Normal Ideals in Generalized Almost Distributive Lattices

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Abstract: In this paper, we introduced the concept of an annihilator preserving homomorphism from a GADL $L$ into a GADL $L'$ and studied some basic properties of these homomorphisms. We derived a sufficient condition for a homomorphism to be annihilator preserving homomorphism. We introduced the concept of a normal ideal in a GADL $L$ and proved that the set $\mathcal{N}(L)$ of all normal ideals of $L$ forms a Boolean algebra.

Keywords: generalized almost distributive lattice (GADL); annihilator preserving homomorphism; disjunctive GADL; normal ideal.

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1 Introduction

The concept of a Generalized Almost Distributive Lattice (GADL) was introduced by Rao et al. [1] as a generalization of an Almost Distributive Lattice (ADL) [2]. The class of GADLs inherit almost all the properties of a distributive lattice except possibly the commutativity of $\land, \lor$, the right distributivity of either of the operations $\lor$ or $\land$ over the other. The class of GADLs include the class of

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ADLs properly and retain many important properties of ADLs. In section 3, we introduce the concept of an annihilator preserving homomorphism from a GADL $L$ into a GADL $L'$ and study some basic properties of these homomorphisms. We derive a sufficient condition for a homomorphism to be annihilator preserving homomorphism. In section 4, we define the notion of a dense element in a GADL and the concept of a disjunctive GADL and we prove that, in a GADL $L$, every left identity element is a dense element and the converse holds when $L$ is disjunctive. We introduce the concept of a normal ideal in a GADL $L$ and prove that the set $\mathcal{N}(L)$ of all normal ideals of $L$ forms a Boolean algebra.

2 Preliminaries

First, we recall certain definitions and properties of GADLs from [1–3] that are required in the paper.

Definition 2.1 ([2]). An Almost Distributive Lattice (ADL) is an algebra $(L, \lor, \land)$ of type $(2, 2)$ satisfying

1) $(x \lor y) \land z = (x \land z) \lor (y \land z)$;
2) $x \land (y \lor z) = (x \land y) \lor (x \land z)$;
3) $(x \lor y) \land y = y$;
4) $(x \lor y) \land x = x$;
5) $x \lor (x \land y) = x$.

If there is an element $0 \in L$ such that $0 \land a = 0$ for all $a \in L$, then $(L, \lor, \land, 0)$ is called an ADL with 0.

Definition 2.2 ([2]). Let $X$ be a non-empty set. Fix some element $x_0 \in X$. Then, for any $x, y \in X$ define $\lor$ and $\land$ on $X$ by,

$$x \lor y = \begin{cases} x, & \text{if } x \neq x_0 \\ y, & \text{if } x = x_0 \end{cases}, \quad x \land y = \begin{cases} y, & \text{if } x \neq x_0 \\ x_0, & \text{if } x = x_0 \end{cases}.$$

Then $(X, \lor, \land, x_0)$ is an ADL, with $x_0$ as its zero element. This ADL is called a discrete ADL.

Definition 2.3 ([1]). An algebra $(L, \lor, \land)$ of type $(2, 2)$ is called a Generalized Almost Distributive Lattice if it satisfies the following axioms:

1) $\text{(As}\land) \quad (x \land y) \land z = x \land (y \land z)$;
2) $\text{(LD}\land) \quad x \land (y \lor z) = (x \land y) \lor (x \land z)$;
3) $\text{(LD}\lor) \quad x \lor (y \land z) = (x \lor y) \land (x \lor z)$;
4) $\text{(A}_1) \quad x \land (x \lor y) = x$;
(A2) \((x \lor y) \land x = x\);
(A3) \((x \land y) \lor y = y\).

**Example 2.4.** Let \(L = \{a, b, c\}\). Define two binary operations \(\lor\) and \(\land\) on \(L\) as follows:

\[
\begin{array}{ccc}
\lor & a & b & c \\
a & a & b & a \\
b & b & b & b \\
c & c & c & c \\
\end{array}
\quad
\begin{array}{ccc}
\land & a & b & c \\
a & a & a & c \\
b & b & a & c \\
c & c & a & c \\
\end{array}
\]

Hence the algebra \((L, \lor, \land)\) is a Generalized Almost Distributive Lattice.

For brevity, we will refer to this Generalized Almost Distributive Lattice as GADL. The GADL \((L, \lor, \land)\) in Example 2.4 is not an ADL for \((c \lor b) \land b \neq b\). Let \((L, \lor, \land)\) be a GADL. For any \(a, b \in L\) define \(a \leq b\) if and only if \(a \land b = a\) or, equivalently, \(a \lor b = b\). Then \(\leq\) is a partial ordering on \(L\). In this section, \(L\) stands for a GADL unless otherwise mentioned.

