Meromorphic Functions That Share One Finite Value DM with Their First Derivative

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Abstract: This paper has studied the uniqueness of meromorphic functions that share one finite value DM (different multiplicities) with first derivatives and obtains some results which improve a result given by Zhang [1].

Keywords: Nevanlinna theory; uniqueness theorem; share DM; meromorphic function.

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1 Introduction

We say that two nonconstant meromorphic functions \( f \) and \( g \) share the finite value \( a \) IM (ignoring multiplicities), if \( f - a \) and \( g - a \) have the same zeros. If \( f - a \) and \( g - a \) have the same zeros with the same multiplicities, we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities). If \( f - a \) and \( g - a \) have the same zeros with the different multiplicities, we say that \( f \) and \( g \) share the value \( a \) DM (different multiplicities). In this paper the term “meromorphic” will always mean meromorphic in the complex plane. We use the standard notations and results of the Nevanlinna theory (see [2], for example). In particular, \( S(r, f) \) denotes any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \), except possibly for a set \( E \) of \( r \) of finite linear measure. Let \( k \) be a positive integer, we denote by \( N_k(r, f - a) \) the counting function of zeros of \( f - a \) with multiplicity \( \leq k \) and by \( N_{k+1}(r, \frac{1}{f-a}) \) the
counting function of zeros of $f - a$ with multiplicity $> k$. Definitions of the terms $N_k(r,f)$ and $N_{k+1}(r,f)$ can be similarly formulated. Finally $N_2(r,\frac{1}{f})$ denotes the counting function of zeros of $f$ where a zero of multiplicity $k$ is counted with multiplicity $\min\{k,2\}$.

Rubel and Yang \[3\] proved the following result:

**Theorem 1.1.** If a nonconstant entire function $f$ and its derivative $f'$ share two finite values CM, then $f \equiv f'$.

Mues and Steinmetz \[4\] have shown that “CM” can be replaced by “IM” in Theorem 1.1 and Gundersen \[5\] have shown that “entire” can be replaced by “meromorphic” in Theorem 1.1.

On the other hand, the meromorphic function \[4\]

$$f(z) = \left[\frac{1}{2} - \frac{\sqrt{5}}{2} \tan \left(\frac{\sqrt{5}}{4} iz\right)\right]^2$$

(1.1)

shares 0 by DM and 1 by IM (neither CM nor DM) with $f'$, while the meromorphic function \[6\]

$$f(z) = \frac{2a}{1 - ce^{-2z}}$$

(1.2)

shares 0 CM and a DM with $f'$, where $c$ and $a$ are nonzero constants. It immediately yields from (1.1) and (1.2) that $f \not\equiv f'$.

Zhang \[1\] proved the following theorem:

**Theorem 1.2.** Let $f$ be a nonconstant meromorphic function, $a$ be a nonzero finite complex constant. If $f$ and $f'$ share 0 CM, and share a IM, then $f \equiv f'$ or $f$ is given as (1.2).

From example (1.2) we also see that $N(r,\frac{1}{f}) = N(r,\frac{1}{f'}) = 0$.

## 2 Main Results

The purpose of this paper is to prove:

**Theorem 2.1.** Let $f$ be a nonconstant meromorphic function. Suppose that $f$ and $f'$ share the value $a$ ($\neq 0, \infty$) DM. Then either

$$f(z) = \frac{a[1 + b + (b - 1)ce^{2b\ell z}]}{1 - ce^{2b\ell z}},$$

(2.1)

where $b, c, \ell$ are nonzero constants and $b^2\ell = -1$, or

$$T(r,f') \leq 12\bar{N} \left( r, \frac{1}{f'} \right) + S(r,f)$$

(2.2)

and

$$T(r,f) \leq \frac{11}{2} N_2 \left( r, \frac{1}{f} \right) + S(r,f).$$

(2.3)
Proof. Suppose that $a = 1$ (the general case follows by considering $\frac{1}{a} f$ instead of $f$). We consider the following function

$$
\psi = \frac{2f'}{f-1} - \frac{3f''}{2(f'-1)} + \frac{f'''}{f' - f''}.
$$

(2.4)

