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Common Endpoints for Non-Commutative Suzuki Mappings

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Abstract : In this paper, we prove an endpoint theorem for multi-valued Suzuki mappings in uniformly convex hyperbolic spaces. As a consequence, we obtain a common endpoint theorem for a pair of single-valued and multi-valued Suzuki mappings without the commutative condition. Our results extend and improve the results of Espinola et al. (2015), Saejung (2016), Kudtha and Panyanak (2018) and many others.

Keywords : endpoint; fixed point; Suzuki mapping; uniformly convex hyperbolic space.
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1 Introduction

Let (X, d, W) be a hyperbolic space. The *distance* from a point x in X to a nonempty subset E of X is defined by

$$\operatorname{dist}(x, E) := \inf\{d(x, y) : y \in E\}.$$

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We denote by $\mathcal{K}(E)$ the family of nonempty compact subsets of E and by $\mathcal{KC}(E)$ the family of nonempty compact convex subsets of E. The *Pompeiu-Hausdorff distance* on $\mathcal{K}(E)$ is defined by

$$H(A,B) := \max\left\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\right\} \text{ for all } A,B\in\mathcal{K}(E).$$

A multi-valued mapping $T : E \to \mathcal{K}(E)$ is said to be *nonexpansive* [1] if

$$H(T(x),T(y)) \leq d(x,y) \tag{1.1}$$

for all $x, y \in E$. If (1.1) is valid for all $x, y \in E$ with $\frac{1}{2}\operatorname{dist}(x,T(x)) \leq d(x,y)$, then T is called a *Suzuki mapping* [2]. It is known that every nonexpansive mapping is a Suzuki mapping and, in general, the converse is not true. An element x in E is called a *fixed point* of T if $x \in T(x)$. Moreover, if $\{x\} = T(x)$, then x is called an *endpoint* of T. We denote by $Fix(T)$ the set of all fixed points of T and by $End(T)$ the set of all endpoints of T. It is clear that $End(T) \subseteq Fix(T)$ for every multi-valued mapping T and $End(t) = Fix(t)$ for every single-valued mapping t.

Endpoint theory for multi-valued mappings has many useful applications in applied sciences, for instance, in game theory and optimization theory. In particular, in 1986, Corley [3] proved that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of a certain multi-valued mapping.

Let E be a nonempty subset of a metric space (X, d) and $x \in X$. The *radius* of E relative to x is defined by

$$r_x(E) := \sup\{d(x,y) : y \in E\}.$$

The *diameter* of E is defined by

$$\operatorname{diam}(E) := \sup\{d(x,y) : x,y \in E\}.$$

A single-valued mapping $t : E \to E$ and a multi-valued mapping $T : E \to \mathcal{K}(E)$ are said to be *commuting mappings* [4] if for $x, y \in E$ such that $x \in T(y)$, one has $t(x) \in T(t(y))$. A sequence $\{x_n\}$ in E is called *an approximate endpoint sequence* for T [5] if $\lim_{n\to\infty} r_{x_n}(T(x_n)) = 0$.

The existence of endpoints for nonexpansive mappings was first studied by Panyanak [6] in 2015. He showed that a multi-valued nonexpansive mapping on a bounded closed convex subset E of a uniformly convex Banach space X has an endpoint if and only if it has an approximate endpoint sequence in E. It was quickly noted by Espinola et al. [7] that Panyanak's result can be extended to the general setting of Banach spaces with the Dominguez-Lorenzo condition. Since then the endpoint results for some generalized nonexpansive mappings have been rapidly developed and many papers have appeared (see, e.g., [8-12]). Among other things, Kudtha and Panyanak [11] obtained the following result.

Theorem 1.1. *Let X be a uniformly convex hyperbolic space with monotone modulus of uniform convexity and let E be a nonempty bounded closed convex subset*

of X. Let $t : E \to E$ be a single-valued Suzuki mapping and $T : E \to \mathcal{KC}(E)$ be a multi-valued Suzuki mapping. Suppose that the following conditions hold:
(i) *t and T are commuting mappings;*
(ii) *T has an approximate endpoint sequence in $End(t)$.*
Then t and T have a common endpoint in E.

