(\(\tau_1, \tau_2\))\(^*\)-Semi Star Generalized Locally Closed Sets

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Abstract : The aim of this paper is to continue the study of (\(\tau_1, \tau_2\))\(^*\)-semi star generalized closed sets by introducing the concepts of (\(\tau_1, \tau_2\))\(^*\)-semi star generalized locally closed sets and study their basic properties in bitopological spaces.

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1 Introduction

The study of generalization of closed sets [1, 2] has been found to ensure some new separation axioms which have been very useful in the study of certain objects of digital topology [3]. In recent years many generalizations of closed sets have been developed by various authors. Chandrasekhara Rao and Joseph [4] introduced the concepts of semi star generalized open sets and semi star generalized closed sets in unital topological spaces with the help of semi open sets [5].


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Meanwhile Kelly [13] introduced the concept of bitopological spaces. Jelic [14] introduced \( \tau_i \)-closed sets and \( \tau_i \)-continuity in bitopological settings. Chandrasekhara Rao and Kannan [15, 16] introduced the concepts of \( \tau_1 \tau_2 \)-semi star generalized locally closed sets and then they introduced the concepts of \( \tau_1 \tau_2 \)-semi star generalized locally closed sets in bitopological spaces and then they introduced the concepts of \( \tau_1 \tau_2 \)-semi star generalized locally closed sets and \( s^*g \)-submaximal spaces with the help of \( s^*g \)-closed sets and studied their basic properties in bitopological spaces.

Kannan et al introduced \( (\tau_1, \tau_2)^* \)-semi star generalized closed sets [17] in bitopological spaces. In this sequel the aim of this paper is to introduce the concepts of \( (\tau_1, \tau_2)^* \)-semi star generalized locally closed sets and study their basic properties in bitopological spaces. In the next section some prerequisites and abbreviations are established.

2 Preliminaries

Let \( (X, \tau_1, \tau_2) \) or simply \( X \) denote a bitopological space. By \( \tau_1 \tau_2-S^*GO(X, \tau_1, \tau_2) \) \{resp. \( \tau_1 \tau_2-S^*GC(X, \tau_1, \tau_2) \}\}, we shall mean the collection of all \( \tau_1 \tau_2-s^*g \) open sets (resp. \( \tau_1 \tau_2-s^*g \) closed sets) in \( (X, \tau_1, \tau_2) \). For any subset \( A \subseteq X \), \( \tau_1 \)-int\( (A) \) and \( \tau_1 \)-cl\( (A) \) denote the interior and closure of a set \( A \) with respect to the topology \( \tau_1 \) respectively. \( A^C \) denotes the complement of \( A \) in \( X \) unless explicitly stated. We shall require the following known definitions.

**Definition 2.1.** A subset of a bitopological space \( (X, \tau_1, \tau_2) \) is called

(a) \( \tau_1 \tau_2 \)-semi open [18, 19] if there exists a \( \tau_1 \)-open set \( U \) such that \( U \subseteq A \subseteq \tau_2 \)-cl\( (U) \).

(b) \( \tau_1 \tau_2 \)-semi closed [18, 19] if \( X-A \) is \( \tau_1 \tau_2 \)-semi open.

Equivalently, a subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) is called \( \tau_1 \tau_2 \)-semi closed if there exists a \( \tau_1 \)-closed set \( F \) such that \( \tau_2 \)-int \( (F) \subseteq A \subseteq F \).

(c) \( \tau_1 \tau_2 \)-generalized closed (\( \tau_1 \tau_2 \)-g closed) [20] if \( \tau_2 \)-cl\( (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( \tau_1 \)-open in \( X \).

(d) \( \tau_1 \tau_2 \)-generalized open (\( \tau_1 \tau_2 \)-g open) [20] if \( X-A \) is \( \tau_1 \tau_2 \)-g closed.

(e) \( \tau_1 \tau_2 \)-semi star generalized closed (\( \tau_1 \tau_2-s^*g \) closed) [15] if \( \tau_2 \)-cl\( (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-semi open in \( X \).

(f) \( \tau_1 \tau_2 \)-semi star generalized open (\( \tau_1 \tau_2-s^*g \) open) [15] if \( X-A \) is \( \tau_1 \tau_2-s^*g \) closed in \( X \).
(g) \( \tau_1 \tau_2 \)-semi generalized closed \( (\tau_1 \tau_2 \text{-sg closed}) \) if \( \tau_2 \text{-scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-semi open in \( X \).

