Fine Spectrum of the Generalized Difference Operator $\Delta_{uv}$ on the Sequence Space $c_0$

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Abstract: The purpose of this paper is to determine spectrum and fine spectrum of the operator $\Delta_{uv}$ on the sequence space $c_0$. The operator $\Delta_{uv}$ on sequence space $c_0$ is defined as $\Delta_{uv}x = (u_n x_n + v_{n-1} x_{n-1})_{n=0}^\infty$ satisfying certain conditions, where $x_{-1} = 0$ and $x = (x_n) \in c_0$. In this paper we have obtained the results on the spectrum and point spectrum for the operator $\Delta_{uv}$ on the sequence space $c_0$. Further, the results on continuous spectrum, residual spectrum and fine spectrum of the operator $\Delta_{uv}$ on sequence space $c_0$ are also derived.

Keywords: spectrum of an operator; generalized difference operator; sequence space.

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1 Introduction

Let $u = (u_k)$ and $v = (v_k)$ be sequences such that

(i) $u$ is either a constant sequence or sequence of distinct real numbers with $U = \lim_{k \to \infty} u_k$,

(ii) $v$ is a sequence of nonzero real numbers with $V = \lim_{k \to \infty} v_k \neq 0$, and

(iii) $|U - u_k| < |V|$ for each $k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

We define the operator $\Delta_{uv}$ on the sequence space $c_0$ as follows:

$$\Delta_{uv}x = (u_n x_n + v_{n-1} x_{n-1})_{n=0}^\infty$$

with $x_{-1} = 0$, where $x = (x_n) \in c_0$. (1.1)
It is easy to verify that the operator $\Delta_{uv}$ can be represented by the matrix

$$\Delta_{uv} = \begin{pmatrix}
u_0 & 0 & 0 & \cdots \\
v_0 & u_1 & 0 & \cdots \\
0 & v_1 & u_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \quad (1.2)$$

The spectrum of the Cesaro operator on the sequence space $c_0$ is investigated by Reade [1], Akhmedov and Basar [2]. Spectrum of the Cesaro operator on sequence spaces $bv_0$ and $bv_1$ is obtained by Okutoyi [3] and Okutoyi [4], respectively. Furthermore, Coskun [5] studied the spectrum and fine spectrum for $p$-Cesaro operator acting on the space $c_0$. Yildirim [6] and [7] examined fine spectrum of the Rhaly operator on sequence spaces $c_0$ and $c_1$. The spectrum and fine spectrum of the difference operator $\Delta$ over the sequence spaces $c_0$ and $c_1$ is determined by Altay and Basar [8], where $\Delta x = (x_n - x_{n-1})$. The fine spectrum of the Zweier matrix $Z_s$ on sequence spaces $l_1$ and $bv_0$ is obtained by Altay and Karakuş [9], where $s$ is a real number with $s \neq 0, 1$ and $Z_s x = (sx_n + (1 - s)x_{n-1})$. Altay and Basar [10] determined fine spectrum of the operator $B(r, s)$ over sequence spaces $c_0$ and $c_1$, where $B(r, s) x = (rx_n + sx_{n-1})$. Recently, spectrum and fine spectrum of the operator $B(r, s, t)$ on sequence spaces $c_0$ and $c_1$ is studied by Furkan, Bilgic and Altay [11], where $B(r, s, t) x = (rx_n + sx_{n-1} + tx_{n-2})$.

In this paper we determine spectrum, point spectrum, continuous spectrum and residual spectrum of the operator $\Delta_{uv}$ on the sequence space $c_0$. It is easy to verify that by choosing suitably $u$ and $v$ sequences, one can get easily the operators such as $B(r, s)$, $Z_s$ etc. Choosing $u = (r), v = (s)$ and $u = (s), v = (1 - s)$, then the operator $\Delta_{uv}$ reduces to $B(r, s)$ and $Z_s$, respectively. Similarly, if $u = (1), v = (-1)$ and $u = (0), v = (1)$, then the operator $\Delta_{uv}$ reduces to $\Delta$ and right-shift operator, respectively. Thus, the results of this paper generalizes the corresponding results of many operator whose matrix representation has diagonal and post-diagonal elements studied by earlier authors.

2 Preliminaries and Notation

Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. The set of all bounded linear operators on $X$ into itself is denoted by $B(X)$. The adjoint $T^* : X^* \to X^*$ of $T$ is defined by

$$(T^* \phi)(x) = \phi(T x) \text{ for all } \phi \in X^* \text{ and } x \in X.$$ 

Clearly, $T^*$ is a bounded linear operator on the dual space $X^*$.

Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With $T$, we associate the operator $T_\alpha = (T - \alpha I)$, where $\alpha$ is a complex number and $I$ is the identity operator on $D(T)$. The inverse of $T_\alpha$ (if exists) is denoted by $T_\alpha^{-1}$ and known as the resolvent operator of $T$. 
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Since the spectral theory is concerned with many properties of $T_\alpha$ and $T_\alpha^{-1}$, which depend on $\alpha$, so we are interested the set of those $\alpha$ in the complex plane for which $T_\alpha^{-1}$ exists or $T_\alpha^{-1}$ is bounded or domain of $T_\alpha^{-1}$ is dense in $X$.