In [11], the authors also showed that the condition (ii) is necessary for Theorem 1.1. In general, two mappings need not be commute, thus the following question should be of interest.

Question: Is Theorem 1.1 true if the condition (i) is eliminated?

In this paper, we show that the answer is "Yes". To support our result, we also show that there exists a non-commutative pair of single-valued and multi-valued Suzuki mappings which have a common endpoint.

2 Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers.

Definition 2.1. [13] A *hyperbolic space* is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a function such that for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$, we have
(W1) $d(z, W(x, y, \alpha)) \leq (1-\alpha)d(z, x) + \alpha d(z, y)$;
(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;
(W3) $W(x, y, \alpha) = W(y, x, 1-\alpha)$;
(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$.

If $x, y \in X$ and $\alpha \in [0, 1]$, then we use the notation $(1-\alpha)x \oplus \alpha y$ for $W(x, y, \alpha)$. It is easy to see that for any $x, y \in X$ and $\alpha \in [0, 1]$, one has

$$d(x, (1-\alpha)x \oplus \alpha y) = \alpha d(x, y) \text{ and } d(y, (1-\alpha)x \oplus \alpha y) = (1-\alpha)d(x, y).$$

Let $[x, y] := \{(1-\alpha)x \oplus \alpha y : \alpha \in [0, 1]\}$. A nonempty subset E of X is said to be *convex* if $[x, y] \subseteq E$ for all $x, y \in E$.

Definition 2.2. [13] The hyperbolic space (X, d, W) is called *uniformly convex* if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq r\varepsilon$, we have

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \leq (1-\delta)r.$$

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called *a modulus of uniform convexity.* The mapping δ is *monotone* if for every fixed ε it decreases with respect to r.

Obviously, uniformly convex Banach spaces are uniformly convex hyperbolic spaces. CAT(0) spaces are also uniformly convex hyperbolic spaces, see [13, Proposition 8].

Definition 2.3. [14] Let E be a nonempty subset of a metric space (X, d). A multivalued mapping $T : E \to CB(E)$ is said to satisfy *condition* (E_μ) if there exists $\mu \geq 1$ such that for each $x, y \in E$, we have

$$\operatorname{dist}(x, T(y)) \leq \mu \operatorname{dist}(x, T(x)) + d(x, y).$$

The mapping T is said to be *quasi-nonexpansive* if for each $x \in E$ and $y \in Fix(T)$, one has

$$H(T(x), T(y)) \leq d(x, y).$$

Let E be a nonempty subset of a metric space (X, d) and $\{x_n\}$ be a bounded sequence in X. The *asymptotic radius* of $\{x_n\}$ relative to E is defined by

$$r(E, \{x_n\}) = \inf\big\{\limsup_{n\to\infty} d(x_n, x) : x \in E\big\}.$$

The *asymptotic center* of $\{x_n\}$ relative to E is defined by

$$A(E, \{x_n\}) = \big\{x \in E : \limsup_{n\to\infty} d(x_n, x) = r(E, \{x_n\})\big\}.$$

The sequence $\{x_n\}$ is called *regular* relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_k}\})$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. It is known that every bounded sequence in a metric space has a regular subsequence (see [15]; also [16, p. 3690]).

Before proving our main results we collect some basic facts about uniformly convex hyperbolic spaces. From now on, X stands for a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity.