(h) \( \tau_1 \tau_2 \)-semi generalized open \( (\tau_1 \tau_2 \text{-sg open}) \) if \( X - A \) is \( \tau_1 \tau_2 \text{-sg closed} \) in \( X \).

(i) \( \tau_1 \tau_2 \)-generalized semi closed \( (\tau_1 \tau_2 \text{-gs closed}) \) if \( \tau_2 \text{-scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \)-open in \( X \).

(j) \( \tau_1 \tau_2 \)-generalized semi open \( (\tau_1 \tau_2 \text{-gs open}) \) if \( X - A \) is \( \tau_1 \tau_2 \text{-gs closed} \) in \( X \).

(k) \( (\tau_1, \tau_2)^* \)-generalized closed \( \{(\tau_1, \tau_2)^*-g \text{ closed}\} \) if \( \tau_1 \tau_2 \text{-cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \tau_2 \)-open in \( X \).

(l) \( (\tau_1, \tau_2)^* \)-generalized open \( \{(\tau_1, \tau_2)^*-g \text{ open}\} \) if \( X - A \) is \( (\tau_1, \tau_2)^*-g \text{ closed} \).

(m) \( (\tau_1, \tau_2)^* \)-semi open \( \{(\tau_1, \tau_2)^*-g \text{ open}\} \) if \( A \subseteq \tau_1 \tau_2 \text{-cl}[\tau_1 \tau_2 \text{-int}(S)] \).

(n) \( (\tau_1, \tau_2)^* \)-semi closed \( \{(\tau_1, \tau_2)^*-g \text{ closed}\} \) if \( X - A \) is \( (\tau_1, \tau_2)^*-\text{semi open} \).

(o) \( (\tau_1, \tau_2)^* \)-semi generalized closed \( \{(\tau_1, \tau_2)^*-\text{sg closed}\} \) if \( (\tau_1, \tau_2)^*-\text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( (\tau_1, \tau_2)^*-\text{semi open} \) in \( X \).

(p) \( (\tau_1, \tau_2)^* \)-semi generalized open \( \{(\tau_1, \tau_2)^*-\text{sg open}\} \) if \( X - A \) is \( (\tau_1, \tau_2)^*-\text{sg closed} \).

(q) \( (\tau_1, \tau_2)^* \)-generalized semi closed \( \{(\tau_1, \tau_2)^*-\text{gs closed}\} \) if \( (\tau_1, \tau_2)^*-\text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( (\tau_1, \tau_2)^*-\text{open} \) in \( X \).

(r) \( (\tau_1, \tau_2)^* \)-generalized semi open \( \{(\tau_1, \tau_2)^*-\text{gs open}\} \) if \( X - A \) is \( (\tau_1, \tau_2)^*-\text{gs closed} \).

(s) \( (\tau_1, \tau_2)^* \)-semi star generalized closed \( \{(\tau_1, \tau_2)^*-\text{s*g closed}\} \) if \( \tau_1 \tau_2 \text{-cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_1 \tau_2 \)-semi open in \( X \).

(t) \( (\tau_1, \tau_2)^* \)-semi star generalized open \( \{(\tau_1, \tau_2)^*-\text{s*g open}\} \) if \( X - A \) is \( (\tau_1, \tau_2)^*-\text{s*g closed} \).

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Definition 3.1. A subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) is said to be

(a) \( (\tau_1, \tau_2)^*-s* \text{g locally closed set} \) if \( A = G \cap F \) where \( G \) is \( \tau_1 \tau_2 \text{-s*g open set} \) and \( F \) is \( \tau_1 \tau_2 \text{-s*g closed set} \) in \( X \).

(b) \( (\tau_1, \tau_2)^*-s* \text{g locally closed*} \) if \( A = G \cap F \) where \( G \) is \( \tau_1 \tau_2 \text{-s*g open set} \) and \( F \) is \( \tau_2 \)-closed in \( X \).

(c) \( (\tau_1, \tau_2)^*-s* \text{g locally closed**} \) if \( A = G \cap F \) where \( G \) is \( \tau_1 \)-open and \( F \) is \( \tau_1 \tau_2 \text{-s*g closed} \) in \( X \).
Remark 3.2.

(a) The class of all \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed sets in \((X, \tau_1, \tau_2)\) is denoted by 
\((\tau_1, \tau_2)^*\)-s\(^*\)GLC\((X, \tau_1, \tau_2)\).