From the fundamental estimate of logarithmic derivative it follows that

$$
m(r, \psi) = S(r, f).
$$

(2.5)

Since $f$ and $f'$ share $1 ~ DM$, all zeros of $f - 1$ are simple and all zeros of $f' - 1$ with multiplicities not less than two. And so

$$
N\left(r, \frac{1}{f-1}\right) = N_1\left(r, \frac{1}{f-1}\right)
$$

(2.6)

and

$$
N\left(r, \frac{1}{f-1}\right) = \tilde{N}\left(r, \frac{1}{f'-1}\right) = \tilde{N}(2)\left(r, \frac{1}{f'-1}\right).
$$

(2.7)

Suppose that $z_2$ is a zero of $f' - 1$ with multiplicity 2. Since $f$ and $f'$ share $1 ~ DM$, we see from (2.6) and (2.4) that

$$
\psi(z_2) = 0.
$$

(2.8)

If $z_\infty$ is a simple pole of $f$, then an elementary calculation gives that

$$
\psi(z_\infty) = O(1).
$$

(2.9)

It follows from (2.6) - (2.9) that the poles of $\psi$ can only occur at zeros of $f'$, or zeros of $f''$ which are not zeros of $f'(f' - 1)$, zeros of $f' - 1$ with multiplicities not less than three and multiple poles of $f$. Thus

$$
N(r, \psi) \leq \tilde{N}\left(r, \frac{1}{f'}\right) + \tilde{N}_3\left(r, \frac{1}{f'-1}\right) + \tilde{N}(2)(r, f) + \tilde{N}_0\left(r, \frac{1}{f''}\right),
$$

(2.10)

where $\tilde{N}_0(r, \frac{1}{f''})$ denotes the counting function corresponding to the zeros of $f''$ that are not zeros of $f'(f' - 1)$, each zero in this counting function is counted only once.

We distinguish the following two cases

Case 1. $\psi \equiv 0$. Then, by integrating two sides of (2.4) we obtain

$$
\frac{(f - 1)^4}{(f' - 1)^3} = c\left(\frac{f'}{f''}\right)^2,
$$

(2.11)

where $c$ is a nonzero constant. If $z_q$ is a zero of $f' - 1$ with multiplicity $q \geq 3$, then from (2.6) and (2.11) we see that

$$
O((z - z_q)^{2-q}) = c.
$$
This implies that \( q = 2 \), a contradiction. Therefore
\[
N_{13} \left( r, \frac{1}{f' - 1} \right) = 0. \tag{2.12}
\]
Also if \( z_p \) is a pole of \( f \) with multiplicity \( p \geq 2 \), then from (2.11) we find that
\[
O((z - z_p)^{1-p}) = c.
\]
Hence \( p = 1 \), a contradiction. Therefore
\[
N_{22}(r, f) = 0. \tag{2.13}
\]
It follows from \( f \) and \( f' \) share 1 DM, (2.6), (2.7), (2.12) and (2.13) that
\[
\frac{f' - 1}{f - 1} = e^\alpha, \tag{2.14}
\]
where \( \alpha \) is some entire function. Combining (2.11) and (2.14) we get
\[
\left( \frac{f''}{f'} \right) \left( \frac{f''}{f' - 1} - \frac{f''}{f} \right) = ce^{2\alpha}. \tag{2.15}
\]
Consequently,
\[
T(r, e^\alpha) = S(r, f). \tag{2.16}
\]
Also we know from (2.15) that
\[
\tilde{N} \left( r, \frac{1}{f'} \right) = S(r, f). \tag{2.17}
\]
Suppose that \( z_1 \) is a simple zero of \( f - 1 \). Then by (2.7) and (2.12) we have
\[
f(z) - 1 = (z - z_1) + a_3(z - z_1)^3 + \cdots, a_3 \neq 0 \tag{2.18}
\]
Substituting (2.18) into (2.11) and (2.14) we find that
\[
3a_3c = 4 \quad \text{and} \quad 3a_3 = e^{\alpha(z_1)},
\]
which implies
\[
e^{\alpha(z_1)} = \frac{4}{c}. \tag{2.19}
\]
If \( e^\alpha \neq \frac{4}{c} \), then we have from (2.6) and (2.16) that
\[
N \left( r, \frac{1}{f' - 1} \right) \leq N \left( r, \frac{1}{e^\alpha - \frac{1}{c}} \right) \leq T(r, e^\alpha) + O(1) = S(r, f). \tag{2.20}
\]
By (2.7), (2.17), (2.20) and the second fundamental theorem we have
\[
T(r, f') \leq \tilde{N} \left( r, \frac{1}{f'} \right) + \tilde{N} \left( r, \frac{1}{f' - 1} \right) + \tilde{N}(r, f) + S(r, f)
\leq \tilde{N}(r, f) + S(r, f).
Since
\[ T(r, f') = m(r, f') + N(r, f') = m(r, f') + N(r, f) + \bar{N}(r, f), \]
it follows from the last inequality that
\[ m(r, f') + N(r, f) = S(r, f), \]
and so \( T(r, f') = S(r, f) \). From this, (2.17) and (2.14) we get \( T(r, f) = S(r, f) \) which is impossible. Therefore \( e^\alpha \equiv \frac{4}{\psi} \). Together with (2.14) we arrive at the conclusion (2.1).