**Lemma 2.5 ([1]).** Let \(L\) be a GADL with 0. For any \(a, b \in L\), the followings hold:

\begin{enumerate}
\item \(a \lor a = a\);
\item \(a \land a = a\);
\item \(a \lor (a \land b) = a\);
\item \(a \lor (b \land a) = a\);
\item \(a \land b = b \Rightarrow a \lor b = a\);
\item \(a \lor b = b \Leftrightarrow a \land b = a\);
\item \(a \lor (a \lor b) = a \lor b\);
\item \(b \land (a \lor b) = a \land b\);
\item \(a \land (b \land a) = b \land a\);
\item \(a \leq c, b \leq c\) if and only if \(a \land b = b \land a\) and \(a \lor b = b \lor a\);
\item \(a \land b \land c = b \land a \land c\);
\item \(a \land b = 0 \Leftrightarrow b \land a = 0\);
\item \(a \lor 0 = a \lor 0 = a \land 0 = 0\);
\item For any \(m \in L\), \(m\) is maximal with respect partial ordering \(\leq\) if and only if \(m \lor x = m\) for all \(x \in L\).
\end{enumerate}

**Definition 2.6 ([1]).** Let \(L\) be a GADL. An element \(e \in L\) is said to be left identity element in \(L\) if \(e \land x = x\) for all \(x \in L\).
Note that every left identity element is maximal element but converse need not to be true. In Example 2.4, we observe that c is maximal but not a left identity element.

**Definition 2.7 ([3]).** A non-empty subset $I$ of $L$ is said to be an ideal of $L$ if (i) $a, b \in I$ implies $a \lor b \in I$ and (ii) $a \in I, x \in L$ implies $a \land x \in I$.

**Theorem 2.8 ([3]).** Let $(L, \lor, \land, 0)$ be a GADL with 0 and $I$ an ideal of $L$ then for any $a, b \in L$, the followings hold:

(i) $a \land b \in I \iff b \land a \in I$;

(ii) The set $S = \{a \land x \mid x \in L\}$ is the smallest ideal of $L$ containing $a$. We denote by $S = \langle a \rangle$;

(iii) $(a) \cap (b) = (a \land b)$ and $a \in (b) \iff b \land a = a$.

### 3 Annihilator Preserving Homomorphisms

In this section, we introduce the concept of an annihilator preserving homomorphism from a GADL $L$ into a GADL $L'$ and study some basic properties of these homomorphisms. We derive a sufficient condition for a homomorphism to be annihilator preserving homomorphism. In the following we give the definition of a homomorphism between two GADLs with zero and the Kernel of a homomorphism in a natural way.

**Definition 3.1.** Let $L$ and $L'$ be two GADLs with zeros. Then a mapping $f : L \to L'$ is called a homomorphism if it satisfies the following:

1. $f(a \lor b) = f(a) \lor f(b)$;
2. $f(a \land b) = f(a) \land f(b)$;
3. $f(0) = 0'$ (where $0'$ is the zero element of $L'$);

and the set $Ker f = \{x \in L \mid f(x) = 0'\}$ is called the Kernel of the homomorphism $f$.

We first prove the following lemma which is useful in the forthcoming results.

**Lemma 3.2.** Let $L$ and $L'$ be two GADLs with 0 and $0'$, respectively, and $f : L \to L'$ a homomorphism. Then we have the followings:

1. For any ideal $J$ of $L'$, $f^{-1}(J)$ is an ideal of $L$ containing $Ker f$.
2. If $f$ is onto, then for any ideal $I$ of $L$, $f(I)$ is an ideal of $L'$. 
Proof. (1) Let $J$ be an ideal of $L$. Since $f(0) = 0' \in J$, we obtain $0 \in f^{-1}(J)$. Let $a, b \in f^{-1}(J)$ where $a, b \in L$. Then $f(a), f(b) \in J$. Since $J$ is an ideal in $L$, we obtain that $f(a \lor b) = f(a) \lor f(b) \in J$. Therefore $a \lor b \in f^{-1}(J)$. Again, let $x \in f^{-1}(J)$ and $r \in L$. Then $f(x) \in J$. Now $f(x \land r) = f(x) \land f(r) \in J$. Hence $x \land r \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is an ideal of $L$. So that $0' \in J$. Therefore $f = f^{-1}(\{0'\}) \subseteq f^{-1}(J)$.