(b) The class of all \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) sets in \((X, \tau_1, \tau_2)\) is denoted by 
\((\tau_1, \tau_2)^*\)-s\(^*\)GLC\(^*\)\((X, \tau_1, \tau_2)\).

(c) The class of all \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) sets in \((X, \tau_1, \tau_2)\) is denoted by 
\((\tau_1, \tau_2)^*\)-s\(^*\)GLC\(^*\)\(^*\)\((X, \tau_1, \tau_2)\).

Example 3.3. Let \(X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{b, c\}\}\). Then, \(\tau_1\tau_2\)-s\(^*\)g open sets in \((X, \tau_1, \tau_2)\) are \(\phi, X, \{a\}, \{b, c\}\) and \(\tau_1\tau_2\)-s\(^*\)g closed sets in 
\((X, \tau_1, \tau_2)\) are \(X, \phi, \{a\}, \{b, c\}\). Then

(a) \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed sets in \((X, \tau_1, \tau_2)\) are \(\phi, X, \{a\}, \{b, c\}\).

(b) \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) sets in \((X, \tau_1, \tau_2)\) are \(\phi, X, \{a\}, \{b, c\}\).

(c) \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) sets in \((X, \tau_1, \tau_2)\) are \(\phi, X, \{a\}, \{b, c\}\).

Definition 3.4. A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is said to be

(a) \((\tau_1, \tau_2)^*\)-g locally closed set if \(A = G \cap F\) where \(G\) is \(\tau_1\tau_2\)-g open set and \(F\) 
is \(\tau_1\tau_2\)-g closed set in \(X\).

(b) \((\tau_1, \tau_2)^*\)-sg locally closed set if \(A = G \cap F\) where \(G\) is \(\tau_1\tau_2\)-sg open set and 
\(F\) is \(\tau_1\tau_2\)-sg closed set in \(X\).

(c) \((\tau_1, \tau_2)^*\)-gs locally closed set if \(A = G \cap F\) where \(G\) is \(\tau_1\tau_2\)-gs open set and 
\(F\) is \(\tau_1\tau_2\)-gs closed set in \(X\).

Example 3.5. Let \(X = \{a, b, c, d\}, \tau_1 = \{\phi, X, \{a, b\}\}, \tau_2 = \{\phi, X, \{b, c\}, \{c\}, \{b, c\}\}\). 
Then all the subsets of \(X\) are \((\tau_1, \tau_2)^*\)-g locally closed, \((\tau_1, \tau_2)^*\)-sg locally closed 
and \((\tau_1, \tau_2)^*\)-gs locally closed in \(X\).

Theorem 3.6. In any bitopological space \((X, \tau_1, \tau_2)\),

(i) \(A \in (\tau_1, \tau_2)^*\)-s\(^*\)GLC\(^*\)\((X, \tau_1, \tau_2)\) \(\Rightarrow\) \(A \in (\tau_1, \tau_2)^*\)-s\(^*\)GLC\((X, \tau_1, \tau_2)\).

(ii) \(A \in (\tau_1, \tau_2)^*\)-s\(^*\)GLC\(^*\)\(^*\)\((X, \tau_1, \tau_2)\) \(\Rightarrow\) \(A \in (\tau_1, \tau_2)^*\)-s\(^*\)GLC\((X, \tau_1, \tau_2)\).

(iii) \(A \in \tau_1\tau_2\)-s\(^*\)GC\((X, \tau_1, \tau_2)\) \(\Rightarrow\) \(A \in (\tau_1, \tau_2)^*\)-s\(^*\)GLC\((X, \tau_1, \tau_2)\).

(iv) \(A \in \tau_1\tau_2\)-s\(^*\)G\((X, \tau_1, \tau_2)\) \(\Rightarrow\) \(A \in (\tau_1, \tau_2)^*\)-s\(^*\)GLC\((X, \tau_1, \tau_2)\).

Proof. (i) Since \(A\) is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) subset in \((X, \tau_1, \tau_2)\), we have \(A = G \cap F\) 
where \(G\) is \(\tau_1\tau_2\)-s\(^*\)g open set and \(F\) is \(\tau_2\)-closed in \(X\). Since every \(\tau_2\)-closed 
set is \(\tau_1\tau_2\)-s\(^*\)g closed in \((X, \tau_1, \tau_2)\), \(A = G \cap F\) where \(G\) is \(\tau_1\tau_2\)-s\(^*\)g open 
and \(F\) is \(\tau_1\tau_2\)-s\(^*\)g closed in \((X, \tau_1, \tau_2)\). Therefore, \(A \in (\tau_1, \tau_2)^*\)-s\(^*\)GLC\((X, \tau_1, \tau_2)\).

