx_n\}$ converge strongly to zero, then $x - Tx = 0$.

**Lemma 2.4** ([17]). Let $C$ be a nonempty closed convex subset of $H$ and also give a real number $a \in \mathbb{R}$. The set $D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$ is convex and closed.

**Lemma 2.5** ([18]). Assume that $\{a_n\}$ is sequence of nonnegative real numbers such that
\[ a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq 1, \quad (2.5) \]

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ is sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n \to \infty} (\delta_n/\gamma_n) \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$.

For solving the generalized mixed equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$, a continuous monotone $A : C \to H$, and $\varphi : C \to \mathbb{R}$ satisfies the following conditions:
(A1) \( F(x, x) = 0 \) for all \( x \in C \);
(A2) \( F \) is monotone, that is, \( F(x, y) + F(y, x) \leq 0 \) for any \( x, y \in C \);
(A3) For each fixed \( y \in C \), \( x \mapsto F(x, y) \) is weakly upper semicontinuous;
(A4) For each fixed \( x \in C \), \( y \mapsto F(x, y) \) is convex and lower semicontinuous;
(B1) For each \( x \in C \) and \( r > 0 \), there exists a bounded subset \( D_x \subseteq C \) and \( y_x \in C \) such that, for any \( z \in C \setminus D_x \),
\[
F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0;
\]
(B2) \( C \) is a bounded set.

**Lemma 2.6** ([19]). Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). Let \( F : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4), and let \( \varphi : C \to \mathbb{R} \cup \{+\infty\} \) be convex and proper lower semicontinuous function such that \( C \cap \text{dom} \varphi \neq \emptyset \). For \( r > 0 \) and \( x \in H \), define a mapping \( K_r : H \to C \) as follows:
\[
K_r(x) = \left\{ u \in C : F(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0, \, \forall y \in C \right\}
\]  
for all \( x \in H \). Assume that either (B1) or (B2) holds. Then, the following hold:
(i) \( K_r \) is single valued;
(ii) \( K_r \) is firmly nonexpansive, that is, \( \|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y \rangle \) for any \( x, y \in H \);
(iii) \( F(K_r) = \text{MEP}(F, \varphi) \);
(iv) \( \text{MEP}(F, \varphi) \) is closed and convex.

**Definition 2.7** ([20]). Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \), let \( \{S_n\} \) be a family of asymptotically nonexpansive mappings of \( C \) into itself, and let \( \{\beta_{n,k} : n, k \in \mathbb{N}, \, 1 \leq k \leq n\} \) be a real sequence of real numbers such that \( 0 \leq \beta_{i,j} \leq 1 \) for every \( i, j \in \mathbb{N} \) with \( i \geq j \). For any \( n \geq 1 \), define a mapping \( W_n : C \to C \) as follows:
\[
U_{n,n} = \beta_{n,n} S_n^n + (1 - \beta_{n,n}) I,
U_{n,n-1} = \beta_{n,n-1} S_{n-1}^n U_{n,n} + (1 - \beta_{n,n-1}) I,
\]
\[
\vdots
\]
\[
U_{n,k} = \beta_{n,k} S_k^n U_{n,k+1} + (1 - \beta_{n,k}) I,
\]
\[
\vdots
\]
\[
U_{n,2} = \beta_{n,2} S_2^n U_{n,3} + (1 - \beta_{n,2}) I,
W_n = U_{n,1} = \beta_{n,1} S_1^n U_{n,2} + (1 - \beta_{n,1}) I.
\]
Such a mapping \( W_n \) is called the modified \( W \)-mapping generated by \( S_n, S_{n-1}, \ldots, S_1 \) and \( \beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1} \).
Lemma 2.8 ([21]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of $C$ into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n} \|x - y\|$ ($\forall m, n \in \mathbb{N}, \forall x, y \in C$) such that $F = \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, and let $\{\beta_{n,k} : n, k \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers with $0 < a \leq \beta_{n,1} \leq 1$ for all $n \in \mathbb{N}$ and $0 < b \leq \beta_{n,i} \leq c < 1$ for every $n \in \mathbb{N}$ and $i = 2, \ldots, n$ for some $a, b, c \in (0, 1)$. Let $W_n$ be the modified $W$-mappings generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$. Let $r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1} \cdots \beta_{n,n-1}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1} \cdots \beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \cdots\}$. Then, the followings hold:

(i) $\|W_n x - z\|^2 \leq (1 + r_{n,1}) \|x - z\|^2$ for all $n \in \mathbb{N}$, $x \in C$ and $z \in \cap_{i=1}^{\infty} F(S_i)$;

(ii) if $C$ is bounded and $\lim_{n \to \infty} r_{n,1} = 0$ for every sequence $\{z_n\}$ in $C$,

$$\lim_{n \to \infty} \|z_{n+1} - z_n\| = 0, \quad \lim_{n \to \infty} \|z_n - W_n z_n\| = 0$$

imply $\omega(z_n) \subseteq F$; (2.9)

(iii) if $\lim_{n \to \infty} r_{n,1} = 0$, $F = \cap_{i=1}^{\infty} F(W_n)$ and $F$ is closed convex.

3 Main Results

In this section, we prove a strong convergence for the set of common fixed points of an infinite family of asymptotically nonexpansive mappings and the sets of solutions of a generalized mixed equilibrium problem in Hilbert space.

Theorem 3.1. Let $C$ be a nonempty bounded closed convex subset of a real Hilbert space $H$, let $F : C \times C \to \mathbb{R}$ be a bifunction, $A : C \to H$ be an $\alpha$-inverse strongly monotone, and $\varphi : C \to \mathbb{R}$ be a convex and lower semicontinuous function, satisfying the conditions (A1) – (A4), (B1) or (B2), let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of $C$ into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n} \|x - y\|$ ($\forall m, n \in \mathbb{N}, \forall x, y \in C$) such that $F \cap \text{GM EP} \neq \emptyset$, where $F = \cap_{i=1}^{\infty} F(S_i)$, and let $\{\beta_{n,k} : n, k \in \mathbb{N}, 1 \leq k \leq n\}$ be a sequence of real numbers with $0 < a \leq \beta_{n,1} \leq 1$ for all $n \in \mathbb{N}$ and $0 < b \leq \beta_{n,i} \leq c < 1$ for every $n \in \mathbb{N}$ and $2 \leq i \leq n$ for some $a, b, c \in (0, 1)$. Let $W_n$ be the modified $W$-mapping is generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$.

Assume that $r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1} \cdots \beta_{n,n-1}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1} \cdots \beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \cdots\}$. Then, $\lim_{n \to \infty} r_{n,1} = 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences is
generated by the following algorithm:

\[ x_1 \in C \] chosen arbitrarily,

\[ u_n \in C, \]

\[ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in C, \]

\[ y_n = \alpha_n u_n + (1 - \alpha_n)W_n u_n, \]

\[ C_{n+1} = \{ v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 + \theta_n \}, \]

\[ x_{n+1} = P_{C_{n+1}}x_1, \quad n \in \mathbb{N}, \]

where \( C_1 = C \) and \( \theta_n = (1 - \alpha_n)r_{n,1}(\text{diam}C)^2 \) and \( 0 \leq \alpha_n \leq d < 1 \) and \( 0 < e \leq r_n \leq f < 2\alpha \). Then \( \{ x_n \} \) and \( \{ u_n \} \) converge strongly to \( P_{\text{FGMEP}}(x_1) \).

**Proof.** We split the proof into 4 steps.

**Step 1.** Show that the sequences \( \{ x_n \} \) and \( \{ y_n \} \) are well defined.

By Lemma 2.4, we have that \( C_n \) is closed and convex. Let \( x, y \in C \). Since \( A \) is \( \alpha \)-inverse strongly monotone and \( r_n < 2\alpha, \forall n \in \mathbb{N} \), we have

\[
\| (I - r_nA)x - (I - r_nA)y \|^2 = \| x - y - r_n(Ax - Ay) \|^2 \\
= \| x - y \|^2 - 2r_n \langle x - y, Ax - Ay \rangle + r_n^2 \| Ax - Ay \|^2 \\
\leq \| x - y \|^2 - 2\alpha r_n \| Ax - Ay \|^2 + r_n^2 \| Ax - Ay \|^2 \\
= \| x - y \|^2 + r_n (r_n - 2\alpha) \| Ax - Ay \|^2 \\
\leq \| x - y \|^2.
\]

Thus \( I - r_nA \) is nonexpansive. Since

\[ F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in C, \] (3.3)

we obtain

\[ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} (y - u_n, u_n - (I - r_nA)x_n) \geq 0, \forall y \in C. \] (3.4)

It follows from Lemma 2.6 that \( u_n = K_{r_n}(x_n - r_nAx_n) \), for all \( n \in \mathbb{N} \).