Lemma 2.4. *The following statements hold:*

(i) [2, Proposition 2] *if E is a nonempty subset of X and $t : E \to E$ is a single-valued Suzuki mapping with $End(t) \neq \emptyset$, then t is a quasi-nonexpansive mapping;*

(ii) [17, Lemma 3.2] *if E is a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$ is a multi-valued Suzuki mapping, then T satisfies condition (E_3);*

(iii) [6, Proposition 2.4] *Let E be a nonempty subset of X, $\{x_n\}$ be a sequence in E, and $T : E \to \mathcal{K}(E)$ be a multi-valued mapping. Then $r_{x_n}(T(x_n)) \to 0$ if and only if $dist(x_n, T(x_n)) \to 0$ and $diam(T(x_n)) \to 0$.*

(iv) [18, Proposition 3.3] *if E is a nonempty closed convex subset of X and $\{x_n\}$ be a bounded sequence in E, then $A(E, \{x_n\})$ consists of exactly one point.*

3 Main Results

This section is begun by proving an endpoint theorem for multi-valued Suzuki mappings. The proof closely follows the proof of Theorem 3.1 in [11], for for the convenience of readers we include the details.

Theorem 3.1. *Let E be a nonempty closed convex subset of X and $T : E \to \mathcal{K}(E)$ be a multi-valued Suzuki mapping. Let $\{x_n\}$ be a sequence in E which is regular relative to E. Suppose that $\{x_n\}$ is an approximate endpoint sequence for T and $A(E, \{x_n\}) = \{x\}$. Then x is an endpoint of T.*

Proof. Let $r = r(E, \{x_n\})$. For $n \in \mathbb{N}$, we let $y_n \in T(x_n)$ be such that $d(x_n, y_n) = \operatorname{dist}(x_n, T(x_n))$. Since $\{x_n\}$ is an approximate endpoint sequence for T, by Lemma 2.4 (iii) we have

$$\operatorname{dist}(x_n, T(x_n)) \to 0 \text{ and } \operatorname{diam}(T(x_n)) \to 0. \tag{3.1}$$

Case 1. For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\frac{1}{2}d(x_m, y_m) > d(x_m, x)$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{2}d(x_{n_k}, y_{n_k}) > d(x_{n_k}, x) \text{ for all } k \in \mathbb{N}. \tag{3.2}$$

It follows from (3.1) and (3.2) that $\lim_{k\to\infty} x_{n_k} = x$. By Lemma 2.4 (ii), we have

$$\begin{aligned}\operatorname{dist}(x, T(x)) &\leq d(x, x_{n_k}) + \operatorname{dist}(x_{n_k}, T(x)) \\ &\leq 2d(x, x_{n_k}) + 3\operatorname{dist}(x_{n_k}, T(x_{n_k})) \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence $x \in T(x)$. Notice also that $\frac{1}{2}\operatorname{dist}(x, T(x)) = 0 \leq d(x_{n_k}, x)$ for all $k \in \mathbb{N}$. Since T is a Suzuki mapping, we have

$$H(T(x_{n_k}), T(x)) \leq d(x_{n_k}, x) \to 0 \text{ as } k \to \infty. \tag{3.3}$$

Let $v \in T(x)$ and choose $u_{n_k} \in T(x_{n_k})$ so that $d(v, u_{n_k}) = \operatorname{dist}(v, T(x_{n_k}))$. From (3.1) and (3.3) we have

$$\begin{aligned}d(x, v) &\leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) + d(y_{n_k}, u_{n_k}) + d(u_{n_k}, v) \\ &\leq d(x, x_{n_k}) + \operatorname{dist}(x_{n_k}, T(x_{n_k})) + \operatorname{diam}(T(x_{n_k})) + H(T(x_{n_k}), T(x)) \\ &\to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence $v = x$ for all $v \in T(x)$. Therefore $x \in End(T)$.

Case 2. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{2}d(x_n, y_n) \leq d(x_n, x)$ for all $n \geq n_0$. This implies that $\frac{1}{2}\operatorname{dist}(x_n, T(x_n)) \leq d(x_n, x)$ and so $H(T(x_n), T(x)) \leq d(x_n, x)$. For $n \in \mathbb{N}$, select $z_n \in T(x)$ so that $d(y_n, z_n) = \operatorname{dist}(y_n, T(x))$. Since $T(x)$ is compact, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \to w \in T(x)$. For j sufficiently large, we have

$$\begin{aligned}d(x_{n_j}, w) &\leq d(x_{n_j}, y_{n_j}) + d(y_{n_j}, z_{n_j}) + d(z_{n_j}, w) \\ &\leq d(x_{n_j}, y_{n_j}) + H(T(x_{n_j}), T(x)) + d(z_{n_j}, w) \\ &\leq \operatorname{dist}(x_{n_j}, T(x_{n_j})) + d(x_{n_j}, x) + d(z_{n_j}, w).\end{aligned}$$