(ii) Since \(A\) is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) subset in \((X, \tau_1, \tau_2)\), we have \(A = G \cap F\) 
where \(G\) is \(\tau_1\)-open and \(F\) is \(\tau_1\tau_2\)-s\(^*\)g closed in \((X, \tau_1, \tau_2)\). Since every
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\(\tau_1\)-open set is \(\tau_1 \tau_2\)-s\(^*\)g open in \((X, \tau_1, \tau_2), A = G \cap F\) where \(G\) is \(\tau_1 \tau_2\)-s\(^*\)g open and \(F\) is \(\tau_1 \tau_2\)-s\(^*\)g closed in \((X, \tau_1, \tau_2)\). Therefore, \(A \in \tau_1 \tau_2\)-S\(^*\)GLC\((X, \tau_1, \tau_2)\).

(iii) Since \(A = A \cap X\) and \(A\) is \(\tau_1 \tau_2\)-s\(^*\)g closed and \(X\) is \(\tau_1 \tau_2\)-s\(^*\)g open in \((X, \tau_1, \tau_2)\), we have \(A \in (\tau_1, \tau_2)^*\)-S\(^*\)GLC\((X, \tau_1, \tau_2)\).

(iv) Since \(A = A \cap X\) and \(A\) is \(\tau_1 \tau_2\)-s\(^*\)g open and \(X\) is \(\tau_1 \tau_2\)-s\(^*\)g closed in \((X, \tau_1, \tau_2)\), we have \(A \in (\tau_1, \tau_2)^*\)-S\(^*\)GLC\((X, \tau_1, \tau_2)\).

\[ \square \]

**Remark 3.7.** The converses of (i), (ii), (iii) and (iv) of the above theorem are not true in general as can be seen from the following examples.

**Example 3.8.** In Example \ref{ex:3.7}, \{\(b, c, d\)\} is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed in \((X, \tau_1, \tau_2)\), but not \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) in \((X, \tau_1, \tau_2)\).

**Example 3.9.** In Example \ref{ex:3.7}, \{\(b\)\} is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed in \((X, \tau_1, \tau_2)\), but not \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed\(^*\) in \((X, \tau_1, \tau_2)\).

**Example 3.10.** In Example \ref{ex:3.7}, \{\(b, c, d\)\} is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed in \((X, \tau_1, \tau_2)\), but not \((\tau_1, \tau_2)^*\)-s\(^*\)g open in \((X, \tau_1, \tau_2)\) and \{\(a\)\} is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed in \((X, \tau_1, \tau_2)\), but not \((\tau_1, \tau_2)^*\)-s\(^*\)g closed in \((X, \tau_1, \tau_2)\).

Recall that a bitopological space \((X, \tau_1, \tau_2)\) is a pairwise door space \cite{23} if every subset of \((X, \tau_1, \tau_2)\) is either \(\tau_i\)-open or \(\tau_j\)-closed, for \(i, j = 1, 2\) and \(i \neq j\).

**Theorem 3.11.** If \((X, \tau_1, \tau_2)\) is pairwise door space, then every subset of \(X\) is both \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed and \((\tau_2, \tau_1)^*\)-s\(^*\)g locally closed.

**Proof.** Since \((X, \tau_1, \tau_2)\) is pairwise door space, every subset of \((X, \tau_1, \tau_2)\) is either \(\tau_1\)-open or \(\tau_2\)-closed and \(\tau_2\)-open or \(\tau_1\)-closed. Since every \(\tau_1\)-open (resp. \(\tau_2\)-closed) subset of \((X, \tau_1, \tau_2)\) is \(\tau_1 \tau_2\)-s\(^*\)g open (resp. \(\tau_1 \tau_2\)-s\(^*\)g closed), we have every subset of \((X, \tau_1, \tau_2)\) is either \(\tau_1 \tau_2\)-s\(^*\)g open or \(\tau_1 \tau_2\)-s\(^*\)g closed. Since every \(\tau_1 \tau_2\)-s\(^*\)g open and \(\tau_1 \tau_2\)-s\(^*\)g closed subset of \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed, we have every subset of \(X\) is \((\tau_1, \tau_2)^*\)-s\(^*\)g locally closed. Similarly we can prove that every subset of \(X\) is \((\tau_2, \tau_1)^*\)-s\(^*\)g locally closed.

\[ \square \]

**Theorem 3.12.** For a subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\), the following are equivalent.

\(a\) \(A \in (\tau_1, \tau_2)^*\)-S\(^*\)GLC\((X, \tau_1, \tau_2)\).

