Common Endpoints for Non-Commutative Suzuki Mappings

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Abstract: In this paper, we prove an endpoint theorem for multi-valued Suzuki mappings in uniformly convex hyperbolic spaces. As a consequence, we obtain a common endpoint theorem for a pair of single-valued and multi-valued Suzuki mappings without the commutative condition. Our results extend and improve the results of Espinola et al. (2015), Saejung (2016), Kudtha and Panyanak (2018) and many others.

Keywords: endpoint; fixed point; Suzuki mapping; uniformly convex hyperbolic space.

2010 Mathematics Subject Classification: 47H09; 47H10.

1 Introduction

Let \((X,d,W)\) be a hyperbolic space. The distance from a point \(x\) in \(X\) to a nonempty subset \(E\) of \(X\) is defined by

\[ \text{dist}(x,E) := \inf\{d(x,y) : y \in E\}. \]
We denote by \( \mathcal{K}(E) \) the family of nonempty compact subsets of \( E \) and by \( \mathcal{KC}(E) \) the family of nonempty compact convex subsets of \( E \). The Pompeiu-Hausdorff distance on \( \mathcal{K}(E) \) is defined by
\[
H(A, B) := \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}
\]
for all \( A, B \in \mathcal{K}(E) \).

A multi-valued mapping \( T : E \to \mathcal{K}(E) \) is said to be nonexpansive \([1]\) if
\[
H(T(x), T(y)) \leq d(x, y) \quad (1.1)
\]
for all \( x, y \in E \). If (1.1) is valid for all \( x, y \in E \) with \( \frac{1}{2} \text{dist}(x, T(x)) \leq d(x, y) \), then \( T \) is called a Suzuki mapping \([2]\). It is known that every nonexpansive mapping is a Suzuki mapping and, in general, the converse is not true. An element \( x \) in \( E \) is called a fixed point of \( T \) if \( x \in T(x) \). Moreover, if \( \{x\} = T(x) \), then \( x \) is called an endpoint of \( T \). We denote by \( \text{Fix}(T) \) the set of all fixed points of \( T \) and by \( \text{End}(T) \) the set of all endpoints of \( T \). It is clear that \( \text{End}(T) \subseteq \text{Fix}(T) \) for every multi-valued mapping \( T \) and \( \text{End}(t) = \text{Fix}(t) \) for every single-valued mapping \( t \).

Endpoint theory for multi-valued mappings has many useful applications in applied sciences, for instance, in game theory and optimization theory. In particular, in 1986, Corley \([3]\) proved that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of a certain multi-valued mapping.

Let \( E \) be a nonempty subset of a metric space \((X, d)\) and \( x \in X \). The radius of \( E \) relative to \( x \) is defined by
\[
r_x(E) := \sup \{d(x, y) : y \in E\}.
\]
The diameter of \( E \) is defined by
\[
\text{diam}(E) := \sup \{d(x, y) : x, y \in E\}.
\]
A single-valued mapping \( t : E \to E \) and a multi-valued mapping \( T : E \to \mathcal{K}(E) \) are said to be commuting mappings \([4]\) if for \( x, y \in E \) such that \( x \in T(y) \), one has \( t(x) \in T(t(y)) \). A sequence \( \{x_n\} \) in \( E \) is called an approximate endpoint sequence for \( T \) \([5]\) if \( \lim_{n \to \infty} r_{x_n}(T(x_n)) = 0 \).

The existence of endpoints for nonexpansive mappings was first studied by Panyanak \([6]\) in 2015. He showed that a multi-valued nonexpansive mapping on a bounded closed convex subset \( E \) of a uniformly convex Banach space \( X \) has an endpoint if and only if it has an approximate endpoint sequence in \( E \). It was quickly noted by Espinola et al. \([7]\) that Panyanak’s result can be extended to the general setting of Banach spaces with the Dominguez-Lorenzo condition. Since then the endpoint results for some generalized nonexpansive mappings have been rapidly developed and many papers have appeared (see, e.g., [8-12]). Among other things, Kudtha and Panyanak \([11]\) obtained the following result.

**Theorem 1.1.** Let \( X \) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity and let \( E \) be a nonempty bounded closed convex subset...
of $X$. Let $t : E \to E$ be a single-valued Suzuki mapping and $T : E \to \mathcal{KC}(E)$ be a multi-valued Suzuki mapping. Suppose that the following conditions hold:

(i) $t$ and $T$ are commuting mappings;
(ii) $T$ has an approximate endpoint sequence in $\text{End}(t)$.