**Definition 2.1.** ([12], pp. 371) Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \rightarrow X$ be a linear operator with domain $D(T) \subseteq X$. A regular value of $T$ is a complex number $\alpha$ such that

(R1) $T_\alpha^{-1}$ exists,
(R2) $T_\alpha^{-1}$ is bounded,
(R3) $T_\alpha^{-1}$ is defined on a set which is dense in $X$.

Resolvent set $\rho(T, X)$ of $T$ is the set of all regular values $\alpha$ of $T$. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called spectrum of $T$. The spectrum $\sigma(T, X)$ is further partitioned into three disjoint sets namely point spectrum, continuous spectrum and residual continuous as follows:

Point spectrum $\sigma_p(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that $T_\alpha^{-1}$ does not exist, i.e., condition (R1) fails. The element of $\sigma_p(T, X)$ is called eigenvalue of $T$.

Continuous spectrum $\sigma_c(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that conditions (R1) and (R3) hold but condition (R2) fails, i.e., $T_\alpha^{-1}$ exists, domain of $T_\alpha^{-1}$ is dense in $X$ but $T_\alpha^{-1}$ is unbounded.

Residual spectrum $\sigma_r(T, X)$ is the set of all $\alpha \in \mathbb{C}$ such that $T_\alpha^{-1}$ exists but do not satisfy condition (R3), i.e., domain of $T_\alpha^{-1}$ is not dense in $X$. The condition (R2) may or may not holds good.

Goldberg’s classification of operator $T_\alpha$ ([13], pp. 58): Let $X$ be a Banach space and $T_\alpha \in B(X)$, where $\alpha$ is a complex number. Again, let $R(T_\alpha)$ and $T_\alpha^{-1}$ denote the range and inverse of the operator $T_\alpha$, respectively. Then the following possibilities may occur;

(A) $R(T_\alpha) = X$,
(B) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(C) $R(T_\alpha) \neq X$,

and

(1) $T_\alpha$ is injective and $T_\alpha^{-1}$ is continuous,
(2) $T_\alpha$ is injective and $T_\alpha^{-1}$ is discontinuous,
(3) $T_\alpha$ is not injective.

**Remark 2.2.** Combining (A), (B), (C) and (1), (2), (3); we get nine different cases. These are labeled by $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and $C_3$. The notation $\alpha \in A_2\sigma(T, X)$ means the operator $T_\alpha \in A_2$, i.e., $R(T_\alpha) = X$ and $T_\alpha$ is injective but $T_\alpha^{-1}$ is discontinuous. Similarly others.

**Remark 2.3.** If $\alpha$ is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then $\alpha$ belongs to the resolvent set $\rho(T, X)$ of $T$ on $X$. The other classification gives rise to the fine spectrum of $T$.

**Lemma 2.4.** ([14], pp. 129) The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from $c_0$ to itself if and only if
(i) the rows of $A$ in $l_1$ and their $l_1$ norms are bounded, and
(ii) the columns of $A$ are in $c_0$.

Note: The operator norm of $T$ is the supremum of the $l_1$ norms of the rows.

**Lemma 2.5.** ([13], pp. 59) $T$ has a dense range if and only if $T^\times$ is one to one, where $T^\times$ denotes the adjoint operator of the operator $T$.

**Lemma 2.6.** ([13], pp. 60) The adjoint operator $T^\times$ of $T$ is onto if and only if $T$ has a bounded inverse.

### 3 Main Results

#### 3.1 Spectrum and Point Spectrum of the Operator $\Delta_{uv}$ on the Sequence Space $c_0$

In this section we obtain spectrum and point spectrum of the operator $\Delta_{uv}$ on $c_0$.

**Theorem 3.1.** The operator $\Delta_{uv} : c_0 \to c_0$ is a bounded linear operator and

$$\|\Delta_{uv}\|_{B(c_0)} = \sup_k (|u_k| + |v_{k-1}|).$$

**Proof.** Proof is simple. So we omit. $\Box$

**Theorem 3.2.** Spectrum of the operator $\Delta_{uv}$ on the sequence space $c_0$ is given by

$$\sigma(\Delta_{uv}, c_0) = \{ \alpha \in \mathbb{C} : |U - \alpha| \leq |V| \}.$$

**Proof.** The proof of this theorem is divided into two parts. In the first part, we show that $\sigma(\Delta_{uv}, c_0) \subseteq \{ \alpha \in \mathbb{C} : |U - \alpha| \leq |V| \}$, which is equivalent to

$$\alpha \in \mathbb{C} \text{ with } |U - \alpha| > |V| \text{ implies } \alpha \notin \sigma(\Delta_{uv}, c_0), \text{ i.e., } \alpha \in \rho(\Delta_{uv}, c_0).$$

