Existence and Stability Properties of Positive Weak Solutions for a Class of Dirichlet Equations Involving Indefinite Weight Functions Driven by a $(p_1,\ldots,p_n)$-Laplacian Operator

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Abstract: In this note, we prove the existence and stability properties of positive weak solutions to a class of nonlinear equations driven by a $(p_1,\ldots,p_n)$-Laplacian operator and indefinite weight functions. First by using the method of sub-super solution we study the existence of positive weak solution. Next we study the stability properties of positive weak solution.

Keywords: $(p_1,\ldots,p_n)$-Laplacian; sub-super solution; linearized stability.

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1 Introduction and Preliminaries

In this work, we study the existence and stability properties of positive weak solutions to the nonlinear elliptic system

$$
\begin{cases}
-\Delta_{p_i} u_i = \lambda u_i(x) \prod_{j=1}^{n} u_j^{a_{ij}} - c_i & x \in \Omega, \\
 u_i(x) = 0 & x \in \partial \Omega
\end{cases}
$$

(1.1)

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for $1 \leq i \leq n$, where $\Omega$ is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with $C^2$-boundary $\partial \Omega$, $p_i > 1$, $\Delta_{p_i} u_i := \text{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the $p_i$-Laplacian operator, $\lambda_i, c_i$ and $\alpha_i$ are positive parameters for $1 \leq i, j \leq n$, and the weight functions $a_i$ satisfies $a_i \in C(\Omega)$ and $a_i(x) > a^+_i > 0$ for all $x \in \Omega$ for $1 \leq i \leq n$. First by using the method of sub-super solution we study the existence of positive weak solution. Next we study the stability properties of positive weak solution directly by analyzing the linearized system.

Problems involving the $p$-Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see [1]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

Systems of the form

$$
\begin{cases}
-\Delta_{p_i} u = \lambda u^\alpha \quad &x \in \Omega, \\
-\Delta_{q_j} v = \lambda v^\beta \quad &x \in \Omega, \\
u(x) = 0 = v(x) \quad &x \in \partial \Omega,
\end{cases} \tag{1.2}
$$

arise in several context in biology and engineering (see [2, 3]). These systems provide simple models to describe, for instance, the interaction of three diffusing biological species. See [4] for details on the physical models involving more general reaction-diffusion system. Semipositive problems have been of great interest during the past two decades, and they continue to pose mathematically difficult problems in the study of positive solutions (see [5–10]). We refer to [11, 12] for additional results in nonlinear elliptic systems.

Throughout this paper, we let $X$ be the Cartesian product of $n$ spaces $W^{1,p_i}_0(\Omega)$ for $1 \leq i \leq n$, i.e., $X = W^{1,p_1}_0(\Omega) \times \cdots \times W^{1,p_n}_0(\Omega)$. We give the definition of weak solution and sub-super solution of (1.1) as follows.

**Definition 1.1.** We say that $u = (u_1, \ldots, u_n) \in X$ is a *weak solution* of the system (1.1), if we have

$$
\int_{\Omega} \sum_{i=1}^n |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla w_i(x) \, dx - \int_{\Omega} \sum_{i=1}^n \left( \lambda a_i(x) \prod_{j=1}^n u_j^{\alpha_j}(x) - c_i \right) w_i(x) \, dx = 0
$$

for all $w = (w_1, \ldots, w_n) \in X$.

**Definition 1.2.** We say that $\psi = (\psi_1, \ldots, \psi_n)$ and $z = (z_1, \ldots, z_n)$ in $X$ are a *subsolution* and a *supersolution* of the system (1.1), if we have

$$
\int_{\Omega} \sum_{i=1}^n |\nabla \psi_i(x)|^{p_i-2} \nabla \psi_i(x) \nabla w_i(x) \, dx \leq \int_{\Omega} \sum_{i=1}^n \left( \lambda a_i(x) \prod_{j=1}^n \psi_j^{\alpha_j}(x) - c_i \right) w_i(x) \, dx
$$
and
\[ \int_{\Omega} \sum_{i=1}^{n} |\nabla z_i(x)|^{p_i-2} \nabla z_i(x) \nabla w_i(x) dx \geq \int_{\Omega} \sum_{i=1}^{n} \left( \lambda a_i(x) \prod_{j=1}^{n} z_j^{a_j}(x) - c_i \right) w_i(x) dx, \]
respectively, for all \( w = (w_1, \ldots, w_n) \in X \).