Then $t$ and $T$ have a common endpoint in $E$.

In [11], the authors also showed that the condition (ii) is necessary for Theorem 1.1. In general, two mappings need not be commute, thus the following question should be of interest.

Question: Is Theorem 1.1 true if the condition (i) is eliminated?

In this paper, we show that the answer is “Yes”. To support our result, we also show that there exists a non-commutative pair of single-valued and multi-valued Suzuki mappings which have a common endpoint.

2 Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers.

Definition 2.1. [13] A hyperbolic space is a triple $(X, d, W)$ where $(X, d)$ is a metric space and $W : X \times X \times [0, 1] \to X$ is a function such that for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$, we have

(W1) $d(z, W(x, y, \alpha)) \leq (1 - \alpha)d(z, x) + \alpha d(z, y)$;
(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$;
(W3) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$;
(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$.

If $x, y \in X$ and $\alpha \in [0, 1]$, then we use the notation $(1 - \alpha)x \oplus \alpha y$ for $W(x, y, \alpha)$. It is easy to see that for any $x, y \in X$ and $\alpha \in [0, 1]$, one has

$$d(x, (1 - \alpha)x \oplus \alpha y) = \alpha d(x, y) \quad \text{and} \quad d(y, (1 - \alpha)x \oplus \alpha y) = (1 - \alpha)d(x, y).$$

Let $[x, y] := \{ (1 - \alpha)x \oplus \alpha y : \alpha \in [0, 1] \}$. A nonempty subset $E$ of $X$ is said to be convex if $[x, y] \subseteq E$ for all $x, y \in E$.

Definition 2.2. [13] The hyperbolic space $(X, d, W)$ is called uniformly convex if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq r\varepsilon$, we have

$$d \left( \frac{1}{2}x \oplus \frac{1}{2}y, a \right) \leq (1 - \delta)r.$$ 

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called a modulus of uniform convexity. The mapping $\delta$ is monotone if for every fixed $\varepsilon$ it decreases with respect to $r$. 
Obviously, uniformly convex Banach spaces are uniformly convex hyperbolic
spaces. CAT(0) spaces are also uniformly convex hyperbolic spaces, see [13, Proposition 8].

**Definition 2.3.** [14] Let $E$ be a nonempty subset of a metric space $(X, d)$. A
multivalued mapping $T : E \to CB(E)$ is said to satisfy condition $(E_\mu)$ if there
exists $\mu \geq 1$ such that for each $x, y \in E$, we have
\[
\text{dist}(x, T(y)) \leq \mu \text{dist}(x, T(x)) + d(x, y).
\]
The mapping $T$ is said to be quasi-nonexpansive if for each $x \in E$ and $y \in \text{Fix}(T)$, one has
\[
H(T(x), T(y)) \leq d(x, y).
\]

Let $E$ be a nonempty subset of a metric space $(X, d)$ and $\{x_n\}$ be a bounded
sequence in $X$. The asymptotic radius of $\{x_n\}$ relative to $E$ is defined by
\[
r(E, \{x_n\}) = \inf \left\{ \limsup_{n \to \infty} d(x_n, x) : x \in E \right\}.
\]
The asymptotic center of $\{x_n\}$ relative to $E$ is defined by
\[
A(E, \{x_n\}) = \left\{ x \in E : \lim_{n \to \infty} d(x_n, x) = r(E, \{x_n\}) \right\}.
\]
The sequence $\{x_n\}$ is called regular relative to $E$ if $r(E, \{x_n\}) = r(E, \{x_n\})$
for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. It is known that every bounded sequence in
a metric space has a regular subsequence (see [15]; also [16, p. 3690]).

Before proving our main results we collect some basic facts about uniformly
convex hyperbolic spaces. From now on, $X$ stands for a complete uniformly convex
hyperbolic space with monotone modulus of uniform convexity.

**Lemma 2.4.** The following statements hold:

(i) [2 Proposition 2] if $E$ is a nonempty subset of $X$ and $t : E \to E$ is a single-
valued Suzuki mapping with $\text{End}(t) \neq \emptyset$, then $t$ is a quasi-nonexpansive mapping;

(ii) [17] Lemma 3.2 if $E$ is a nonempty closed convex subset of $X$ and $T : E \to K(E)$ is a multi-valued Suzuki mapping, then $T$ satisfies condition $(E_3)$;

(iii) [6] Proposition 2.4 Let $E$ be a nonempty subset of $X$, $\{x_n\}$ be a sequence in $E$, and $T : E \to K(E)$ be a multi-valued mapping. Then $r_{x_n}(T(x_n)) \to 0$ if and
only if $\text{dist}(x_n, T(x_n)) \to 0$ and $\text{diam}(T(x_n)) \to 0$.