In the second part, we establish the reverse inclusion, i.e.,

$$\{ \alpha \in \mathbb{C} : |U - \alpha| \leq |V| \} \subseteq \sigma(\Delta_{uv}, c_0).$$

Part I: Let $\alpha \in \mathbb{C}$ with $|U - \alpha| > |V|$. Clearly, $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}_0$ as it does not satisfy this condition. Further, $(\Delta_{uv} - \alpha I) = (a_{nk})$ reduces to a triangle and hence has an inverse $(\Delta_{uv} - \alpha I)^{-1} = (b_{nk})$, where

$$b_{nk} = \begin{cases} \frac{1}{(u_0 - \alpha)} & \text{for } k = 0, \\ \frac{1}{(u_0 - \alpha)(u_1 - \alpha)} \text{ for } k = 1, \\ \frac{1}{(u_0 - \alpha)(u_1 - \alpha)(u_2 - \alpha)} \text{ for } k = 2, \\ \vdots \end{cases}.$$

(QL.1)
By Lemma 2.4, the operator \((\Delta_{uv} - \alpha I)^{-1}\) is in \(B(\ell_0)\) if

(i) series \(\sum_{k=0}^{\infty} |b_{nk}|\) is convergent for each \(n \in \mathbb{N}_0\) and \(\sup_{n} \sum_{k=0}^{\infty} |b_{nk}| < \infty\), and

(ii) \(\lim_{n\to\infty} |b_{nk}| = 0\) for each \(k \in \mathbb{N}_0\).

In order to show \(\sup_{n} \sum_{k=0}^{\infty} |b_{nk}| < \infty\), first we prove that the series \(\sum_{k=0}^{\infty} |b_{nk}|\) is convergent for each \(n \in \mathbb{N}_0\). For this consider \(S_n = \sum_{k=0}^{\infty} |b_{nk}|\). Clearly, the series

\[
S_n = \left| \frac{v_0 v_1 \cdots v_{n-1}}{(u_0 - \alpha)(u_1 - \alpha) \cdots (u_n - \alpha)} \right| + \cdots + \left| \frac{v_{n-1}}{(u_{n-1} - \alpha)(u_n - \alpha)} \right| + \left| \frac{1}{(u_n - \alpha)} \right| \quad (3.2)
\]

is convergent for each \(n \in \mathbb{N}_0\). Now we claim that \(\sup_{n} S_n\) is finite. For this, suppose

\[
\beta = \lim_{n\to\infty} \left| \frac{v_{n-1}}{u_n - \alpha} \right|, \quad \text{which is equal to} \quad \left| \frac{V}{U - \alpha} \right|.
\]

So, \(0 < \beta < 1\). We choose \(\epsilon > 0\) such that \(\beta + \epsilon < 1\). Since \(\lim_{n\to\infty} \left| \frac{v_{n-1}}{u_n - \alpha} \right| = \beta\), so there exists a positive integer \(n_0\) such that

\[
\left| \frac{v_{n-1}}{u_n - \alpha} \right| < \beta + \epsilon \quad \text{and} \quad \left| \frac{1}{u_n - \alpha} \right| < \frac{\beta + \epsilon}{m} \quad \text{for all} \quad n \geq n_0, \quad (3.3)
\]

where \(m\) is a lower bound of bounded sequence \(v = (v_k)\).

For \(n \geq n_0\), \(S_n\) can be write as

\[
S_n = \left| \frac{v_0 v_1 \cdots v_{n_0-2} v_{n_0-1} \cdots v_{n-1}}{(u_0 - \alpha)(u_1 - \alpha)(u_2 - \alpha) \cdots (u_{n_0-1} - \alpha)(u_{n_0} - \alpha) \cdots (u_n - \alpha)} \right| + \cdots + \left| \frac{v_{n_0-1} \cdots v_{n-1}}{(u_{n_0-1} - \alpha)(u_{n_0} - \alpha) \cdots (u_n - \alpha)} \right| + \left| \frac{v_{n_0} \cdots v_{n-1}}{(u_{n_0} - \alpha)(u_{n_0+1} - \alpha) \cdots (u_n - \alpha)} \right| + \cdots + \left| \frac{1}{u_n - \alpha} \right|.
\]

Take

\[
M = \max \left\{ \left| \frac{1}{u_0 - \alpha} \right|, \cdots, \left| \frac{1}{u_{n_0-1} - \alpha} \right|, \left| \frac{v_0}{u_1 - \alpha} \right|, \cdots, \left| \frac{v_{n_0-2}}{u_{n_0-1} - \alpha} \right| \right\}.
\]

Using inequalities in (3.3), we have

\[
S_n < M^{n_0} (\beta + \epsilon)^{n-n_0+1} + \cdots + M (\beta + \epsilon)^{n-n_0+1} \frac{(\beta + \epsilon)^{n-n_0+1}}{m} + \cdots + (\beta + \epsilon) \frac{(\beta + \epsilon)^{n-n_0+1}}{m}
\]

\[
= (\beta + \epsilon)^{n-n_0+1} \left[ M^{n_0} + \cdots + M \right] + \frac{(\beta + \epsilon)}{m} \left[ 1 + \cdots + (\beta + \epsilon)^{n-n_0} \right]
\]

\[
< [M^{n_0} + \cdots + M] + \frac{1}{m} \left[ \frac{1}{1 - (\beta + \epsilon)} \right] < \infty.
\]
Thus, $S_n < \infty$ for each $n \in \mathbb{N}$ and hence $\sup_n S_n < \infty$.