**Case 2.** \( \psi \neq 0 \). Then from (2.8), (2.5) and (2.10) we conclude that
\[
\bar{N}(2 \left( r, \frac{1}{f' - 1} \right)) - \bar{N}(3 \left( r, \frac{1}{f' - 1} \right)) \leq N \left( r, \frac{1}{\psi} \right) \leq T(r, \psi) + O(1)
\leq N(r, \psi) + m(r, \psi) + O(1)
\leq \bar{N} \left( r, \frac{1}{f'} \right) + \bar{N}(3 \left( r, \frac{1}{f' - 1} \right)) + \bar{N}(2(r, f))
+ \bar{N}_0 \left( r, \frac{1}{f''} \right) + S(r, f). \tag{2.21}
\]
Since \( N(r, f') = N(r, f) + \bar{N}(r, f) \), from the second fundamental theorem for \( f' \)
\[ T(r, f') \leq \bar{N} \left( r, \frac{1}{f'} \right) + \bar{N} \left( r, \frac{1}{f' - 1} \right) + \bar{N}(r, f) - \bar{N}_0 \left( r, \frac{1}{f''} \right) + S(r, f), \tag{2.22}\]
we have
\[ N(r, f) \leq \bar{N} \left( r, \frac{1}{f'} \right) + \bar{N} \left( r, \frac{1}{f' - 1} \right) - \bar{N}_0 \left( r, \frac{1}{f''} \right) + S(r, f). \tag{2.23}\]
Also, we know from (2.22) that
\[ N \left( r, \frac{1}{f' - 1} \right) \leq \bar{N} \left( r, \frac{1}{f'} \right) + \bar{N} \left( r, \frac{1}{f' - 1} \right) + \bar{N}(r, f) - \bar{N}_0 \left( r, \frac{1}{f''} \right) + S(r, f). \]
Combining this with (2.23) we obtain
\[
N(r, f) - \bar{N}(r, f) + N \left( r, \frac{1}{f' - 1} \right) - 2\bar{N} \left( r, \frac{1}{f' - 1} \right) + 2\bar{N}_0 \left( r, \frac{1}{f''} \right)
\leq 2\bar{N} \left( r, \frac{1}{f'} \right) + S(r, f). \tag{2.24}
\]
Obviously,
\[ N(r, f) - \bar{N}(r, f) \geq \bar{N}(2(r, f)), \tag{2.25} \]
and
\[ N \left( r, \frac{f'}{f' - 1} \right) - 2 \tilde{N} \left( r, \frac{1}{f' - 1} \right) \geq \tilde{N} \left( r, \frac{1}{f' - 1} \right), \tag{2.26} \]
by (2.7). Thus from (2.24) - (2.26) we obtain
\[ \tilde{N}_2(r, f) + \tilde{N}_3 \left( r, \frac{1}{f' - 1} \right) + 2 \tilde{N}_0 \left( r, \frac{1}{f'} \right) \leq 2 \tilde{N} \left( r, \frac{1}{f'} \right) + S(r, f). \]
From this and (2.21) we deduce that
\[ \tilde{N}_2 \left( r, \frac{1}{f' - 1} \right) \leq 5 \tilde{N} \left( r, \frac{1}{f'} \right) + S(r, f). \]
Together with (2.7) we have
\[ \tilde{N} \left( r, \frac{1}{f' - 1} \right) \leq 5 \tilde{N} \left( r, \frac{1}{f'} \right) + S(r, f). \tag{2.27} \]
From (2.27) and (2.23), it follows that
\[ \tilde{N}(r, f) \leq 6 \tilde{N} \left( r, \frac{1}{f'} \right) + S(r, f). \tag{2.28} \]
Finally, Combining (2.22), (2.27) and (2.28) we find that
\[ T(r, f') \leq 12 \tilde{N} \left( r, \frac{1}{f'} \right) + S(r, f). \]
This is the conclusion (2.2).