\(b\) \(A = G \cap [\tau_2\text{-cl}(A)]\) for some \(\tau_1 \tau_2\)-s\(^*\)g open set \(G\).

\(c\) \(A \cup \{X - [\tau_2\text{-cl}(A)]\}\) is \(\tau_1 \tau_2\)-s\(^*\)g open.

\(d\) \([\tau_2\text{-cl}(A)] - A\) is \(\tau_1 \tau_2\)-s\(^*\)g closed.
Proof. (a) \(\Rightarrow\) (b) : Since \(A\) is \((\tau_1, \tau_2)^{-s}*g\) locally closed set in \((X, \tau_1, \tau_2)\), we have \(A = G \cap F\) where \(G\) is \(\tau_1\tau_2^{-s}*g\) open set and \(F\) is \(\tau_2\)-closed in \(X\). Since \(A \subseteq \tau_2\)-cl \((A)\) and \(A \subseteq G\), we have

\[
A \subseteq G \cap [\tau_2\text{-cl } (A)]
\]

(1)

Since \(A \subseteq F\) and \(F\) is \(\tau_2\)-closed in \(X\), we have \(\tau_2\)-cl \((A) \subseteq F\). Therefore, \(G \cap [\tau_2\text{-cl } (A)] \subseteq G \cap F = A\). Hence

\[
G \cap [\tau_2\text{-cl } (A)] \subseteq A
\]

(2)

From(1) and (2), we have \(A = G \cap [\tau_2\text{-cl } (A)]\) for some \(\tau_1\tau_2^{-s}*g\) open set \(G\) in \((X, \tau_1, \tau_2)\).

(b) \(\Rightarrow\) (a) : Suppose that \(A = G \cap [\tau_2\text{-cl } (A)]\) for some \(\tau_1\tau_2^{-s}*g\) open set \(G\) in \((X, \tau_1, \tau_2)\). Since \(\tau_2\)-cl \((A)\) is \(\tau_2\)-closed in \((X, \tau_1, \tau_2)\) and \(G\) is \(\tau_1\tau_2^{-s}*g\) closed in \((X, \tau_1, \tau_2)\), we have \(A \in (\tau_1, \tau_2)^{-s}*GLC^*(X, \tau_1, \tau_2)\)

(c) \(\Rightarrow\) (b) : Suppose that \(A \cup \{X - [\tau_2\text{-cl } (A)]\}\) is \(\tau_1\tau_2^{-s}*g\) open in \((X, \tau_1, \tau_2)\). Let \(G = A \cup \{X - [\tau_2\text{-cl } (A)]\}\). Then \(G\) is \(\tau_1\tau_2^{-s}*g\) open in \((X, \tau_1, \tau_2)\). Now, \(G \cap [\tau_2\text{-cl } (A)] = \{A \cap \{X - [\tau_2\text{-cl } (A)]\}\} \cap [\tau_2\text{-cl } (A)] = \{A \cap [\tau_2\text{-cl } (A)]\} \cup \{\tau_2\text{-cl } (A)\} = \{A \cap [\tau_2\text{-cl } (A)]\} = A\). Therefore, \(A = G \cap [\tau_2\text{-cl } (A)]\) for some \(\tau_1\tau_2^{-s}*g\) open set \(G\) in \((X, \tau_1, \tau_2)\).

(d) \(\Rightarrow\) (c) : Suppose that \(\tau_2\)-cl \((A) - A\) is \(\tau_1\tau_2^{-s}*g\) closed in \((X, \tau_1, \tau_2)\). Let \(F = \tau_2\text{-cl } (A) - A\). Then \(F\) is \(\tau_1\tau_2^{-s}*g\) closed in \((X, \tau_1, \tau_2)\). Now, \(X - F = X - \{\tau_2\text{-cl } (A)\} = \{X - \{\tau_2\text{-cl } (A)\}\} = \{X - \{\tau_2\text{-cl } (A)\}\} \cup \{X \cap \{\tau_2\text{-cl } (A)\}\} = \tau_2\text{-cl } (A)\). Hence \(A \cup \{X - [\tau_2\text{-cl } (A)]\}\) \(\tau_1\tau_2^{-s}*g\) open in \((X, \tau_1, \tau_2)\).

\[\square\]

**Theorem 3.13.** In a bitopological space \((X, \tau_1, \tau_2)\), the following are equivalent.

(a) \(A - [\tau_1\text{-int } (A)]\) is \(\tau_1\tau_2^{-s}*g\) open in \((X, \tau_1, \tau_2)\).