Again, since $\beta < 1$, therefore $|v_{n-1}| \over |u_n - \alpha| < 1$ for large $n$ and consequently,

$$\lim_{n \to \infty} |b_{n0}| = \lim_{n \to \infty} \frac{v_0 v_1 \cdots v_{n-1}}{(u_0 - \alpha)(u_1 - \alpha) \cdots (u_n - \alpha)} = 0.$$  

Similarly, we can show that $\lim_{n \to \infty} |b_{nk}| = 0$ for all $k = 1, 2, 3, \cdots$.

Thus,

$$(\Delta_{uv} - \alpha I)^{-1} \in B(c_0) \text{ for } \alpha \in \mathbb{C} \text{ with } |U - \alpha| > |V|. \quad (3.4)$$

Next, we show that domain of the operator $(\Delta_{uv} - \alpha I)^{-1}$ is dense in $c_0$, which follows if the operator $(\Delta_{uv} - \alpha I)$ is onto. Suppose $(\Delta_{uv} - \alpha I)x = y$, which gives

$$x = (\Delta_{uv} - \alpha I)^{-1} y, \text{ i.e., } x_n = ((\Delta_{uv} - \alpha I)^{-1} y)_n, \ n \in \mathbb{N}.0.$$  

Thus for every $y \in c_0$, we can find $x \in c_0$ such that $(\Delta_{uv} - \alpha I)x = y$. Hence we have

$$\sigma(\Delta_{uv}, c_0) \subseteq \{\alpha \in \mathbb{C} : |U - \alpha| \leq |V|\}.$$  

Part II: Conversely it is required to show

$$\{\alpha \in \mathbb{C} : |U - \alpha| \leq |V|\} \subseteq \sigma(\Delta_{uv}, c_0). \quad (3.6)$$

We first prove inclusion [3.6] under the assumption $\alpha \neq U$ and $\alpha \neq u_k$ for each $k \in \mathbb{N}$. Let $\alpha \in \mathbb{C}$ with $|U - \alpha| \leq |V|$. Clearly, $(\Delta_{uv} - \alpha I)$ is a triangle and hence $(\Delta_{uv} - \alpha I)^{-1}$ exists. So condition (R1) is satisfied but condition (R2) fails as can be seen below:

Suppose $\alpha \in \mathbb{C}$ with $|U - \alpha| < |V|$. Then $\beta > 1$. This means that $|v_{n-1}| > 1$ for large $n$ and consequently, $\lim_{n \to \infty} |b_{n0}| \neq 0$. Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha \in \mathbb{C} \text{ with } |U - \alpha| < |V|. \quad (3.7)$$

Next, we consider $\alpha \in \mathbb{C}$ with $|U - \alpha| = |V|$. Proof is by contradiction. Equality [3.2] can be write as

$$S_n = \frac{v_{n-1}}{|u_n - \alpha|} S_{n-1} + \frac{1}{|u_n - \alpha|}.$$  

Taking limit both sides of equality [3.8] and using condition $|U - \alpha| = |V|$, we get

$$\lim_{n \to \infty} \frac{1}{V} = 0,$$

which is not possible. Thus, $\lim_{n \to \infty} S_n$ does not exist and consequently, $\sup_n S_n$ is unbounded. Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0) \text{ for } \alpha \in \mathbb{C} \text{ with } |U - \alpha| = |V|. \quad (3.9)$$
Finally, we prove the inclusion (3.6) under the assumption $\alpha = U$ and $\alpha = u_k$ for all $k \in \mathbb{N}_0$. For this, we consider

$$(\Delta_{uv} - \alpha I) x = \begin{pmatrix} (u_0 - \alpha)x_0 \\ v_0x_0 + (u_1 - \alpha)x_1 \\ \vdots \\ -v_{k-1}x_{k-1} + (u_k - \alpha)x_k \\ \vdots \end{pmatrix}.$$ \hspace{1cm} (3.13)

Case (i): If $(u_k)$ is a constant sequence, say $u_k = U$ for all $k \in \mathbb{N}_0$, then for $\alpha = U$

$$(\Delta_{uv} - UI)x = 0 \Rightarrow x_0 = 0, x_1 = 0, x_2 = 0, \ldots$$

This shows that the operator $(\Delta_{uv} - UI)$ is one to one, but $R(\Delta_{uv} - UI)$ is not dense in $c_0$. So condition (R3) fails. Hence $U \in \sigma(\Delta_{uv}, c_0)$.