(2) Since $f$ is a homomorphism and $0 \in I$, we obtain that $0' = f(0) = f(I)$. Let $f(a), f(b) \in f(I)$, where $a, b \in I$. Since $I$ is an ideal, $a \lor b \in I$ and hence $f(a \lor b) \in f(I)$. Therefore $f(a) \lor f(b) \in f(I)$. Again, let $f(a) \in f(I)$ and $r \in L'$, where $a \in I$. Since $f$ is onto, there exists $s \in L$ such that $f(s) = r$. Now $f(a) \land r = f(a) \land f(s) = f(a \land s) \in f(I)$. Therefore $f(I)$ is an ideal in $L'$. □

Definition 3.3. For any non-empty subset $A$ of a GADL $L$ with 0, define

$$A^* = \{x \in L \mid x \land a = 0, \text{ for all } a \in A\}.$$ 

This $A^*$ is an ideal of $L$ and is called the annihilator ideal of $A$. For any $a \in L$, we write $[a]^*$ for $\{a\}^*$ and is called annulet of $L$.

It can be easily observed that, for any subset $A$ of $L$, $A \cap A^* = \{0\}$. In the following we prove some properties of annihilator ideals.

Lemma 3.4. For any ideals $I, J$ of a GADL $L$ with 0, we have the following:

1. $I^* = \bigcap_{a \in I} [a]^*$;
2. If $I \subseteq J$ then $J^* \subseteq I^*$;
3. $I \subseteq I^{**}$;
4. $I^{***} = I^*$;
5. $I \cap J = \{0\} \iff I \subseteq J^*$.

Proof. (1) Clearly $I^* \subseteq \bigcap_{a \in I} [a]^*$. Let $x \in [a]^*$ for all $a \in I$. Let $b \in I$. Then $x \in [b]^*$ and hence $x \land b = 0$. Therefore $x \in I^*$.

(2) Let $I \subseteq J$. Let $a \in J^*$ and $b \in I$. Then $b \in J$ and $a \land b = 0$. Hence $a \in I^*$. Therefore $J^* \subseteq I^*$.

(3) Let $x \in I$ and $a \in I^*$. Then $x \land a \in I$ and $x \land a \in I^*$. So that $x \land a \in I \cap I^* = \{0\}$. Hence $x \land a = 0$ for all $a \in I^*$. We get $x \in I^{**}$. Therefore $I \subseteq I^{**}$.

(4) Since $I \subseteq I^{**}$, we have, by (2), $I^{***} \subseteq I^*$. Now, let $x \in I^*$ and $a \in I^{**}$. Then $x \land a \in I^* \cap I^{**} = \{0\}$. Hence $x \land a = 0$ for all $a \in I^{**}$. Therefore $x \in I^{***}$, we get $I^* \subseteq I^{***}$. Thus $I^{***} = I^*$.

(5) Suppose $I \cap J = \{0\}$. Let $x \in I$ and $a \in J$. Then $x \land a \in I$ and $x \land a \in J$ and hence $x \land a \in I \cap J = \{0\}$. Therefore $x \land a = 0$, we get $x \in J^*$. Thus $I \subseteq J^*$. Conversely, assume that $I \subseteq J^*$. Let $x \in I \cap J$. Then $x \in I$ implies that $x \in J^*$ and hence $x = x \land x = 0$. Therefore $I \cap J = \{0\}$. □

The following lemma can be verified easily.
Lemma 3.5. Let \( L \) be a GADL with 0. For any \( x, y \in L \), we have the following:

\[
\begin{align*}
(1) \ [x \wedge y]^* &= [y \wedge x]^*; \\
(2) \ x \leq y &\Rightarrow [y]^* \subseteq [x]^*; \\
(3) \ [x \wedge y]^{**} &= [x]^{**} \cap [y]^{**}; \\
(4) \ [x]^{***} &= [x]^*; \\
(5) \ [x \lor y]^* &= [x]^* \cap [y]^* = [y]^* \cap [x]^* = [y \lor x]^*.
\end{align*}
\]

Now we prove the following.

Lemma 3.6. Let \( L \) and \( L' \) be two GADLs with 0, 0', respectively. If \( f : L \rightarrow L' \) is a homomorphism, then for any non-empty subset \( A \) of \( L \), we have

\[
f(A)^* \subseteq \{f(A)\}^*.
\]

Proof. Let \( x \in f(A)^* \) and \( y \in f(A) \). Then there exists \( a \in A^* \) and \( b \in A \) such that \( x = f(a) \) and \( y = f(b) \). Now \( x \wedge y = f(a) \wedge f(b) = f(a \wedge b) = f(0) \) (\( \because a \in A^* \) and \( b \in A \)) = 0'. That is \( x \wedge y = 0' \) for all \( y \in f(A) \). Hence \( x \in \{f(A)\}^* \). Therefore \( f(A)^* \subseteq \{f(A)\}^* \).