Let \( p \in F \cup \text{GMEP} \), by Lemma 2.6, we have \( p = K_{r_n}(p - r_nAp) \), for all \( n \in \mathbb{N} \).

Since \( I - r_nA \) and \( K_{r_n} \) are nonexpansive, we have

\[ \| u_n - p \| \leq \| K_{r_n}(x_n - r_nAx_n) - K_{r_n}(p - r_nAp) \| \leq \| x_n - p \|, \quad \forall n \in \mathbb{N}. \] (3.5)

By Lemma 2.8 and the convexity of \( \| . \|^2 \), we have

\[
\| y_n - p \|^2 = \| \alpha_n(u_n - p) + (1 - \alpha_n)(W_n u_n - p) \|^2 \\
\leq \alpha_n \| u_n - p \|^2 + (1 - \alpha_n) \| W_n u_n - p \|^2 \\
\leq \alpha_n \| u_n - p \|^2 + (1 - \alpha_n)(1 + r_n, 1) \| u_n - p \|^2 \\
= \| u_n - p \|^2 + (1 - \alpha_n) r_n, 1 \| u_n - p \|^2 \\
\leq \| u_n - p \|^2 + \theta_n \\
\leq \| x_n - p \|^2 + \theta_n.
\] (3.6)
So, \( p \in C_n \) for all \( n \) and \( F \cup \text{GMEP} \subset C_n \) for all \( n \). This implies that \( \{x_n\} \) is well defined and by Lemma 2.6, we have that \( \{u_n\} \) is also well defined.

**Step 2.** We show that \( \|x_{n+1} - x_n\| \to 0, \|x_n - u_n\| \to 0, \|u_{n+1} - u_n\| \to 0, \|u_n - W_n u_n\| \to 0 \) as \( n \to \infty \). From \( x_n = P_{C_n} x_1 \), we have that

\[
\langle x_1 - x_n, x_n - v \rangle \geq 0, \quad \text{for each } v \in F \cap \text{GMEP} \subset C_n, \; n \in \mathbb{N}.
\] (3.7)

So, for \( p \in F \cap \text{GMEP} \), we have

\[
0 \leq \langle x_1 - x_n, x_n - p \rangle = -\langle x_n - x_1, x_n - x_1 \rangle + \langle x_1 - x_n, x_1 - p \rangle \\
\leq \|x_n - x_1\|^2 + \|x_n - x_1\|\|x_1 - p\|.
\] (3.8)

This implies that

\[
\|x_n - x_1\|^2 \leq \|x_n - x_1\|\|x_1 - p\|,
\] (3.9)

and hence

\[
\|x_n - x_1\| \leq \|x_1 - p\|.
\] (3.10)

Since \( C \) is bounded, then \( \{x_n\} \) and \( \{u_n\} \) are bounded. From \( x_n = P_{C_n} x_0 \) and \( x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n \), we have

\[
\langle x_1 - x_n, x_n - x_{n+1} \rangle \geq 0, \; \forall n \in \mathbb{N}.
\] (3.11)

So,

\[
0 \leq \langle x_1 - x_n, x_n - x_{n+1} \rangle = -\langle x_n - x_1, x_n - x_1 \rangle + \langle x_1 - x_n, x_1 - x_{n+1} \rangle \\
\leq -\|x_n - x_1\|^2 + \|x_n - x_1\|\|x_1 - x_{n+1}\|.
\] (3.12)

This implies that

\[
\|x_n - x_1\| \leq \|x_1 - x_{n+1}\|, \; \forall n \in \mathbb{N}.
\] (3.13)

Hence, \( \{\|x_n - x_1\|\} \) is nondecreasing, it follows that \( \lim_{n \to \infty} \|x_n - x_1\| \) exists. From (2.1) and (3.11), we have

\[
\|x_{n+1} - x_n\|^2 = \|(x_{n+1} - x_1) - (x_n - x_1)\|^2 \\
= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_{n+1} - x_n, x_n - x_1 \rangle \\
\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.
\] (3.14)