Case (ii): If $(u_k)$ is a sequence of distinct real numbers, then the series $S_k$ is divergent for each $\alpha = u_k$ from equality (3.2) and consequently, $\sup_n S_n$ is unbounded.

Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0)$$ for $\alpha = u_k$. \hspace{1cm} (3.10)

So condition (R2) fails. Hence $u_k \in \sigma(\Delta_{uv}, c_0)$ for all $k \in \mathbb{N}_0$.

Again, taking limit both sides of equality (3.9), we see that $\lim_{n \to \infty} S_n$ does not exist for $\alpha = U$. So $\sup_n S_n$ is unbounded. Hence

$$(\Delta_{uv} - \alpha I)^{-1} \notin B(c_0)$$ for $\alpha = U$. \hspace{1cm} (3.11)

So condition (R2) fails. Hence $U \in \sigma(\Delta_{uv}, c_0)$. Thus, in this case also $u_k \in \sigma(\Delta_{uv}, c_0)$ for all $k \in \mathbb{N}_0$ and $U \in \sigma(\Delta_{uv}, c_0)$. Hence we have

$${\alpha \in \mathbb{C} : |U - \alpha| < |V|} \subseteq \sigma(\Delta_{uv}, c_0).$$ \hspace{1cm} (3.12)

From inclusions (3.5) and (3.12), we get

$$\sigma(\Delta_{uv}, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}.$$ This completes the proof. \hspace{1cm} $\square$

**Theorem 3.3.** Point spectrum of the operator $\Delta_{uv}$ on the sequence space $c_0$ is

$$\sigma_p(\Delta_{uv}, c_0) = \emptyset.$$ 

**Proof.** For the point spectrum of the operator $\Delta_{uv}$, we find those $\alpha$ in $\mathbb{C}$ such that the matrix equation $\Delta_{uv}x = \alpha x$ is satisfy for non-zero vector $x = (x_k)$ in $c_0$.

Consider $\Delta_{uv}x = \alpha x$ for $x \neq 0 = (0, 0, \cdots)$ in $c_0$, which gives system of equations

$$\begin{align*}
u_0x_0 &= \alpha x_0 \\
v_0x_0 + u_1x_1 &= \alpha x_1 \\
\vdots \\
v_{k-1}x_{k-1} + u_kx_k &= \alpha x_k \\
\vdots \end{align*}$$ \hspace{1cm} (3.13)
The proof of this Theorem is divided into two cases.

Case (i): Suppose \((u_k)\) is a constant sequence, say \(u_k = U\) for all \(k \in \mathbb{N}_0\). Let \(x_i\) be the first nonzero entry of the sequence \(x = (x_n)\). Then equation \(v_{i-1}x_{i-1} + Ux_t = \alpha x_t\) gives \(\alpha = U\), and from the equation \(v_i x_t + U x_{i+1} = \alpha x_{i+1}\), we get \(x_i = 0\), which is a contradiction to our assumption. Hence \(\sigma_p(\Delta_{uv}, c_0) = \emptyset\).

Case (ii): Suppose \((u_k)\) is a sequence of distinct real numbers. Clearly, \(x_k = (v_k - 1)\alpha - u_k \) for all \(k \geq 1\).

If \(\alpha = u_0\), then \(\lim_{k \to \infty} \frac{|x_k|}{|x_{k-1}|} > 1\) because \(|U - u_0| < |V|\).

So \(x \notin l_1\) and hence \(x \notin c_0\) for \(x_0 \neq 0\).

Similarly, if \(\alpha = u_k\) for all \(k \geq 1\), then \(x_{k-1} = 0\), \(x_{k-2} = 0\), \(\cdots\), \(x_0 = 0\) and

\[ x_{n+1} = \left(\frac{v_n}{u_k - u_{n+1}}\right) x_n \text{ for all } n \geq k, \]

This implies \(\lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|} > 1\) because \(|U - u_k| < |V|\) for all \(k \geq 1\).

So \(x \notin l_1\) and hence \(x \notin c_0\) for \(x_0 \neq 0\). If \(x_0 = 0\), then \(x_k = 0\) for all \(k \geq 1\). Only possibility is \(x = 0 = (0, 0, \cdots)\). Hence \(\sigma_p(\Delta_{uv}, c_0) = \emptyset\). \(\square\)

### 3.2 Residual and Continuous Spectrum of the Operator \(\Delta_{uv}\) on the Sequence Space \(c_0\)

Let \(T : X \to X\) be a bounded linear operator having matrix representation \(A\) and the dual space of \(X\) denoted by \(X^*\). Again, let \(T^*\) be its adjoint operator on \(X^*\). Then the matrix representation of \(T^*\) is the transpose of the matrix \(A\).