If \( L \) is a GADL with 0, then for any \( A \subseteq L \), \( \{f(A)\}^* = f(A)^* \) is not true in general. Consider the following example.

Example 3.7. Let \( L = \{0, a, b, c\} \) be a discrete ADL. Define a mapping \( f : L \rightarrow L \) by \( f(x) = 0 \) for all \( x \in L \). Then clearly \( f \) is a homomorphism on \( L \). Now take \( A = \{a, b\} \). Then clearly \( A^* = \{0\} \) and \( f(A) = \{0\} \). Hence \( f(A)^* = \{0\} \) and \( \{f(A)\}^* = L \). Therefore \( \{f(A)\}^* \neq f(A)^* \).

This motivates us to introduce the concept of annihilator preserving homomorphism in the following.

Definition 3.8. Let \( L \) and \( L' \) be two GADLs with 0 and 0', respectively. Then a homomorphism \( f : L \rightarrow L' \) is called annihilator preserving if

\[
f(A)^* = \{f(A)\}^*
\]

for any set \( A \) such that \( (0) \subseteq A \subseteq L \).

Example 3.9. Let \( A = \{0, a\} \) and \( B = \{0, b_1, b_2\} \) be two discrete ADLs. Write \( L = A \times B = \{(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\} \). Then \( (L, \lor, \wedge, 0) \) is an GADL with \( 0 = (0, 0) \), under point-wise operations. Let \( L' = \{0', a', b', c'\} \) be another GADL in which the operations \( \lor', \wedge' \) are defined as follows:

\[
\begin{array}{c|cccc}
\lor' & 0' & a' & b' & c' \\
\hline
0' & 0' & a' & b' & c' \\
a' & a' & a' & c' & c' \\
b' & b' & c' & b' & c' \\
c' & c' & c' & c' & c'
\end{array}
\quad
\begin{array}{c|cccc}
\wedge' & 0' & a' & b' & c' \\
\hline
0' & 0' & 0' & 0' & 0' \\
a' & 0' & a' & 0' & a' \\
b' & 0' & 0' & b' & b' \\
c' & 0' & a' & b' & c'
\end{array}
\]
Now, define the mapping \( f : L \rightarrow L' \) as follows:

\[
\begin{align*}
    f((0,0)) &= 0' : f((a,0)) = a' \\
    f((0,b_1)) &= f((0,b_2)) = b' : f((a,b_1)) = f((a,b_2)) = c'.
\end{align*}
\]

It can be easily verified that \( f \) is a homomorphism from \( L \) onto \( L' \).

(i) For \( A = \{(0,0)\} \), clearly we get \( f(A^*) = L' = \{f(A)\}^* \).

(ii) For \( A = \{(a,0)\} \), we get \( A^* = \{(0,0),(0,b_1),(0,b_2)\} \) and \( f(A) = \{a'\} \).

Hence \( f(A^*) = \{0',b'\} = \{f(A)\}^* \).

(iii) For \( A = \{(0,b_i), i = 1,2\} \), we get \( A^* = \{(0,0),(a,0)\} \) and \( f(A) = \{b'\} \).

Hence \( f(A^*) = \{0',a'\} = \{f(A)\}^* \).

Similarly, for \( A = \{(0,0),(a,0)\} \) and \( A = \{(0,0),(0,b_i)\} \), we get \( f(A^*) = \{f(A)^\prime\} \). In the remaining cases, \( f(A^*) = \{0'\} = \{f(A)\}^* \). Therefore \( f \) is an annihilator preserving homomorphism.

If \( f \) is a homomorphism of a GADL \( L \) with 0 into another GADL \( L' \) with 0' such that \( Kef = \{0\} \) and \( f \) is onto, then \( f \) need not be an isomorphism. It may be seen in the following example.

**Example 3.10.** Let \( L = \{0,a,b\} \) and \( L' = \{0',c\} \) be two discrete ADLs. Define a mapping \( f : L \rightarrow L' \) by \( f(0) = 0' \) and \( f(a) = f(b) = c \). Then clearly \( f \) is a homomorphism from \( L \) into \( L' \) and also \( f \) is onto. Also \( Kef = \{0\} \). But \( f \) is not one-one. Hence \( f \) is not an isomorphism.

However, we have the following.