We set
\[ G = \frac{1}{f} \left( \frac{f''}{f'} - 2 \frac{f'}{f' - 1} \right). \tag{2.29} \]
Then
\[
m(r, G) \leq m \left( r, \frac{f'}{f} \left( \frac{f''}{f'(f' - 1)} \right) \right) + m \left( r, \frac{f'}{f(f - 1)} \right) + O(1)
\leq 2m \left( r, \frac{f'}{f} \right) + m \left( r, \frac{f''}{f'} \right) + m \left( r, \frac{f''}{f' - 1} \right) + m \left( r, \frac{f'}{f - 1} \right) + O(1)
= S(r, f). \tag{2.30} \]
Suppose \( z_2 \) be a zero of \( f' - 1 \) with multiplicity 2. Since \( f \) and \( f' \) share 1 DM, we see from (2.29), (2.6) and (2.7) that
\[ G(z_2) = O(1). \tag{2.31} \]
If \( z_\infty \) is a pole of \( f \) with multiplicity \( p \geq 1 \), then an elementary calculation gives that
\[ G(z) = O((z - z_\infty)), \quad if \quad p = 1 \tag{2.32} \]
\[ G(z) = O((z - z_{\infty})^{p-1}), \quad \text{if} \quad p \geq 2. \tag{2.33} \]

It follows from (2.6), (2.7), (2.31), (2.32) and (2.33) that the pole of \( G \) can only occur at zeros of \( f' - 1 \) with multiplicities not less than three and zeros of \( f \). Thus

\[ N(r, G) \leq N_2 \left( r, \frac{1}{f} \right) + \tilde{N}_3 \left( \frac{1}{f' - 1} \right), \]

Together with (2.30) we have

\[ T(r, G) \leq N_2 \left( r, \frac{1}{f} \right) + \tilde{N}_3 \left( \frac{1}{f' - 1} \right) + S(r, f). \tag{2.34} \]

We consider two cases:

**Case I.** \( G \equiv 0 \). Then (2.29) becomes

\[ \frac{f''}{f' - 1} - 2 \frac{f'}{f - 1} = 0. \]

By integration, we get \( f' - 1 = \ell(f - 1)^2 \). We rewrite this in the form

\[ \frac{f'}{f - 1 - b} - \frac{f'}{f - 1 + b} = 2b\ell, \tag{2.35} \]

where \( b^2\ell = -1 \). Integrating this once we arrive at the conclusion (2.1).

**Case II.** \( G \not\equiv 0 \). From (2.32), (2.33) and (2.34) we see that

\[
N(r, f) - \tilde{N}_2(r, f) \leq N \left( r, \frac{1}{G} \right) \leq -m \left( r, \frac{1}{G} \right) + T(r, G) + O(1) \\
\leq -m \left( r, \frac{1}{G} \right) + N_2 \left( r, \frac{1}{f} \right) + \tilde{N}_3 \left( r, \frac{1}{f' - 1} \right) + S(r, f). \tag{2.36}
\]

By rewriting (2.29) we have

\[ f = \frac{1}{G} \left( \frac{f''}{f' - 1} - 2 \frac{f'}{f - 1} \right). \]

