New Inequalities of Ostrowski Type for Mappings whose Derivatives are \((\alpha, m)\)-Convex via Fractional Integrals

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Abstract : New identity similar to an identity of \([1]\) for fractional integrals have been defined. Then making use of this identity, some new Ostrowski type inequalities for Riemann-Liouville fractional integral have been developed. Our results have some relationships with the results of Özdemir et. al., proved in \([1]\) and the analysis used in the proofs is simple.

Keywords : \((\alpha, m)\)-Convex mappings; Hermite-Hadamard inequality; Ostrowski inequality.

1 Introduction

In 1938, A.M. Ostrowski proved the following interesting and useful integral inequality (see also \([2]\) page 468):

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Theorem 1.1. Let \( f : I \to \mathbb{R} \), where \( I \subset \mathbb{R} \) is an interval, be a mapping differentiable in the interior \( I^\circ \) of \( I \), and let \( a, b \in I^\circ \) with \( a < b \). If \( |f'(x)| \leq M \) for all \( x \in [a, b] \), then the following inequality holds:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq M (b-a) \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]
\]

for all \( x \in [a, b] \).

The constant \( \frac{1}{4} \) is the best possible in the sense that it cannot be replaced by a smaller one.

This inequality gives an upper bound for the approximation of the integral average \( \frac{1}{b-a} \int_a^b f(t) \, dt \) by the value \( f(x) \) at point \( x \in [a, b] \). In recent years, such inequalities were studied extensively by many researchers and numerous generalizations, extensions and variants of them appeared in a number of papers, see [1,3–5].

In [6], V.G. Miheșan defined \( (\alpha,m) \)-convexity as the following:

Definition 1.2. The function \( f : [0,b] \to \mathbb{R} \), \( b > 0 \), is said to be \( (\alpha,m) \)-convex, where \( (\alpha,m) \in [0,1]^2 \), if we have

\[
f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)
\]

for all \( x, y \in [0,b] \) and \( t \in [0,1] \).

Denote by \( K_\alpha^m(b) \) the class of all \( (\alpha,m) \)-convex functions on \([0,b]\) for which \( f(0) \leq 0 \).

It can be easily seen that for \( (\alpha,m) = (1,m) \), \( (\alpha,m) \)-convexity reduces to \( m \)-convexity; \( (\alpha,m) = (\alpha,1) \), \( (\alpha,m) \)-convexity reduces to \( \alpha \)-convexity and for \( (\alpha,m) = (1,1) \), \( (\alpha,m) \)-convexity reduces to the concept of usual convexity defined on \([0,b], b > 0 \). For recent results and generalizations concerning \( (\alpha,m) \)-convex functions, see [1,7,8].

In order to prove our results we need the following equality which was given in [1, page 372] by Özdemir et al.:

\[
mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u) \, du = \frac{(x-ma)^2}{b-a} \int_0^1 tf'(tx + ma(1-t)) \, dt - \frac{(mb-x)^2}{b-a} \int_0^1 tf'(tx + mb(1-t)) \, dt \quad (1.1)
\]

which is a special case of Lemma 1 in [9] with \( ma \to a \) and \( mb \to b \).

Using the inequality in (1.1), Özdemir et al. in [1] established the following results which holds for \( (\alpha,m) \)-convex functions.

Theorem 1.3. Let \( I \) be an open interval such that \([0,\infty) \subset I \) and \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L[ma,mb] \) where \( ma,mb \in I \) with
If $|f|^q$ is $(\alpha, m)$-convex on $[ma, mb]$ for $(\alpha, m) \in [0, 1]^2$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'(x)| \leq M$, $x \in [ma, mb]$, then the following inequality holds:

$$
\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u) \, du \right| \leq M \left( \frac{\alpha m + 1}{\alpha + 1} \right)^{\frac{1}{q}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \frac{(x - ma)^2 + (mb - x)^2}{b - a}
$$

for each $x \in [ma, mb]$.

**Theorem 1.4.** Let $I$ be an open interval such that $[0, \infty) \subset I$ and $f : I \rightarrow \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L[ma, mb]$ where $ma, mb \in I$ with $a < b$. If $|f|^q$ is $(\alpha, m)$-convex on $[ma, mb]$ for $(\alpha, m) \in [0, 1]^2$, $q \in [1, \infty)$ and $|f'(x)| \leq M$, then the following inequality holds:

$$
\left| mf(x) - \frac{1}{b-a} \int_{ma}^{mb} f(u) \, du \right| \leq M \left( \frac{2 + \alpha m}{\alpha + 2} \right)^{\frac{1}{q}} \frac{(x - ma)^{\alpha + 1} + (mb - x)^{\alpha + 1}}{(b - a)^2}
$$

for each $x \in [ma, mb]$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 1.5.** Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a^+}^\alpha (f)$ and $J_{b^-}^\alpha (f)$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) \, dt, \quad x > a
$$

and

$$
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) \, dt, \quad x < b
$$

where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha - 1} \, du$. Here $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Some recent results and properties concerning this operator can be found in [10–16].