**Theorem 3.4.** Point spectrum of the adjoint operator \(\Delta_{uv}^*\) on \(c_0^*\) is

\[ \sigma_p(\Delta_{uv}^*, c_0^*) = \{ \alpha \in \mathbb{C} : |U - \alpha| < |V| \} . \]

**Proof.** For the point spectrum of the operator \(\Delta_{uv}^*\), we find those \(\alpha\) in \(\mathbb{C}\) such that the matrix equation \(\Delta_{uv}^* f = \alpha f\) is satisfy for non-zero vector \(f = (f_k)\) in \(c_0^* \cong l_1\). Consider \(\Delta_{uv}^* f = \alpha f\), which gives system of equations

\[
\begin{align*}
    u_0 f_0 + v_0 f_1 & = \alpha f_0 \\
    u_1 f_1 + v_1 f_2 & = \alpha f_1 \\
    & \vdots \\
    u_{k-1} f_{k-1} + v_{k-1} f_k & = \alpha f_{k-1} \\
    & \vdots
\end{align*}
\]

This gives

\[ |f_k| = \frac{|\alpha - u_{k-1}|}{|v_{k-1}|} |f_{k-1}| \text{ for all } k \geq 1. \] (3.14)
Now, we take those $\alpha \in \mathbb{C}$ which satisfy the condition $|U - \alpha| < |V|$.

From equality (3.14), $\lim_{k \to \infty} \frac{|f_k|}{|f_{k-1}|} < 1$. So, series $\sum_{k=0}^{\infty} |f_k|$ converges and hence $f \in l_1$. Thus, $\alpha \in \mathbb{C}$ satisfying the condition $|U - \alpha| < |V|$ implies $f \in l_1$.

Conversely, we show that

$$\sum_{k=0}^{\infty} |f_k| < \infty$$ implies $\alpha \in \mathbb{C}$ satisfy the condition $|U - \alpha| < |V|$ or equivalently for $\alpha \in \mathbb{C}$ satisfy the condition $|U - \alpha| \geq |V|$ implies $\sum_{k=0}^{\infty} |f_k|$ diverges. We first consider $\alpha \in \mathbb{C}$ which satisfy the condition $|U - \alpha| > |V|$. From equality (3.14), $\lim_{k \to \infty} \frac{|f_k|}{|f_{k-1}|} > 1$. So, series $\sum_{k=0}^{\infty} |f_k|$ diverges.

Next, we consider $\alpha \in \mathbb{C}$ such that $|U - \alpha| = |V|$, i.e., $\lim_{k \to \infty} \left| \frac{u_k - \alpha}{v_k} \right| = 1$. So for each $\epsilon > 0$, there exists a positive integer $k_0$ such that

$$1 - \epsilon < \left| \frac{u_k - \alpha}{v_k} \right| < 1 + \epsilon \quad \text{for all } k \geq k_0. \quad (3.15)$$

Take

$$m = \min \left\{ \left| \frac{u_0 - \alpha}{v_0} \right|, \left| \frac{u_1 - \alpha}{v_1} \right|, \ldots, \left| \frac{u_{k_0-1} - \alpha}{v_{k_0-1}} \right| \right\}. \quad (3.16)$$

Using equality (3.14), the series $\sum_{k=0}^{\infty} |f_k|$ can be write as

$$\sum_{k=0}^{\infty} |f_k| = |f_0| + \left| \frac{u_0 - \alpha}{v_0} \right| |f_0| + \cdots + \left| \frac{u_{k_0-1} - \alpha}{v_{k_0-1}} \right| |f_0| + \left| \frac{u_0 - \alpha}{v_0} \right| \cdots \left| \frac{u_{k_0-1} - \alpha}{v_{k_{0-1}}} \right| |f_0| + \cdots > |f_0| + m |f_0| + \cdots + m^k |f_0| + m^k (1 - \epsilon) |f_0| + m^k (1 - \epsilon)^2 |f_0| + \cdots,$$

(3.15) and (3.16)

$$= (1 + m + \cdots + m^{k-1}) |f_0| + \frac{m^k |f_0|}{\epsilon} \to \infty \text{ as } \epsilon \to 0.$$ 

So, in this case also series $\sum_{k=0}^{\infty} |f_k|$ diverges. Thus, $f \in l_1$ implies $\alpha \in \mathbb{C}$ satisfying the condition $|U - \alpha| < |V|$.

This means that $f \in c_0^*$ if and only if $f_0 \neq 0$ and $\alpha \in \mathbb{C}$ such that $|U - \alpha| < |V|$. Hence
\[ \sigma_p(\Delta_{uv}, c_0^*) = \{ \alpha \in \mathbb{C} : |U - \alpha| < |V| \}. \]

**Theorem 3.5.** Residual spectrum of the operator \( \Delta_{uv} \) on the sequence space \( c_0 \) is
\[ \sigma_r(\Delta_{uv}, c_0) = \{ \alpha \in \mathbb{C} : |U - \alpha| < |V| \}. \]

**Proof.** The proof of this theorem is divided into two cases.

Case(i): Suppose \((u_k)\) is a constant sequence, say \(u_k = U\) for all \(k \in \mathbb{N}_0\). For \(\alpha \in \mathbb{C}\) with \(|U - \alpha| < |V|\), the operator \((\Delta_{uv} - \alpha I)\) is a triangle except \(\alpha = U\) and consequently, the operator \((\Delta_{uv} - \alpha I)\) has an inverse. Further by Theorem 3.3, the operator \((\Delta_{uv} - \alpha I)\) is one to one for \(\alpha = U\) and hence has an inverse.