Since \( \lim_{n \to \infty} \|x_n - x_1\| \) exists, we have

\[
\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0.
\] (3.15)

On the other hand, it follows from \( x_{n+1} \in C_{n+1} \) that

\[
\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n \to 0, \; \text{as } n \to \infty.
\] (3.16)
It follows that
\[
\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0, \quad \text{as } n \to \infty.
\] (3.17)

Next, we claim that \(\lim_{n \to \infty} \|x_n - u_n\| = 0\). Let \(p \in F \cap GMEP\), it follows from (3.6) that
\[
\|y_n - p\|^2 \leq \|x_n - p\|^2 + \theta_n
\]
\[
= \|K_{r_n}(I - r_nA)x_n - K_{r_n}(I - r_nA)p\|^2 + \theta_n
\]
\[
\leq \|x_n - p\|^2 + r_n(2\alpha)\|Ax_n - Ap\|^2 + \theta_n.
\] (3.18)

This implies that
\[
e(2\alpha - f)\|Ax_n - Ap\|^2 \leq \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n
\]
\[
\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \theta_n.
\] (3.19)

It follows from (3.17) that
\[
\lim_{n \to \infty} \|Ax_n - Ap\| = 0.
\] (3.20)

From Lemma 2.6, we have
\[
\|u_n - p\|^2 = \|K_{r_n}(I - r_nA)x_n - K_{r_n}(I - r_nA)p\|^2
\]
\[
\leq \langle (x_n - r_nAx_n) - (p - r_nAp), u_n - p \rangle
\]
\[
= \frac{1}{2} \{\|x_n - r_nAx_n - (p - r_nAp)\|^2 + \|u_n - p\|^2
\]
\[
- \|x_n - r_nAx_n - (p - r_nAp) - (u_n - p)\|^2 \}
\]
\[
\leq \frac{1}{2} \{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Ax_n - Ap)\|^2 \}
\]
\[
= \frac{1}{2} \{\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2
\]
\[
+ 2r_n \langle x_n - u_n, Ax_n - Ap \rangle - r_n^2 \|Ax_n - Ap\|^2 \}. \] (3.21)

This implies that
\[
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n \langle x_n - u_n, Ax_n - Ap \rangle - r_n^2 \|Ax_n - Ap\|^2
\]
\[
\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Ap\| + \theta_n
\] (3.22)

By (3.21) and (3.22), we obtain
\[
\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Ap\| + \theta_n.
\] (3.23)

which implies that
\[
\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|y_n - p\|^2 + 2r_n \|x_n - u_n\| \|Ax_n - Ap\| + \theta_n
\]
\[
\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + 2r_n \|x_n - u_n\| \|Ax_n - Ap\| + \theta_n.
\] (3.24)
This implies by (3.17) and (3.24) that
\[
\lim_{n \to \infty} \|u_n - x_n\| = 0. \tag{3.25}
\]
From (3.15) and (3.25), we have
\[
\|u_n - u_{n+1}\| \leq \|u_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - u_{n+1}\| \to 0, \text{ as } n \to \infty. \tag{3.26}
\]
Similarly, from (3.17) and (3.25), we have
\[
\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\| \to 0, \text{ as } n \to \infty. \tag{3.27}
\]
Since
\[
(1 - \alpha_n)\|u_n u_n - u_n\| = \|y_n - u_n\|,
\]
it implies by \(0 \leq \alpha_n \leq d < 1\) that
\[
\|W_n u_n - u_n\| = \frac{\|y_n - u_n\|}{1 - \alpha_n} < \frac{\|y_n - u_n\|}{1 - d} \to 0, \text{ as } n \to \infty. \tag{3.29}
\]

**Step 3.** We show that there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) which converge weakly to \(z\), where \(z \in F \cap GMEP\).