We establish new Ostrowski type inequalities for $(\alpha, m)$-convex functions via Riemann-Liouville fractional integral. An interesting feature of our results is that they provide new estimates on these types of inequalities for fractional integrals.
2 Ostrowski Type Inequalities for Fractional Integrals

In order to prove our main results we need the following identity:

Lemma 2.1. Let $I$ be an open real interval such that $[0, \infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L([ma, mb])$, where $[ma, mb] \in I$ with $a < b$ then for all $x \in (ma, mb)$ and $\alpha > 0$ we have:

$$\frac{(x - ma)^\alpha + (mb - x)^\alpha}{b - a} f(x) - \frac{\Gamma(\alpha + 1)}{b - a} \left[ J_x^a f(ma) + J_x^b f(mb) \right] = \frac{(x - ma)^{\alpha+1}}{b - a} \int_0^1 t^\alpha f'(tx + m(1 - t)a) \, dt - \frac{(mb - x)^{\alpha+1}}{b - a} \int_0^1 t^\alpha f'(tx + m(1 - t)b) \, dt$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt$.

Proof. Integration by parts we have

$$\int_0^1 t^\alpha f'(tx + m(1 - t)a) \, dt = \left[ \frac{t^\alpha f(tx + m(1 - t)a)}{x - ma} \right]_0^1 - \int_0^1 \frac{\alpha t^{\alpha-1} f(tx + m(1 - t)a)}{x - ma} \, dt$$

$$= \frac{f(x)}{x - ma} - \frac{\alpha}{x - ma} \int_{ma}^x \left( \frac{u - ma}{x - ma} \right)^{\alpha-1} f(u) \, \frac{du}{x - ma}$$

$$= \frac{f(x)}{x - ma} - \frac{\Gamma(\alpha + 1)}{(x - ma)^{\alpha+1}} \int_{ma}^x (u - ma)^{\alpha-1} f(u) \, du$$

$$= \frac{f(x)}{x - ma} - \frac{\Gamma(\alpha + 1)}{(x - ma)^{\alpha+1}} J_x^a f(ma) \quad (2.1)$$

and similarly

$$\int_0^1 t^\alpha f'(tx + m(1 - t)b) \, dt = \left[ \frac{t^\alpha f(tx + m(1 - t)b)}{x - mb} \right]_0^1 - \int_0^1 \frac{\alpha t^{\alpha-1} f(tx + m(1 - t)b)}{x - mb} \, dt$$

$$= \frac{f(x)}{x - mb} + \frac{\alpha}{(mb - x)^2} \int_x^{mb} \left( \frac{mb - u}{mb - x} \right)^{\alpha-1} f(u) \, du$$

$$= \frac{f(x)}{x - mb} + \frac{\Gamma(\alpha + 1)}{(mb - x)^{\alpha+1}} \int_x^{mb} (mb - u)^{\alpha-1} f(u) \, du$$

$$= \frac{f(x)}{x - mb} + \frac{\Gamma(\alpha + 1)}{(mb - x)^{\alpha+1}} J_x^b f(mb). \quad (2.2)$$
Multiplying the both sides of (2.1) and (2.2) by \(\frac{(x-ma)^{\alpha+1}}{b-a}\) and \(\frac{(mb-x)^{\alpha+1}}{b-a}\), respectively, we have

\[
\frac{(x-ma)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + m(1-t)a) dt = \frac{(x-ma)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a} J_{x}^{\alpha} f(ma)
\]

(2.3)

and

\[
\frac{(mb-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha f'(tx + m(1-t)b) dt = \frac{(mb-x)^{\alpha}}{b-a} f(x) + \frac{\Gamma(\alpha+1)}{b-a} J_{b}^{\alpha} f(mb).
\]

(2.4)

If we add the inequalities in (2.3) and (2.4) we get the desired result.

Using Lemma 2.1 we can obtain the following fractional integral inequalities:

**Theorem 2.2.** Let \(I\) be an open interval such that \([0, \infty) \subset I\) and \(f : I \to \mathbb{R}\) be a differentiable function on \(I\) such that \(f' \in L[ma, mb]\) where \(ma, mb \in I\) with \(a < b\). If \(|f'|\) is \((\alpha, m)\)-convex on \([ma, mb]\) for \((\alpha, m) \in [0, 1]^2\) and \(|f'(x)| \leq M\), then the following inequality holds:

\[
\left| \frac{(x-ma)^{\alpha} + (mb-x)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a} [J_{x}^{\alpha} f(ma) + J_{b}^{\alpha} f(mb)] \right| \leq M \left( \frac{1 + ma}{2\alpha + 1} \right) \frac{(x-ma)^{\alpha+1} + (mb-x)^{\alpha+1}}{b-a}
\]

for all \(x \in [ma, mb]\).