Therefore

\[ m(r, f) \leq m \left( r, \frac{1}{G} \right) + m \left( r, \frac{f''}{f' - 1} \right) + m \left( r, \frac{f'}{f - 1} \right) + O(1) \]

\[ \leq m \left( r, \frac{1}{G} \right) + S(r, f). \]

Combining this with (2.36) we have

\[ T(r, f) \leq N_2 \left( r, \frac{1}{f} \right) + \tilde{N}_3 \left( \frac{1}{f' - 1} \right) + \tilde{N}_2(r, f) + S(r, f). \tag{2.37} \]
From (2.37) and (2.36), we obtain

\[
N\left(r, \frac{1}{f'^{-1}}\right) \leq T(r, f') + O(1) = m(r, f') + N(r, f') + O(1)
\]

\[
\leq m\left(r, \frac{f'}{f}\right) + m(r, f) + N(r, f) + \bar{N}(r, f) + O(1)
\]

\[
\leq T(r, f) + \bar{N}(r, f) + S(r, f)
\]

\[
\leq 2N_2\left(r, \frac{1}{f'}\right) + 2\bar{N}_3\left(\frac{1}{f'^{-1}}\right) + \bar{N}_{(2)}(r, f) + S(r, f). \quad (2.38)
\]

Set

\[
W = \frac{1}{f'} \left(\frac{f''}{f'^{-1}} - 3\frac{f'}{f'-1}\right). \quad (2.39)
\]

Proceeding as above, we have

\[
m(r, W) = S(r, f), \quad (2.40)
\]

\[
W(z_3) = O(1), \quad (2.41)
\]

\[
W(z) = O((z - z_\infty)^{p-1}), \quad (2.42)
\]

where \(z_3\) is a zero of \(f'-1\) with multiplicity 3 and \(z_\infty\) is a pole of \(f\) with multiplicity \(p \geq 1\). Thus

\[
N(r, W) \leq N_2\left(r, \frac{1}{f'}\right) + \bar{N}_2\left(\frac{1}{f'^{-1}}\right) + \bar{N}_{(4)}\left(\frac{1}{f'}\right).
\]

Together with (2.40) we find

\[
T(r, W) \leq N_2\left(r, \frac{1}{f'}\right) + \bar{N}_2\left(\frac{1}{f'^{-1}}\right) + \bar{N}_{(4)}\left(\frac{1}{f'}\right) + S(r, f). \quad (2.43)
\]

If \(W \equiv 0\), then

\[
\frac{f''}{f'-1} - 3\frac{f'}{f'-1} = 0.
\]

Therefore, we get \(f' - 1 = c(f - 1)^3\). This imply that

\[
N(r, f) = 0, \quad (2.44)
\]

and \(m(r, f') = 3m(r, f) + O(1)\). Hence \(m(r, f) = S(r, f)\). This together with (2.44) gives the contradiction \(T(r, f) = S(r, f)\). Therefore \(W \neq 0\). From this, (2.42) and (2.43) we see that

\[
\bar{N}_{(2)}(r, f) \leq N\left(r, \frac{1}{W}\right) \leq T(r, W) + O(1)
\]

\[
\leq N_2\left(r, \frac{1}{f'}\right) + \bar{N}_2\left(\frac{1}{f'^{-1}}\right) + \bar{N}_{(4)}\left(r, \frac{1}{f'-1}\right) + S(r, f). \quad (2.45)
\]
It follows from (2.7), (2.38) and (2.45) that
\[ N\left(r, \frac{1}{f' - 1}\right) = \bar{N}\left(r, \frac{1}{f' - 1}\right) \leq 3N_2\left(r, \frac{1}{f}\right) + S(r, f). \] (2.46)

Also, from (2.37), (2.45) and (2.7) we find that
\[ m\left(r, \frac{1}{f' - 1}\right) \leq 2N_2\left(r, \frac{1}{f}\right) + \bar{N}_{(r,1/f' - 1)} + S(r, f). \] (2.47)

Set
\[ L = \frac{f''}{f(f-1)}. \] (2.48)

It is clear that
\[ m(r, L) \leq m\left(r, \frac{f''}{f'} \left(\frac{f}{f(f-1)}\right)\right) = S(r, f). \] (2.49)