**Theorem 3.11.** Let \( L \) and \( L' \) be two GADLs with zeroes 0 and 0' respectively and \( f : L \rightarrow L' \) a homomorphism. If \( Kef = \{0\} \) and \( f \) is onto, then \( f \) is annihilator preserving.

**Proof.** Assume that \( f \) is onto and \( Kef = \{0\} \). Let \( A \) be a non-empty subset of \( L \). We have always \( f(A^*) \subseteq \{f(A)\}^* \). Let \( x \in \{f(A)\}^* \subseteq L' \). Since \( f \) is onto, there exists \( y \in L \) such that \( f(y) = x \). By knowing that \( f(y) \in \{f(A)\}^* \).

Then \( f(y) \wedge m = 0 \) for all \( m \in f(A) \). Let \( a \in A \). Then \( f(y) \wedge f(a) = 0' \). That is \( f(y \wedge a) = 0' \) which means \( y \wedge a \in Kef = \{0\} \). Then \( y \wedge a = 0 \). Hence \( y \in A^* \).

Therefore \( \{f(A)\}^* \subseteq f(A^*) \). Thus \( \{f(A)\}^* = f(A^*) \). Therefore \( f \) is annihilator preserving.

**Theorem 3.12.** Let \( L \) and \( L' \) be two GADLs with 0 and 0', respectively and \( f : L \rightarrow L' \) an epimorphism. If \( Kef = \{0\} \), then

\[
A^* = B^* \iff \{f(A)\}^* = \{f(B)\}^*
\]

for any two non-empty subsets \( A, B \) of \( L \).
Proof. Since \( f \) is an epimorphism and \( \text{Ker}\ f = \{0\} \), by Theorem 3.11, \( f \) is annihilator preserving. Let \( A, B \) be two non-empty subsets of \( L \). Assume that \( A^* = B^* \). Then clearly \( f(A^*) = f(B^*) \). Hence \( \{f(A)\}^* = \{f(B)\}^* \). Conversely, assume that \( \{f(A)\}^* = \{f(B)\}^* \). Let \( t \in A^* \) and \( b \in B \). Then \( t \land a = 0 \) for all \( a \in A \). Now,

\[
\begin{align*}
t \in A^* & \Rightarrow f(t) \in f(A^*) \\
& \Rightarrow f(t) \in \{f(A)\}^* \quad \text{(by Theorem 3.11)} \\
& \Rightarrow f(t) \in \{f(B)\}^* \quad \text{(since } f(A^*) = f(B^*)) \\
& \Rightarrow f(t) \land f(b) = 0' \quad \text{(since } f(b) \in f(B)) \\
& \Rightarrow f(t \land b) = 0' \\
& \Rightarrow t \land b \in \text{Ker}f = \{0\} \\
& \Rightarrow t \land b = 0 \\
& \Rightarrow t \in B^*.
\end{align*}
\]

Hence \( A^* \subseteq B^* \). Similarly, we can obtain that \( B^* \subseteq A^* \). Therefore \( A^* = B^* \).

4 Normal Ideals

In this section we introduce the concept of a dense element in a GADL, disjunctive GADL and normal ideal in a GADL and we prove that the set of all normal ideals of a GADL forms a Boolean algebra. First we begin with the following.

Definition 4.1. An element \( a \) of \( L \) is called dense element if \( [a]^* = \{0\} \).

Theorem 4.2. In a GADL, every left identity element is a dense element.

Proof. Let \( m \) be a maximal element in \( L \) and \( x \in [m]^* \). Then \( x \land m = 0 \). So that \( 0 = x \land m = x \). Hence \( x = 0 \). Therefore \( [m]^* = \{0\} \). Thus \( m \) is a dense element.

In the following example we show that a dense element needs not to be a left identity element.

Example 4.3. Let \( L = \{0, a, b, c\} \) and define \( \lor \) and \( \land \) on \( L \) as follows:

\[
\begin{array}{c|ccc}
\lor & 0 & a & b & c \\
\hline
0 & 0 & a & b & c \\
a & a & a & a & a \\
b & b & a & b & b \\
c & c & c & c & c
\end{array}
\quad
\begin{array}{c|ccc}
\land & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & b & b & c \\
c & 0 & b & b & c
\end{array}
\]

Then clearly \( (L, \lor, \land, 0) \) is a GADL with 0. Clearly \( a, b, c \) are dense elements but \( b \) and \( c \) are not left identity elements.

Now we define the notion of a disjunctive GADL.
Definition 4.4. A GADL $L$ with 0, is called disjunctive iff for all $a, b \in L$, $[a]^* = [b]^*$ implies $a = b$.