**Proof.** From Lemma 2.1 and using the \((\alpha, m)\)-convexity of \(|f'|\), we have

\[
\left| \frac{(x-ma)^{\alpha} + (mb-x)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{b-a} [J_{x}^{\alpha} f(ma) + J_{b}^{\alpha} f(mb)] \right| \leq \frac{(x-ma)^{\alpha+1}}{b-a} \int_0^1 t^\alpha |f'(tx + m(1-t)a)| dt
\]

\[+ \frac{(mb-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha |f'(tx + m(1-t)b)| dt\]

\[
\leq \frac{(x-ma)^{\alpha+1}}{b-a} \int_0^1 t^\alpha [t^\alpha |f'(x)| + m(1-t^\alpha) |f'(a)|] dt
\]

\[+ \frac{(mb-x)^{\alpha+1}}{b-a} \int_0^1 t^\alpha [t^\alpha |f'(x)| + m(1-t^\alpha) |f'(b)|] dt\]

\[
\leq M \left( \frac{1 + ma}{2\alpha + 1} \right) \frac{(x-ma)^{\alpha+1} + (mb-x)^{\alpha+1}}{b-a}
\]
where we have used the fact that
\[
\int_0^1 \left[ t^{2\alpha} + m (t^\alpha - t^{2\alpha}) \right] dt = \frac{1 + ma}{2\alpha + 1}.
\]
The proof is completed.

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following results:

**Theorem 2.3.** Let \( I \) be an open interval such that \([0, \infty) \subset I \) and \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L [ma, mb] \) where \( ma, mb \in I \) with \( a < b \). If \( |f'|^q \) is \((\alpha, m)\)-convex on \([ma, mb] \) for \((\alpha, m) \in [0, 1]^2 \), \( q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( |f'(x)| \leq M \), then the following inequality holds:

\[
\left| \frac{(x - ma)^\alpha + (mb - x)^\alpha}{b - a} f(x) - \frac{\Gamma (\alpha + 1)}{b - a} \left[ J_{x^-}^\alpha f(ma) + J_x^\alpha f(mb) \right] \right| \leq M \left( \frac{1 + ma}{\alpha + 1} \right)^\frac{1}{q} \left( \frac{1}{\alpha p + 1} \right)^\frac{1}{q} \frac{(x - ma)^{\alpha+1} + (mb - x)^{\alpha+1}}{b - a}.
\]

for all \( x \in [ma, mb] \).

**Proof.** Suppose that \( p > 1 \). From Lemma 2.1 and using the Hölder inequality, we have

\[
\left| \frac{(x - ma)^\alpha + (mb - x)^\alpha}{b - a} f(x) - \frac{\Gamma (\alpha + 1)}{b - a} \left[ J_{x^-}^\alpha f(ma) + J_x^\alpha f(mb) \right] \right| \\
\leq \frac{(x - ma)^{\alpha+1}}{b - a} \int_0^1 t^\alpha |f'(tx + m(1 - t)a)| dt \\
+ \frac{(mb - x)^{\alpha+1}}{b - a} \int_0^1 t^\alpha |f'(tx + m(1 - t)b)| dt \\
\leq \frac{(x - ma)^{\alpha+1}}{b - a} \left( \int_0^1 t^{\alpha p} dt \right)^\frac{1}{p} \left( \int_0^1 |f'(tx + m(1 - t)a)|^q dt \right)^\frac{1}{q} \\
+ \frac{(mb - x)^{\alpha+1}}{b - a} \left( \int_0^1 t^{\alpha p} dt \right)^\frac{1}{p} \left( \int_0^1 |f'(tx + m(1 - t)b)|^q dt \right)^\frac{1}{q}.
\]

Since \( |f'|^q \) is \((\alpha, m)\)-convex function and \( |f'(x)| \leq M \), then we have

\[
\left( \int_0^1 |f'(tx + m(1 - t)a)|^q dt \right)^\frac{1}{q} \leq \left( \int_0^1 \left[ t^{\alpha} |f'(x)|^q + m (1 - t^\alpha) |f'(a)|^q \right] dt \right)^\frac{1}{q} = M \left( \frac{1 + ma}{\alpha + 1} \right)^\frac{1}{q}.
\]
and similarly
\[
\left( \int_0^1 |f'(tx + m(1 - t)b)|^{q} \ dt \right)^{\frac{1}{q}} \leq \left( \int_0^1 \left[ t^\alpha |f'(x)|^{q} + m(1 - t^\alpha) |f'(b)|^{q} \right] dt \right)^{\frac{1}{q}} = M \left( \frac{1 + am}{\alpha + 1} \right)^{\frac{1}{q}}.
\]