If \( z_\infty \) is a pole of \( f \) with multiplicity \( p \geq 1 \), then from (2.48) we see that \[ L(z) = O\left((z-z_\infty)^{p-2}\right). \] (2.50)

Also, if \( z_q \) is a zero of \( f' - 1 \) with multiplicity \( q \geq 2 \), then from (2.48) we get
\[ L(z) = O\left((z-z_q)^{q-2}\right). \] (2.51)

Therefore from (2.48), (2.50) and (2.51) we conclude that
\[ N(r, L) \leq N_2\left(r, \frac{1}{f}\right) + N_{(r,f)}. \]

Together with (2.49) we have
\[ T(r, L) \leq N_2\left(r, \frac{1}{f}\right) + N_{(r,f)} + S(r, f). \] (2.52)

If \( L \equiv 0 \), then \( f \) is a linear function. So \( f \) and \( f' \) can not share 1 DM which contradicts the condition of Theorem 2.1. Next we assume that \( L \neq 0 \). From this, (2.51) and (2.52) we see that
\[ N_{(3)}\left(r, \frac{1}{f'-1}\right) - 2\bar{N}_{(3)}\left(r, \frac{1}{f'-1}\right) \leq N\left(r, \frac{1}{L}\right) \leq T(r, L) + O(1) \leq N_2\left(r, \frac{1}{f}\right) + N_{(r,f)} + S(r, f). \]

That is
\[ N_{(3)}\left(r, \frac{1}{f'-1}\right) + \bar{N}_{(2)}(r, f) \leq N_2\left(r, \frac{1}{f}\right) + 2\bar{N}_{(3)}\left(r, \frac{1}{f'-1}\right) + \bar{N}(r, f) + S(r, f). \] (2.53)
Hence from this and (2.36) we obtain
\[ \bar{N}_4 \left( r, \frac{1}{f' - 1} \right) + \bar{N}_{(2)}(r, f) \leq 2N_2 \left( r, \frac{1}{f} \right) + S(r, f), \]
and eliminating \( \bar{N}_{(2)}(r, f) \) between this and (2.37) gives
\[ m \left( r, \frac{1}{f - 1} \right) + \bar{N}_4 \left( r, \frac{1}{f' - 1} \right) \leq 3N_2 \left( r, \frac{1}{f} \right) + S(r, f), \tag{2.54} \]
and eliminating \( \bar{N}_4(r, \frac{1}{f-1}) \) between (2.54) and (2.47) leads to
\[ m \left( r, \frac{1}{f - 1} \right) \leq \frac{5}{2} N_2 \left( r, \frac{1}{f} \right) + S(r, f). \]
Combining this with (2.46) we will arrive at the conclusion (2.3). This completes the proof of Theorem 2.1.

Remark 2.2. From (2.1) we find that
\begin{enumerate}
\item If \( \ell = -1 \), then \( b = \pm 1 \). Hence (2.1) becomes \( f(z) = \frac{-2a}{1-ce^{-r}} \). This is (1.2).
\item If \( c = 1 \), then \( f(z) = a[1 - b \coth(b\ell z)] \).
\item If \( c = -1 \), then \( f(z) = a[1 - b \tanh(b\ell z)] \).
\item If \( b \neq \pm 1 \), then \( T(r, f) = N(r, \frac{1}{f}) + S(r, f) \).
\item \( N(r, \frac{1}{f'}) = 0 \).
\end{enumerate}

From Theorem 2.1 and Remarks 2.2 (3), we deduce readily the following corollaries:

Corollary 2.3. Let \( f \) be a nonconstant meromorphic function. If \( f \) and \( f' \) share the value \( a \) \((\neq 0, \infty) \) DM and if \( \bar{N}(r, \frac{1}{f'}) = S(r, f) \), then \( f \) is given as (2.1).

Corollary 2.4. Let \( f \) be a nonconstant meromorphic function. If \( f \) and \( f' \) share the value \( a \) \((\neq 0, \infty) \) DM and if \( \bar{N}(r, \frac{1}{f}) = S(r, f) \), then \( f \) is given as (1.2).

It is obvious that Corollary 2.3 is extension and improvement for Theorem 1.2 and Corollary 2.4 is improvement for Theorem 1.2.

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