(b) \([\tau_1\text{- int } (A)] \cup [X - A]\) is \(\tau_1\tau_2^{-s}*g\) closed in \((X, \tau_1, \tau_2)\).

(c) \(G \cup [\tau_1\text{-int } (A)] = A\) for some \(\tau_1\tau_2^{-s}*g\) open set \(G\) in \((X, \tau_1, \tau_2)\).
Proof. (a) ⇒ (b): Now, $X - \{A - [\tau_1 \text{-int}(A)]\} = X \cap \{A - [\tau_1 \text{-int}(A)]\}^C = X \cap [A \cap \{\tau_1 \text{-int}(A)\}]^C = X \cap \{A^C \cup \{\tau_1 \text{-int}(A)\}\}^C = X \cap \{A^C \cup \{\tau_1 \text{-int}(A)\}\}^C = \{\tau_1 \text{-int}(A)\} \cup \{X - A\}$. Since $A - [\tau_1 \text{-int}(A)]$ is $\tau_1 \tau_2$-g open, we have $X - \{A - [\tau_1 \text{-int}(A)]\} = [\tau_1 \text{-int}(A)] \cup \{X - A\}$ is $\tau_1 \tau_2$-g closed in $(X, \tau_1, \tau_2)$.

(b) ⇒ (a): Suppose that $[\tau_1 \text{-int}(A)] \cup \{X - A\}$ is $\tau_1 \tau_2$-g closed. Since $[\tau_1 \text{-int}(A)] \cup \{X - A\}$ is $\tau_1 \tau_2$-g closed, we have $X - \{\tau_1 \text{-int}(A)\} \cup \{X - A\}$ is $\tau_1 \tau_2$-g open. Now, $X - \{\tau_1 \text{-int}(A)\} \cup \{X - A\} = X \cap \{\tau_1 \text{-int}(A)\} \cup \{X - A\}\}^C = X \cap \{\tau_1 \text{-int}(A)\}^C \cup \{X - A\}\}^C = X \cap \{\tau_1 \text{-int}(A)\}^C \cap \{X - A\}\}^C = A \cap [\tau_1 \text{-int}(A)]^C = A - [\tau_1 \text{-int}(A)]$. Therefore, $A - [\tau_1 \text{-int}(A)]$ is $\tau_1 \tau_2$-g open in $(X, \tau_1, \tau_2)$.

(c) ⇒ (b): Suppose that $A = G \cup [\tau_1 \text{-int}(A)] = A$ for some $\tau_1 \tau_2$-g open set $G$ in $(X, \tau_1, \tau_2)$. Now, $[\tau_1 \text{-int}(A)] \cup \{X - A\} = \tau_1 \text{-int}(A) \cup A^C = \tau_1 \text{-int}(A) \cup \{G \cup \{\tau_1 \text{-int}(A)\}\}^C = \tau_1 \text{-int}(A) \cup \{G^C \cap \{\tau_1 \text{-int}(A)\}\}^C = \tau_1 \text{-int}(A) \cup \{G^C \cap \{\tau_1 \text{-int}(A)\}\}^C \cap \{X - A\} = [\tau_1 \text{-int}(A)] \cup G^C = X - G$. Since $G$ is $\tau_1 \tau_2$-g open in $(X, \tau_1, \tau_2)$, we have $X - G$ is $\tau_1 \tau_2$-g closed in $(X, \tau_1, \tau_2)$. Therefore, $[\tau_1 \text{-int}(A)] \cup \{X - A\}$ is $\tau_1 \tau_2$-g closed in $(X, \tau_1, \tau_2)$.

Remark 3.14. Even $A$ and $B$ are not $(\tau_1, \tau_2)^{-s^g}$ locally closed sets in $(X, \tau_1, \tau_2)$, $A \cup B$ is $(\tau_1, \tau_2)^{-s^g}$ locally closed in general as can be seen from the following example.