(iv) [13] Proposition 3.3 if $E$ is a nonempty closed convex subset of $X$ and $\{x_n\}$ be a bounded sequence in $E$, then $A(E, \{x_n\})$ consists of exactly one point.
3 Main Results

This section is begun by proving an endpoint theorem for multi-valued Suzuki mappings. The proof closely follows the proof of Theorem 3.1 in [11], for the convenience of readers we include the details.

Theorem 3.1. Let \( E \) be a nonempty closed convex subset of \( X \) and \( T : E \to \mathcal{K}(E) \) be a multi-valued Suzuki mapping. Let \( \{x_n\} \) be a sequence in \( E \) which is regular relative to \( E \). Suppose that \( \{x_n\} \) is an approximate endpoint sequence for \( T \) and \( A(E, \{x_n\}) = \{x\} \). Then \( x \) is an endpoint of \( T \).

Proof. Let \( r = r(E, \{x_n\}) \). For \( n \in \mathbb{N} \), we let \( y_n \in T(x_n) \) be such that \( d(x_n, y_n) = \text{dist}(x_n, T(x_n)) \). Since \( \{x_n\} \) is an approximate endpoint sequence for \( T \), by Lemma 2.4 (iii) we have

\[
\text{dist}(x_n, T(x_n)) \to 0 \quad \text{and} \quad \text{diam}(T(x_n)) \to 0. \tag{3.1}
\]

**Case 1.** For each \( n \in \mathbb{N} \) there exists \( m \in \mathbb{N} \) such that \( m \geq n \) and \( \frac{1}{2}d(x_m, y_m) > d(x_m, x) \). Then there is a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that

\[
\frac{1}{2}d(x_{n_k}, y_{n_k}) > d(x_{n_k}, x) \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.2}
\]

It follows from (3.1) and (3.2) that \( \lim_{k \to \infty} x_{n_k} = x \). By Lemma 2.4 (ii), we have

\[
\text{dist}(x, T(x)) \leq d(x, x_{n_k}) + \text{dist}(x_{n_k}, T(x)) \leq 2d(x, x_{n_k}) + 3\text{dist}(x_{n_k}, T(x_{n_k})) \to 0 \quad \text{as} \quad k \to \infty.
\]

Hence \( x \in T(x) \). Notice also that \( \frac{1}{2}\text{dist}(x, T(x)) = 0 \leq d(x_{n_k}, x) \) for all \( k \in \mathbb{N} \). Since \( T \) is a Suzuki mapping, we have

\[
H(T(x_{n_k}), T(x)) \leq d(x_{n_k}, x) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.3}
\]

Let \( v \in T(x) \) and choose \( u_{n_k} \in T(x_{n_k}) \) so that \( d(v, u_{n_k}) = \text{dist}(v, T(x_{n_k})) \). From (3.1) and (3.3) we have

\[
d(x, v) \leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) + d(y_{n_k}, u_{n_k}) + d(u_{n_k}, v) \leq d(x, x_{n_k}) + \text{dist}(x_{n_k}, T(x_{n_k})) + \text{diam}(T(x_{n_k})) + H(T(x_{n_k}), T(x)) \to 0 \quad \text{as} \quad k \to \infty.
\]

Hence \( v = x \) for all \( v \in T(x) \). Therefore \( x \in \text{End}(T) \).

**Case 2.** There exists \( n_0 \in \mathbb{N} \) such that \( \frac{1}{2}d(x_n, y_n) \leq d(x_n, x) \) for all \( n \geq n_0 \). This implies that \( \frac{1}{2}\text{dist}(x_n, T(x_n)) \leq d(x_n, x) \) and so \( H(T(x_n), T(x)) \leq d(x_n, x) \). For \( n \in \mathbb{N} \), select \( z_n \in T(x) \) so that \( d(y_n, z_n) = \text{dist}(y_n, T(x)) \). Since \( T(x) \) is compact, there exists a subsequence \( \{z_{n_j}\} \) of \( \{z_n\} \) such that \( z_{n_j} \to w \in T(x) \). For \( j \) sufficiently large, we have