Example 4.5. Let $L = \{0, a, b, c\}$ be a set. Define $\lor$ and $\land$ on $L$ as follows:

\[
\begin{array}{cccc}
\lor & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & a & a & a \\
b & b & a & b & a \\
c & c & a & a & c \\
\end{array}
\quad
\begin{array}{cccc}
\land & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & b & b & 0 \\
c & 0 & c & 0 & c \\
\end{array}
\]

Then clearly $(L, \lor, \land, 0)$ is a GADL with 0. Now, $[a]^* = [0]$, $[b]^* = \{0, c\}$ and $[c]^* = \{0, b\}$. We can see that $x \neq y$ and $[x]^* \neq [y]^*$ for all $x, y \in L$. Hence $L$ is disjunctive.

In a GADL $L$ with 0, we know that a left identity element is always a dense element (Theorem 4.2). Now we prove the converse in disjunctive GADL.

Theorem 4.6. If $L$ is a disjunctive GADL, then every dense element of $L$ is a left identity element.

Proof. Assume that $L$ is disjunctive. Let $m$ be a dense element of $L$. That is $[m]^* = \{0\}$. For any $x \in L$, $[m \land x]^* = [m]^* \land [x]^* = L \land [x]^* = [x]^*$ and hence $[m \land x]^* = [x]^*$. So that $[m \land x]^* = [x]^*$. Since $L$ is disjunctive, we get that $m \land x = x$. Therefore $m$ is a left identity element of $L$. \qed

Now we define the concept of a normal ideal in a GADL.

Definition 4.7. Let $L$ be a GADL with 0. An ideal $I$ of $L$ is called a normal ideal if $I = I^*$, or equivalently, $I = S^* = \{y \in L \mid y \land s = 0, \text{ for all } s \in S\}$ for some non-empty subset $S$ of $L$. We denote the set of all normal ideals of $L$ by $\mathcal{N}(L)$.

Example 4.8. Let $L = \{0, a, b, c\}$ and define $\lor$ and $\land$ on $L$ as follows:

\[
\begin{array}{cccc}
\lor & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & a & a & a \\
b & b & a & b & a \\
c & c & a & a & c \\
\end{array}
\quad
\begin{array}{cccc}
\land & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c \\
b & 0 & b & b & 0 \\
c & 0 & c & 0 & c \\
\end{array}
\]

Then clearly $(L, \lor, \land, 0)$ is a GADL with 0. Consider the set $I = \{0, b\} \subseteq L$. Then clearly $I$ is an ideal in $L$. Now $I^* = \{0, c\}$ and also $I^{**} = \{0, b\} = I$. Thus $I$ is normal ideal in $L$. Similarly, the ideal $J = \{0, c\}$ of $L$, is another normal ideal in $L$.

Now, we prove the following

Theorem 4.9. Let $L$ and $L'$ be two GADLs with zeroes 0 and $0'$, respectively and $f : L \to L'$ a homomorphism. Then we have the following:
Lemma 4.12. If $f$ is annihilator preserving and onto, then $f(I)$ is a normal ideal of $L'$ for every normal ideal $I$ of $L$.

(2) If in addition $\text{Ker} f = \{0\}$, then $f^{-1}(J)$ is a normal ideal of $L$, for every normal ideal $J$ of $L'$.

Proof. (1) Let $I$ be a normal ideal of $L$. Then by Lemma 3.2(2), $f(I)$ is an ideal of $L'$. Since $f$ is annihilator preserving, $\{f(I)\}^{**} = f(I^{**}) = f(I)$. Therefore $f(I)$ is a normal ideal in $L'$.

(2) Let $J$ be a normal ideal of $L'$. Then by Lemma 3.2(1), $f^{-1}(J)$ is an ideal of $L$. Let $x \in f^{-1}(\{f^{-1}(J)\}^{**})$ and $y \in J$. Then $y = f(t)$ for some $t \in L$. Let $s \in f^{-1}(J)$. Then $f(s) \in J$ and hence $y \wedge f(s) = 0$. Therefore $t \wedge s = 0$. Hence $t \in f^{-1}(J)$. Therefore $x \wedge t = 0$ and hence $f(x) \wedge f(t) = 0'$. Thus $f(x) \in J^{**} = J$. Hence $x \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is a normal ideal in $L$.

Corollary 4.10. Let $L$ and $L'$ be two GADLs with zeros $0$ and $0'$, respectively, and $f : L \to L'$ is annihilator preserving homomorphism. Assume that $\text{Ker} f = \{0\}$ and $f$ is onto. Then $J$ is normal ideal of $L'$ if and only if $f^{-1}(J)$ is a normal ideal of $L$.