This implies by the regularity of $\{x_n\}$ that $\limsup_{j\to\infty} d(x_{n_j}, w) \leq \limsup_{j\to\infty} d(x_{n_j}, x) = r$. Hence $w \in A(E, \{x_{n_j}\}) = \{x\}$. Therefore $x = w \in T(x)$. Let $v \in T(x)$ and choose $u_{n_j} \in T(x_{n_j})$ so that $d(v, u_{n_j}) = \operatorname{dist}(v, T(x_{n_j}))$. Thus

$$\begin{aligned} d(x_{n_j}, v) &\leq d(x_{n_j}, y_{n_j}) + d(y_{n_j}, u_{n_j}) + d(u_{n_j}, v) \\ &\leq d(x_{n_j}, y_{n_j}) + \operatorname{diam}(T(x_{n_j})) + H(T(x), T(x_{n_j})) \\ &\leq \operatorname{dist}(x_{n_j}, T(x_{n_j})) + \operatorname{diam}(T(x_{n_j})) + d(x_{n_j}, x). \end{aligned}$$

It follows from (3.1) that $\limsup_{j\to\infty} d(x_{n_j}, v) \leq \limsup_{j\to\infty} d(x_{n_j}, x) = r$. Hence $v \in A(E, \{x_{n_j}\}) = \{x\}$, and so $v = x$ for all $v \in T(x)$. Therefore $x \in End(T)$. □

Now, we are ready to prove our main theorem. In contrast to Theorem 1.1, it does not need the convexity of $T(x)$.

Theorem 3.2. *Let E be a nonempty bounded closed convex subset of X, $t : E \to E$ be a single-valued mapping and $T : E \to \mathcal{K}(E)$ be a multi-valued mapping. Suppose that t and T are Suzuki mappings such that T has an approximate endpoint sequence in $End(t)$. Then t and T have a common endpoint in E.*

Proof. Let $\{x_n\}$ be an approximate endpoint sequence for T in $\operatorname{End}(t)$. By passing to a subsequence, we may assume that $\{x_n\}$ is regular relative to E. Let $A(E, \{x_n\}) = \{x\}$. By Theorem 3.1, $x \in \operatorname{End}(T)$. It follows from Lemma 2.4 (i) that

$$\limsup_{n\to\infty} d(x_n, t(x)) \leq \limsup_{n\to\infty} d(x_n, x).$$

This implies $t(x) \in A(E, \{x_n\}) = \{x\}$, and hence $x \in \operatorname{End}(t)$. Therefore x is a common endpoint of t and T. □

The following example shows that there exists a non-commutative pair of single-valued and multi-valued Suzuki mappings which have a common endpoint.

Example 3.3. Let $X = \mathbb{R}$, $E = [0, 3]$ and $t : E \to E$ be defined by

$$t(x) = \begin{cases} 0 & \text{if} \quad x \neq 3, \\ 1 & \text{if} \quad x = 3. \end{cases}$$

Then t is a single-valued Suzuki mapping (see [2]). Let $T : E \to \mathcal{K}(E)$ be defined by

$$T(x) = \left[\frac{x}{2}, x\right] \text{ for all } x \in E.$$

Then $H(T(x), T(y)) = |x - y|$ for all $x, y \in E$. Therefore T is nonexpansive and hence it is a Suzuki mapping. If $x = 3/2$ and $y = 3$, then $x \in T(y)$ but $t(x) = 0 \notin \left[\frac{1}{2}, 1\right] = T(t(y))$. Therefore t and T are not commuting, hence we cannot apply Theorem 1.1. However, by Theorem 3.2, we can conclude that t and T have a common endpoint in E.

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