But by Theorem 3.4, the operator \((\Delta_{uv} - \alpha I)^\ast\) is not one to one for \(\alpha \in \mathbb{C}\) with \(|U - \alpha| < |V|\). Hence by Lemma 2.5, the range of the operator \((\Delta_{uv} - \alpha I)\) is not dense in \(c_0\). Thus, \(\sigma_r(\Delta_{uv}, c_0) = \{ \alpha \in \mathbb{C} : |U - \alpha| < |V| \}\).

Case(ii): Suppose \((u_k)\) is a sequence of distinct real numbers. For \(\alpha \in \mathbb{C}\) such that \(|U - \alpha| < |V|\), the operator \((\Delta_{uv} - \alpha I)\) is a triangle except \(\alpha = u_k\) for all \(k \in \mathbb{N}_0\) and consequently, the operator \((\Delta_{uv} - \alpha I)\) has an inverse. Further by Theorem 3.3, the operator \((\Delta_{uv} - u_k I)\) is one to one and hence \((\Delta_{uv} - u_k I)^{-1}\) exists for all \(k \in \mathbb{N}_0\).

On the basis of argument as given in Case(i), it is easy to verify that the range of the operator \((\Delta_{uv} - \alpha I)\) is not dense in \(c_0\). Thus, \(\sigma_r(\Delta_{uv}, c_0) = \{ \alpha \in \mathbb{C} : |U - \alpha| < |V| \}\).

**Theorem 3.6.** Continuous spectrum of the operator \(\Delta_{uv}\) on the sequence space \(c_0\) is
\[ \sigma_c(\Delta_{uv}, c_0) = \{ \alpha \in \mathbb{C} : |U - \alpha| = |V| \}. \]

**Proof.** The proof of this theorem is divided into two cases.

Case(i): Suppose \((u_k)\) is a constant sequence. For \(\alpha \in \mathbb{C}\) with \(|U - \alpha| = |V|\), the operator \((\Delta_{uv} - \alpha I)\) is a triangle because \(\alpha \neq U\) and has an inverse. The operator \((\Delta_{uv} - \alpha I)^{-1}\) is discontinuous by condition (3.9). Therefore, the operator \((\Delta_{uv} - \alpha I)\) has an unbounded inverse.

As the operator \((\Delta_{uv} - \alpha I)^\ast\) is one to one for \(\alpha \in \mathbb{C}\) satisfying \(|U - \alpha| = |V|\) follows from Theorem 3.3. So, the range of the operator \((\Delta_{uv} - \alpha I)\) is dense in \(c_0\) by Lemma 2.5. Hence
\[ \sigma_c(\Delta_{uv}, c_0) = \{ \alpha \in \mathbb{C} : |U - \alpha| = |V| \}. \]

Case(ii): Suppose \((u_k)\) is a sequence of distinct real numbers. For \(\alpha \in \mathbb{C}\) with \(|U - \alpha| = |V|\), the operator \((\Delta_{uv} - \alpha I)\) is a triangle because \(\alpha \neq u_k\) for each \(k \in \mathbb{N}\) and consequently, the operator \((\Delta_{uv} - \alpha I)\) has an inverse. The operator \((\Delta_{uv} - \alpha I)^{-1}\) is discontinuous by condition (3.9). Therefore, \((\Delta_{uv} - \alpha I)\) has an unbounded inverse.
On the basis of argument as given in Case (i), it is easy to verify that the range of the operator \((\Delta uv - \alpha I)\) is dense in \(c_0\). Hence
\[
\sigma_c(\Delta uv, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| = |V|\}. \quad \square
\]