Therefore, we have
\[
\left| (x - ma)^\alpha + (mb - x)^\alpha \over b - a \right| f(x) - \frac{\Gamma(\alpha + 1)}{b - a} [J_x^\alpha f(ma) + J_x^\alpha f(mb)] \leq \frac{(x - ma)^{\alpha+1}}{b - a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{q}} \left( M^q \frac{1 + am}{\alpha + 1} \right)^{\frac{1}{q}} + \frac{(mb - x)^{\alpha+1}}{b - a} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{q}} \left( M^q \frac{1 + am}{\alpha + 1} \right)^{\frac{1}{q}} \right)
\]
\[
= M \left( \frac{1 + am}{\alpha + 1} \right)^{\frac{1}{q}} \left( \frac{1 + am}{\alpha + 1} \right)^{\frac{1}{q}} (x - ma)^{\alpha+1} + (mb - x)^{\alpha+1} + (mb - x)^{\alpha+1} \over b - a \right)
\]

\[ \tag{2.5} \]

\[ \text{Theorem 2.4.} \text{ Let } I \text{ be an open interval such that } [0, \infty) \subset I \text{ and } f : I \to \mathbb{R} \text{ be a differentiable function on } I \text{ such that } f' \in L[ma, mb] \text{ where } ma, mb \in I \text{ with } a < b. \text{ If } |f'|^q \text{ is } (\alpha, m) \text{-convex on } [ma, mb] \text{ for } (\alpha, m) \in [0, 1]^2, q \geq 1 \text{ and } |f'(x)| \leq M, \text{ then the following inequality holds:}
\]
\[
\left| (x - ma)^\alpha + (mb - x)^\alpha \over b - a \right| f(x) - \frac{\Gamma(\alpha + 1)}{b - a} [J_x^\alpha f(ma) + J_x^\alpha f(mb)] \leq M \left( \frac{\alpha (m + 1) + 1}{2\alpha + 1} \right)^{\frac{1}{q}} (x - ma)^{\alpha+1} + (mb - x)^{\alpha+1} \over b - a \right) \]
\[
\text{for all } x \in [ma, mb].
\]
\[ \text{Proof.} \text{ From Lemma 2.1 and using the well known power mean inequality, we have}
\]
\[
\left| (x - ma)^\alpha + (mb - x)^\alpha \over b - a \right| f(x) - \frac{\Gamma(\alpha + 1)}{b - a} [J_x^\alpha f(ma) + J_x^\alpha f(mb)] \leq \frac{(x - ma)^{\alpha+1}}{b - a} \left( \int_0^1 t^\alpha |f'(tx + m(1 - t)a)| \ dt \right)^{\frac{1}{\alpha}} + \frac{(mb - x)^{\alpha+1}}{b - a} \left( \int_0^1 t^\alpha |f'(tx + m(1 - t)b)| \ dt \right)^{\frac{1}{\alpha}}
\]
\[
\leq \frac{(x - ma)^{\alpha+1}}{b - a} \left( \int_0^1 t^\alpha dt \right)^{\frac{1}{\alpha}} \left( \int_0^1 t^\alpha |f'(tx + m(1 - t)a)|^q \ dt \right)^{\frac{1}{q}} + \frac{(mb - x)^{\alpha+1}}{b - a} \left( \int_0^1 t^\alpha dt \right)^{\frac{1}{\alpha}} \left( \int_0^1 t^\alpha |f'(tx + m(1 - t)b)|^q \ dt \right)^{\frac{1}{q}}
\]
\[
\tag{2.5}
\]
Since \(|f'|^q\) is \((\alpha, m)\)-convex function and \(|f'(x)| \leq M\), then we have

\[
\int_0^1 t^\alpha |f'(tx + m(1-t)a)|^q dt = \int_0^1 t^\alpha |f'(tx + m(1-t)b)| dt 
\leq M^q \frac{\alpha (m + 1)}{(2\alpha + 1)(\alpha + 1)}.
\] (2.6)

Then using the inequality (2.6) in (2.5) and computing the integrals in (2.5), we get the desired result.

\textbf{Remark 2.5.}

(i) In Theorem 2.3, if we choose \(\alpha = 1\) we get the result in Theorem 1.3 with \(\alpha = 1\).

(ii) In Theorem 2.4, if we choose \(\alpha = 1\) we get the result in Theorem 1.4 with \(\alpha = 1\).

\textbf{References}


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