Example 3.15. In Example 3.3, $A = \{b\}, B = \{c\}$ are not $(\tau_1, \tau_2)^{-s^g}$ locally closed sets in $(X, \tau_1, \tau_2)$, but $A \cup B = \{b, c\}$ is $(\tau_1, \tau_2)^{-s^g}$ locally closed set in $(X, \tau_1, \tau_2)$.

Remark 3.16. Since every $(\tau_1, \tau_2)^{-s^g}$ locally closed set is the intersection of a $\tau_1 \tau_2$-g open set and $\tau_1 \tau_2$-g closed set, we can conclude the following.

Theorem 3.17. A subset $A$ of $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2)^{-s^g}$ locally closed if and only if $A^C$ is the union of a $\tau_1 \tau_2$-g open set and $\tau_1 \tau_2$-g closed set.

Remark 3.18. Every $\tau_1$-open set (resp. $\tau_2$-closed set) is $\tau_1 \tau_2$-g open (resp. $\tau_1 \tau_2$-g closed). Accordingly, we conclude the following.

Theorem 3.19.

(a) Every $\tau_1$-open set is $(\tau_1, \tau_2)^{-s^g}$ locally closed and every $\tau_2$-closed set is $(\tau_1, \tau_2)^{-s^g}$ locally closed.
(b) Every $\tau_1\tau_2$-locally closed set is $(\tau_1, \tau_2)^*s^*g$ locally closed, $(\tau_1, \tau_2)^*s^*g$ locally closed and $(\tau_1, \tau_2)^*s^*g$ locally closed$^*$. 

Remark 3.20. But the converses of the assertions of above theorem are not true in general as can be seen in the following examples.

Example 3.21.

(a) In Example 3.3, $\{b, c\}$ is $(\tau_1, \tau_2)^*s^*g$ locally closed set in $(X, \tau_1, \tau_2)$, but $\{b, c\}$ is not $\tau_2$-closed in $(X, \tau_1, \tau_2)$.

(b) In Example 3.3, $\{b, c\}$ is $(\tau_1, \tau_2)^*s^*g$ locally closed set in $(X, \tau_1, \tau_2)$, but $\{b, c\}$ is not $\tau_1$-open in $(X, \tau_1, \tau_2)$.

(c) In Example 3.3, $\{b\}$ is a $(\tau_1, \tau_2)^*s^*g$ locally closed set, $(\tau_1, \tau_2)^*s^*g$ locally closed$^*$ set and $(\tau_1, \tau_2)^*s^*g$ locally closed$^{**}$ set but not $\tau_1\tau_2$-locally closed in $(X, \tau_1, \tau_2)$.

Remark 3.22. Since every $\tau_1\tau_2$-$s^*g$ closed set is $\tau_1\tau_2$-$g$ closed, $\tau_1\tau_2$-$sg$ closed and $\tau_1\tau_2$-$gs$ closed, we conclude the following.

Theorem 3.23.

(a) Every $(\tau_1, \tau_2)^*s^*g$ locally closed is $(\tau_1, \tau_2)^*g$ locally closed.

(b) Every $(\tau_1, \tau_2)^*s^*g$ locally closed is $(\tau_1, \tau_2)^*sg$ locally closed.

(c) Every $(\tau_1, \tau_2)^*s^*g$ locally closed is $(\tau_1, \tau_2)^*gs$ locally closed.

Remark 3.24. But none of the assertions of the above theorem are reversible in general as can be seen in the following example.

Example 3.25. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{a, c\}\}$. Then $\{a, b\}$ is a $(\tau_1, \tau_2)^*g$ locally closed set, $(\tau_1, \tau_2)^*sg$ locally closed set and $(\tau_1, \tau_2)^*gs$ locally closed set, but not $(\tau_1, \tau_2)^*s^*g$ locally closed in $(X, \tau_1, \tau_2)$.

From the above results we conclude the following.

![Diagram showing relationships between several locally closed sets](image-url)

Fig 1. Relationship between several locally closed sets
Since the finite intersection of $\tau_1$-open sets is $\tau_1$-open and the intersection of two $\tau_1\tau_2$-$s^*g$ closed sets is $\tau_1\tau_2$-$s^*g$ closed, we immediately get the following theorem.

**Theorem 3.26.** In any bitopological space $(X, \tau_1, \tau_2)$, intersection of two $(\tau_1, \tau_2)^{*s^*g}$ locally closed sets is $(\tau_1, \tau_2)^{*s^*g}$ locally closed.