### 3.3 Fine Spectrum of the Operator \(\Delta uv\) on the Sequence Space \(c_0\)

**Theorem 3.7.** If \(\alpha\) satisfies \(|U - \alpha| > |V|\), then \((\Delta uv - \alpha I)\) ∈ \(A_1\).

**Proof.** It is required to show that the operator \((\Delta uv - \alpha I)\) is bijective and has a continuous inverse for \(\alpha \in \mathbb{C}\) with \(|U - \alpha| > |V|\). Since \(\alpha \neq U\) and \(\alpha \neq u_k\) for each \(k \in \mathbb{N}_0\), therefore the operator \((\Delta uv - \alpha I)\) is a triangle. Hence it has an inverse. The operator \((\Delta uv - \alpha I)^{-1}\) is continuous for \(\alpha \in \mathbb{C}\) with \(|U - \alpha| > |V|\) by statement (3.4). Also the equation
\[
(\Delta uv - \alpha I)x = y \quad \text{gives} \quad x = (\Delta uv - \alpha I)^{-1}y, \quad \text{i.e.,}
\]
\[
x_n = ((\Delta uv - \alpha I)^{-1}y)_n, \quad n \in \mathbb{N}_0.
\]
Thus, for every \(y \in c_0\), we can find \(x \in c_0\) such that
\[
(\Delta uv - \alpha I)x = y, \quad \text{since} \quad (\Delta uv - \alpha I)^{-1} \in B(c_0).
\]
This shows that the operator \((\Delta uv - \alpha I)\) is onto and hence \((\Delta uv - \alpha I)\) ∈ \(A_1\). \quad \square

**Theorem 3.8.** Let \(u\) be constant sequence, say \(u_k = U\) for all \(k \in \mathbb{N}_0\). Then \(U \in C_1\sigma(\Delta uv, c_0)\).

**Proof.** We have \(\sigma_r(\Delta uv, c_0) = \{\alpha \in \mathbb{C} : |U - \alpha| < |V|\}\). Clearly, \(U \in \sigma_r(\Delta uv, c_0)\).

It is sufficient to show that the operator \((\Delta uv - UI)^{-1}\) is continuous. By Lemma 2.6, it is enough to show that \((\Delta uv - UI)^{\times}\) is onto, i.e., for given \(y = (y_n) \in c_0^{\times}\), we have to find \(x = (x_n) \in c_0^{\times}\) such that \((\Delta uv - UI)^{\times}x = y\). Now \((\Delta uv - UI)^{\times}x = y\), i.e.,
\[
\begin{align*}
v_0x_1 &= y_0 \\
v_1x_2 &= y_1 \\
\vdots \\
v_{i-1}x_i &= y_{i-1} \\
\vdots
\end{align*}
\]
Thus, \(v_nx_n = y_{n-1}\) for all \(n \geq 1\) which implies \(\sum_{n=0}^{\infty} |x_n| < \infty\), since \(y \in l_1\) and \(v = (v_k)\) is a convergent sequence. This shows that operator \((\Delta uv - UI)^{\times}\) is onto and hence \(U \in C_1\sigma(\Delta uv, c_0)\). \quad \square
Theorem 3.9. Let \( u \) be constant sequence, say \( u_k = U \) for all \( k \in \mathbb{N}_0 \) and \( \alpha \neq U \) but \( \alpha \in \sigma_r(\Delta_{uv},c_0) \). Then \( \alpha \in C_2\sigma(\Delta_{uv},c_0) \).

Proof. It is sufficient to show that the operator \( (\Delta_{uv} - \alpha I)^{-1} \) is discontinuous for \( \alpha \neq U \) and \( \alpha \in \sigma_r(\Delta_{uv},c_0) \). The operator \( (\Delta_{uv} - \alpha I)^{-1} \) is discontinuous by statement (3.7) for \( U \neq \alpha \in \mathbb{C} \) with \( |U - \alpha| < |V| \).

Theorem 3.10. Let \( u \) be a sequence of distinct real numbers and \( \alpha \in \sigma_r(\Delta_{uv},c_0) \). Then \( \alpha \in C_2\sigma(\Delta_{uv},c_0) \).

Proof. It is sufficient to show that the operator \( (\Delta_{uv} - \alpha I)^{-1} \) is discontinuous for \( \alpha \in \sigma_r(\Delta_{uv},c_0) \). The operator \( (\Delta_{uv} - \alpha I)^{-1} \) is discontinuous by statements (3.7), (3.10) and (3.11) for \( \alpha \in \mathbb{C} \) with \( |U - \alpha| < |V| \).

Theorem 3.11. Let \( u \) and \( v \) be constant sequences and \( \alpha \in \sigma_c(\Delta_{uv},c_0) \). Then \( \alpha \in B_2\sigma(\Delta_{uv},c_0) \).

Proof. It is sufficient to show that the operator \( (\Delta_{uv} - \alpha I) \) is not onto, i.e., there is no sequence \( x = (x_n) \) in \( c_0 \) such that \( (\Delta_{uv} - \alpha I)x = y \) for some \( y \in c_0 \). Clearly, \( y = (1,0,0,\cdots) \in c_0 \). We have

\[
(\Delta_{uv} - \alpha I)x = y \implies x_n = (-1)^n \frac{V^n}{(U - \alpha)^{n+1}} \text{ for each } n \geq 0.
\]

Therefore, \( |x_n| = \left| \frac{1}{V} \right| \) for each \( n \geq 0 \) because \( |U - \alpha| = |V| \). Consequently, \( \lim_{n \to \infty} |x_n| = \left| \frac{1}{V} \right| > 0 \). This shows that \( x \notin c_0 \) and hence the operator \( (\Delta_{uv} - \alpha I) \) is not onto.

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References


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