Theorem 4.11. Let $L$ be a GADL with $0$. Then the set $\mathcal{N}(L)$ of all normal ideals of $L$ forms a Boolean algebra.

Proof. For $I, J \in \mathcal{N}(L)$, define $I \wedge J = I \cap J$ and $I \vee J = (I^{*} \cap J^{*})^{*}$. Let $I, J \in \mathcal{N}(L)$. Then $I^{**} = I$ and $J^{**} = J$. Hence $(I \cap J)^{**} = I^{**} \cap J^{**} = I \cap J$. Thus $I \cap J \in \mathcal{N}(L)$. We have also $I \vee J \in \mathcal{N}(L)$. It can be easily observed that $(\mathcal{N}(L), \wedge, \vee)$ is a lattice. Since $\{0\}^{*} = L$ and $L^{*} = \{0\}$, we get that $\{0\}, L \in \mathcal{N}(L)$ are the least and the greatest elements of $\mathcal{N}(L)$, respectively. Therefore, $(\mathcal{N}(L), \wedge, \vee)$ is a bounded lattice. Let $I \in \mathcal{N}(L)$. Then clearly $I^{*} \in \mathcal{N}(L)$ and $I \cap I^{*} = I \cap I^{*} = (0, 0)$. Therefore $I^{*} = (I^{*} \cap I)^{*} = (0)^{*} = 0^{*} = L$. Thus $I^{*}$ is the complement of $I$ for any $I \in \mathcal{N}(L)$. Therefore $(\mathcal{N}(L), \wedge, \vee, *, \{0\}, L)$ is a complemented lattice. Let $I, J, K \in \mathcal{N}(L)$. We prove that $I \vee (J \wedge K) = (I \vee J) \wedge (I \vee K)$. We first prove that $(I \vee J) \wedge K \subseteq I \vee (J \wedge K)$. We have $I \cap K \cap [I^{*} \cap (J \wedge K)^{*}] = \{0\}$, so that $K \cap I^{*} \cap (J \wedge K)^{*} \subseteq I^{*}$. Similarly $J \cap K \cap [I^{*} \cap (J \wedge K)^{*}] = \{0\}$ implies that $K \cap I^{*} \cap (J \wedge K)^{*} \subseteq I^{*}$. Hence $K \cap I^{*} \cap (J \wedge K)^{*} \subseteq I^{*} \cap J^{*}$. Thus, by Lemma 3.4, we get that $[K \cap I^{*} \cap (J \wedge K)^{*}] \cap (I^{*} \cap J^{*})^{*} = \{0\}$. That is, $I \cap (J \wedge K) \cap [I^{*} \cap (J \wedge K)^{*}] = \{0\}$. Thus $K \cap (I^{*} \cap J^{*})^{*} \subseteq [I^{*} \cap (J \wedge K)^{*}]^{*}$. Hence $(I \vee J) \wedge K \subseteq I \vee (J \wedge K)$.

We prove the distributivity. $(I \vee J) \cap (I \vee K) \subseteq I \vee (J \cap K) = I \vee ([I \vee K] \cap J) \subseteq I \vee ([I \vee (K \cap J)] = I \vee (J \cap K)$. Clearly, $I \vee (J \cap K) \subseteq (I \vee J) \cap (I \vee K)$. Thus $(\mathcal{N}(L), \wedge, \vee, *, \{0\}, L)$ is a Boolean algebra.

It can be easily observed that every annulet, for any $x \in L$, $[x]^{*}$ is a normal ideal in $L$. We denote the set of all annulets of $L$ by $\mathcal{N}_{0}(L)$. That is, $\mathcal{N}_{0}(L) = \{[x]^{*} | x \in L\}$. Annulets have many important properties. We give some of them in the following lemma which can be easily verified.

Lemma 4.12. Let $L$ be a GADL with $0$ and $x, y \in L$. Then we have:
(1) \( x \leq y \Rightarrow [y]^* \subseteq [x]^* \);
(2) \([x \land y]^* = [y \land x]^* \);
(3) \([x \lor y]^* = [y \lor x]^* \);
(4) \([x \lor y]^* = [x]^* \cap [y]^* \).

Since each annulet is a normal ideal, we can have the following:
\[
[x]^* \lor [y]^* = [(x)^* \lor (y)^*]^* = [(x \land y)^*]^* = [x \land y]^*.
\[
[x]^* \land [y]^* = [x]^* \cap [y]^* = [x \lor y]^*.
\]

Then we prove in the following theorem that the set \(N_0(L)\) of all annulets of a GADL \(L\) forms a distributive lattice.