\[
d(x_{n_j}, w) \leq d(x_{n_j}, y_{n_j}) + d(y_{n_j}, z_{n_j}) + d(z_{n_j}, w) \leq d(x_{n_j}, y_{n_j}) + H(T(x_{n_j}), T(x)) + d(z_{n_j}, w) \leq \text{dist}(x_{n_j}, T(x_{n_j})) + d(x_{n_j}, x) + d(z_{n_j}, w).
\]
This implies by the regularity of \( \{x_n\} \) that \( \limsup_{j \to \infty} d(x_{n_j}, w) \leq \limsup_{j \to \infty} d(x_{n_j}, x) = r \). Hence \( w \in A(E, \{x_{n_j}\}) = \{x\} \). Therefore \( x = w \in T(x) \). Let \( v \in T(x) \) and choose \( u_{n_j} \in T(x_{n_j}) \) so that \( d(v, u_{n_j}) = \text{dist}(v, T(x_{n_j})) \). Thus
\[
\begin{align*}
d(x_{n_j}, v) & \leq d(x_{n_j}, y_{n_j}) + d(y_{n_j}, u_{n_j}) + d(u_{n_j}, v) \\
& \leq d(x_{n_j}, y_{n_j}) + \text{diam}(T(x_{n_j})) + H(T(x), T(x_{n_j})) \\
& \leq \text{dist}(x_{n_j}, T(x_{n_j})) + \text{diam}(T(x_{n_j})) + d(x_{n_j}, x).
\end{align*}
\]

It follows from (3.1) that \( \limsup_{j \to \infty} d(x_{n_j}, v) \leq \limsup_{j \to \infty} d(x_{n_j}, x) = r \). Hence \( v \in A(E, \{x_{n_j}\}) = \{x\} \), and so \( v = x \) for all \( v \in T(x) \). Therefore \( x \in \text{End}(T) \). \( \square \)

Now, we are ready to prove our main theorem. In contrast to Theorem 1.1, it does not need the convexity of \( T(x) \).

**Theorem 3.2.** Let \( E \) be a nonempty bounded closed convex subset of \( X \), \( t : E \to E \) be a single-valued mapping and \( T : E \to \mathcal{K}(E) \) be a multi-valued mapping. Suppose that \( t \) and \( T \) are Suzuki mappings such that \( T \) has an approximate endpoint sequence in \( \text{End}(t) \). Then \( t \) and \( T \) have a common endpoint in \( E \).

**Proof.** Let \( \{x_n\} \) be an approximate endpoint sequence for \( T \) in \( \text{End}(t) \). By passing to a subsequence, we may assume that \( \{x_n\} \) is regular relative to \( E \). Let \( A(E, \{x_n\}) = \{x\} \). By Theorem 3.1 \( x \in \text{End}(T) \). It follows from Lemma 2.4 (i) that
\[
\limsup_{n \to \infty} d(x_n, t(x)) \leq \limsup_{n \to \infty} d(x_n, x).
\]
This implies \( t(x) \in A(E, \{x_n\}) = \{x\} \), and hence \( x \in \text{End}(t) \). Therefore \( x \) is a common endpoint of \( t \) and \( T \). \( \square \)

The following example shows that there exists a non-commutative pair of single-valued and multi-valued Suzuki mappings which have a common endpoint.

**Example 3.3.** Let \( X = \mathbb{R} \), \( E = [0, 3] \) and \( t : E \to E \) be defined by
\[
t(x) = \begin{cases} 
0 & \text{if } x \neq 3, \\
1 & \text{if } x = 3.
\end{cases}
\]
Then \( t \) is a single-valued Suzuki mapping (see [2]). Let \( T : E \to \mathcal{K}(E) \) be defined by
\[
T(x) = \left[ \frac{x}{2}, x \right] \quad \text{for all } x \in E.
\]
Then \( H(T(x), T(y)) = |x - y| \) for all \( x, y \in E \). Therefore \( T \) is nonexpansive and hence it is a Suzuki mapping. If \( x = 3/2 \) and \( y = 3 \), then \( x \in T(y) \) but \( t(x) = 0 \notin [\frac{3}{2}, 1] = T(t(y)) \). Therefore \( t \) and \( T \) are not commuting, hence we cannot apply Theorem 1.1. However, by Theorem 3.2 we can conclude that \( t \) and \( T \) have a common endpoint in \( E \).
Acknowledgement: This research was supported by Chiang Mai University.

References


(Received 7 February 2019)
(Accepted 20 March 2019)