Since \(\{x_n\}\) is bounded and \(C\) is closed, there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) which converges weakly to \(z \in C\). From (3.25), it follows by (3.26), (3.29) and Lemma 2.8 that \(z \in F\). Next, we prove that \(z \in GMEP\). Indeed, we observe that \(u_n = K_r(x_n - r_n A x_n)\) and
\[
F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C.
\]
(3.30)

By \((A_2)\), we get
\[
\varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_n).
\]
(3.31)

Replacing \(n\) by \(n_i\), we obtain
\[
\varphi(y) - \varphi(u_{n_i}) + \langle A x_{n_i}, y - u_{n_i} \rangle + \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq F(y, u_{n_i}).
\]
(3.32)

Put \(z_t = ty + (1 - t)z\) for all \(t \in (0, 1]\) and \(y \in C\). Then, we have \(z_t \in C\). So, we have
\[
\langle z_t - u_{n_i}, A z_t \rangle \geq \langle z_t - u_{n_i}, A z_t \rangle - \langle A x_{n_i}, z_t - u_{n_i} \rangle - \langle z_t - u_{n_i}, u_{n_i} - x_{n_i} \rangle
\]
\[+ F(z_t, u_{n_i}) - \varphi(z_t) + \varphi(u_{n_i})
\]
\[= \langle z_t - u_{n_i}, A z_t - A u_{n_i} \rangle + \langle z_t - u_{n_i}, A u_{n_i} - A x_{n_i} \rangle - \langle z_t - u_{n_i}, u_{n_i} - x_{n_i} \rangle
\]
\[+ F(z_t, u_{n_i}) - \varphi(z_t) + \varphi(u_{n_i}).
\]
(3.33)
Since \( \|u_{n_i} - x_n\| \to 0 \), we have \( \|Au_{n_i} - Ax_{n_i}\| \to 0 \). Further, from monotonicity of \( A \), we have \( \langle z_i - u_{n_i}, Az_i - Au_{n_i} \rangle \geq 0 \). So, by (A4) and the weakly lower semicontinuity of \( \varphi \), we have
\[
\langle z_i - z, Az_i \rangle \geq F(z_i, z) - \varphi(z_i) + \varphi(z).
\]  
(3.34)

It follows by (A1) and (A4) that
\[
0 = F(z_i, z) - \varphi(z_i) + \varphi(z_i)
\leq tF(z_i, y) + (1 - t)F(z_i, z) + t\varphi(y) + (1 - t)\varphi(z) - \varphi(z_i)
= t(F(z_i, y) + \varphi(y) - \varphi(z_i)) + (1 - t)(F(z_i, z) + \varphi(z) - \varphi(z_i))
\leq t(F(z_i, y) + \varphi(y) - \varphi(z_i)) + (1 - t)(z_i - z, Az_i)
= t(F(z_i, y) + \varphi(y) - \varphi(z_i)) + (1 - t)(y - z, Az_i)
\]  
(3.35)

and hence
\[
0 \leq F(z_i, y) + \varphi(y) - \varphi(z_i) + (1 - t)(y - z, Az_i).
\]  
(3.36)

Letting \( t \to 0 \), we have, for each \( y \in C \), that
\[
0 \leq F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Az \rangle.
\]  
(3.37)

This implies that \( z \in GMEP \).

**Step 4.** We prove that \( x_n \to z \), \( u_n \to z \), where \( z = P_{F \cap GMEP}x_1 \).

Putting \( z' = P_{F \cap GMEP}x_1 \) and consider the sequence \( \{x_1 - x_n\} \). Then we have \( x_1 - x_n \to x_1 - z \) and by the weak lower semicontinuity of the norm and \( \|x_1 - x_{n+1}\| \leq \|x_1 - z'\| \) for all \( n \in \mathbb{N} \) which is implied by the fact that \( x_{n+1} = P_{C_{n+1}}x_1 \), we have
\[
\|x_1 - z'\| \leq \|x_1 - z\|
\leq \liminf_{n \to \infty} \|x_1 - x_n\|
\leq \limsup_{n \to \infty} \|x_1 - x_n\|
\leq \|x_1 - z'\|.
\]  
(3.38)

This implies that \( \|x_1 - z'\| = \|x_1 - z\| \). By the uniqueness of the nearest point projection of \( x_1 \) onto \( F \cap GMEP \) that
\[
\|x_1 - x_n\| \to \|x_1 - z'\|.
\]  
(3.39)

This implies that \( x_n \to z' \). Since \( \{x_n\} \) is an arbitrary sequence of \( C \), we can conclude that \( x_n \to z' \). By (3.25), we have that \( u_n \to z' \) also. This proof is completed. \( \square \)

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