**Theorem 4.13.** Let \( L \) be a GADL with 0. Then \((N_0(L), \cap, \lor)\) is a sublattice of the Boolean algebra \((N(L), \cap, \lor, 0), L)\) of normal ideals of \(L\) and hence it is a distributive lattice. \(N_0(L)\) has the same greatest element \(L = [0]^*\) as \(N(L)\). \(N_0(L)\) has the smallest element if and only if \(L\) possesses a dense element.

**Proof.** Let \([x]^*, [y]^* \in N_0(L), \) where \(x, y \in L\). Then
1. \( [x]^* \land [y]^* = [x]^* \cap [y]^* = [x \land y]^* \in N_0(L) \) and
2. \( [x]^* \lor [y]^* = [x \lor y]^* \in N_0(L) \).

Hence \(N_0(L)\) is a sublattice of \(N(L)\). Since \(N(L)\) is distributive, we have that \(N_0(L)\) is also distributive. Clearly, \([0]^*\) is the greatest element of \(N(L)\). Now for any \([x]^* \in N_0(L)\), we get \([x]^* \land [0]^* = [x \land 0]^* = [x]^*\) and \([x]^* \lor [0]^* = [x \lor 0]^* = [0]^*\).

It shows that \([0]^*\) is the greatest element in \(N_0(L)\). Now, it remains to prove the final condition of the theorem. Assume \(N_0(L)\) has the smallest element, say \([d]^*\) where \(d \in L\). Suppose \(x \in [d]^*\). Then \(x \land d = 0\). Since \([d]^*\) is the least element, we get \([x]^* = [x]^* \lor [d]^* = [x \land d]^* = [0]^* = L\). Hence \(x = 0\). Thus \([d]^* = (0\]. Therefore \(d\) is a dense element in \(L\).

Conversely, suppose that \(L\) possesses a dense element, say \(d\). So \([d]^* = (0\]. Clearly, \([d]^* \in N_0(L)\). Now for any \(x \in L\), consider \([x]^* \land [d]^* = [x]^* \land (0) = [x]^* \land (0) = (0\]. Also \([x]^* \lor [d]^* = \{[x]^* \lor [d]^*\}^* = \{[x]^* \lor [0]^*\}^* = \{[x]^* \lor [L]\}^* = [x]^* = [x]^*\). Hence \([d]^*\) is the smallest element in \(N_0(L)\).

In general, the mapping \(x \mapsto [x]^*\) of \(L\) into \(N_0(L)\) is a dual onto homomorphism. In fact, we have the following result.

**Theorem 4.14.** A disjunctive GADL \(L\) is dually isomorphic to \(N_0(L)\).

**Proof.** Let \(L\) be a disjunctive GADL. Define a mapping \(\Phi : L \rightarrow N_0(L)\) by \(\Phi(x) = [x]^*, \) for all \(x \in L\). Clearly, \(\Phi\) is well-defined. Let \(x, y \in L\) such that \(\Phi(x) = [y]^*\). Then \([x]^* = [y]^*\). Since \(L\) is disjunctive, we obtain that \(x = y\).

Therefore \(\Phi\) is one to one. Let \(I \in N_0(L)\). Then \(I = [x]^*, \) for some \(x \in L\). Hence \(\Phi(x) = [x]^* = I\). Therefore \(\Phi\) is onto.

Let \([x]^*, [y]^* \in N_0(L)\), where \(x, y \in L\). Then \(\Phi(x \land y) = [x \land y]^* = [x]^* \land [y]^* = \Phi(x) \land \Phi(y)\) and \(\Phi(x \lor y) = [x \lor y]^* = [x]^* \lor [y]^* = \Phi(x) \lor \Phi(y)\). Hence \(\Phi\) is a dual isomorphism.
Conclusion and Future Work

In this paper we have introduced the concept of an annihilator preserving homomorphism and studied some basic properties of these homomorphisms. We derived a sufficient condition for a homomorphism to be annihilator preserving homomorphism. We introduced the concept of a normal ideal in a GADL $L$ and proved that the set $\mathcal{N}(L)$ of all normal ideals of $L$ forms a Boolean algebra. In [3], we have proved that the set of all ideals of a GADL with 0 forms a complete lattice under set inclusion but we were unable to characterize the nature of the supremum of the ideals in this lattice. Also, the ideal generated by any nonempty subset $S$ of a GADL, except the case when $S$ contains only one element, was not characterized. Investigations in this direction are going on in order to give a topological characterization and sheaf representation of a GADL.

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References


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