In this sequel our next result exhibits the intersection of a $(\tau_1, \tau_2)^{*s^*g}$ locally closed set and a $\tau_2$-closed set in a bitopological space.

**Theorem 3.27.** If $A \in (\tau_1, \tau_2)^{*s^*GLC}(X, \tau_1, \tau_2)$ and $B$ is $\tau_2$-closed in $X$, then $A \cap B \in (\tau_1, \tau_2)^{*s^*GLC}(X, \tau_1, \tau_2)$.

*Proof.* It is obvious since every $\tau_2$-closed set is $\tau_1\tau_2$-$s^*g$ closed and the intersection of two $\tau_1\tau_2$-$s^*g$ closed sets is $\tau_1\tau_2$-$s^*g$ closed. \(\square\)

Our next result is an immediate consequence of the above theorem.

**Theorem 3.28.** If $A \in (\tau_1, \tau_2)^{*s^*GLC}(X, \tau_1, \tau_2)$ and $B$ is $\tau_1\tau_2$-$s^*g$ closed in $X$, then $A \cap B \in (\tau_1, \tau_2)^{*s^*GLC}(X, \tau_1, \tau_2)$.

**Remark 3.29.** The complement of a $(\tau_1, \tau_2)^{*s^*g}$ locally closed set in $(X, \tau_1, \tau_2)$ is not $(\tau_1, \tau_2)^{*s^*g}$ locally closed in general and hence the finite union of $(\tau_1, \tau_2)^{*s^*g}$ locally closed sets need not be $(\tau_1, \tau_2)^{*s^*g}$ locally closed in $(X, \tau_1, \tau_2)$. The next examples show the claim.

**Example 3.30.** In Example 3.28, $\{a\}$ is a $(\tau_1, \tau_2)^{*s^*g}$ locally closed set, but its complement $\{b, c\}$ is not $(\tau_1, \tau_2)^{*s^*g}$ locally closed in $(X, \tau_1, \tau_2)$.

**Example 3.31.** Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{b, c\}\}$, $\tau_2 = \{\phi, X, \{b\}, \{a, c\}\}$. Then, $A = \{b\}$, $B = \{c\}$ are $(\tau_1, \tau_2)^{*s^*g}$ locally closed sets, but $A \cup B = \{b, c\}$ is not $(\tau_1, \tau_2)^{*s^*g}$ locally closed in $(X, \tau_1, \tau_2)$.

**Theorem 3.32.** In a bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent.

(a) $A$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed if and only if $A^C$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed.

(b) $(\tau_1, \tau_2)^{*s^*g}$ locally closed sets are closed under finite union.

*Proof.* $(a) \Rightarrow (b)$: Suppose that $A$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed if and only if $A^C$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed. Let $A, B$ be $(\tau_1, \tau_2)^{*s^*g}$ locally closed. Then by our assumption, $A^C, B^C$ are $(\tau_1, \tau_2)^{*s^*g}$ locally closed. Consequently, $(A \cup B)^C = A^C \cap B^C$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed. Therefore, $A \cup B$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed.

$(b) \Rightarrow (a)$: Suppose that $(\tau_1, \tau_2)^{*s^*g}$ locally closed sets are closed under finite union. Let $A$ be $(\tau_1, \tau_2)^{*s^*g}$ locally closed. Then $A = G \cap F$ where $G$ is $\tau_1\tau_2$-$s^*g$ open and $F$ is $\tau_1\tau_2$-$s^*g$ closed in $X$. Since $G^C$ is $\tau_1\tau_2$-$s^*g$ closed and $F^C$ is $\tau_1\tau_2$-$s^*g$ open in $X$ and every $\tau_1\tau_2$-$s^*g$ open and $\tau_1\tau_2$-$s^*g$ closed set is $(\tau_1, \tau_2)^{*s^*g}$ locally closed, we have $A^C$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed by our assumption. Similarly, we can prove if $A^C$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed then $A$ is $(\tau_1, \tau_2)^{*s^*g}$ locally closed. \(\square\)
Conclusion

Thus we have studied properties of \((\tau_1,\tau_2)^*-s^*g\) locally closed sets in the context of bitopological spaces. Borges [24] showed that locally closed sets play an important role in the context of simple extensions and some results of Engelking [25] indicate that locally closed subsets are of some interest in the setting of local compactness, Čech-Stone compactifications, or Čech complete spaces. Further research can be undertaken in this direction.

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References

(τ_1, τ_2)∗-Semi Star Generalized Locally Closed Sets


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