Now if there exist a subsolution \( \psi = (\psi_1, \ldots, \psi_n) \) and a supersolution \( z = (z_1, \ldots, z_n) \) such that \( 0 \leq \psi_i(x) \leq z_i(x) \) for all \( x \in \Omega \) for \( 1 \leq i \leq n \), then the system (1.1) has a positive solution \( u = (u_1, \ldots, u_n) \in X \) such that \( \psi_i(x) \leq u_i(x) \leq z_i(x) \) for all \( x \in \Omega \) for \( 1 \leq i \leq n \) (see [12]).

2 Existence Results

In this section, we shall prove that if \( 0 < \alpha_j^i < 1 \) for \( 1 \leq i, j \leq n \), then there exist positive constants \( c_0 \) and \( \lambda^* \) such that the system (1.1) has a positive solution when \( c_i \leq c_0 \) for \( 1 \leq i \leq n \) and \( \lambda \geq \lambda^* \). We will obtain the existence of positive weak solution to the system (1.1) by constructing a positive subsolution \( \psi = (\psi_1, \ldots, \psi_n) \) and a positive supersolution \( z = (z_1, \ldots, z_n) \).

To precisely state our theorem, for \( 1 \leq i \leq n \) we first consider the eigenvalue problem
\[ \begin{cases} -\Delta_{p_i} \phi_i = \lambda_i |\phi_i|^{p_i-2} \phi_i & x \in \Omega, \\ \phi_i = 0 & x \in \partial \Omega. \end{cases} \quad (2.1) \]

Let \( \lambda_{1,p_i} \) be the respective first eigenvalue of \( \Delta_{p_i} \) with Dirichlet boundary condition and \( \phi_{1,p_i} \) the corresponding eigenfunction with
\[ \phi_{1,p_i} > 0, \quad ||\phi_{1,p_i}||_\infty = 1, \quad \text{for } 1 \leq i \leq n. \]

It can be shown that \( |\nabla \phi_{1,p_i}| \neq 0 \) on \( \partial \Omega \) for \( 1 \leq i \leq n \), and hence, depending on \( \Omega \), there exist positive constants \( k, \eta \) and \( \mu \) such that
\[ \begin{cases} |\nabla \phi_{1,p_i}|^{p_i} - \lambda_{1,p_i} |\phi_{1,p_i}|^{p_i} \geq k & \text{in } \Omega, \\ |\phi_{1,p_i}| \geq \mu & \text{in } \Omega \setminus \overline{\Omega}_\eta, \end{cases} \quad (2.2) \]

where \( \overline{\Omega}_\eta = \{ x \in \Omega | d(x, \partial \Omega) \leq \eta \} \).

For \( 1 \leq i \leq n \), we will also consider the unique solution, \( \zeta_i \in C^1(\overline{\Omega}) \) of the boundary value problem
\[ \begin{cases} -\Delta_{p_i} \zeta_i = 1 & x \in \Omega, \\ \zeta_i = 0 & \text{for } 1 \leq i \leq n, \quad x \in \partial \Omega. \end{cases} \]

To discuss our existence result, it is known that \( \zeta_i > 0 \) in \( \Omega \) and \( \partial \zeta_i / \partial n < 0 \) on \( \partial \Omega \) where \( n \) denotes the outward unit normal to \( \partial \Omega \) for \( 1 \leq i \leq n \) (see [13]). Now we state our main result as follows.
Theorem 2.1. Let $0 < \alpha_j^i < 1$ for $1 \leq i, j \leq n$. Then there exist positive constants $c_0$ and $\lambda^*$ such that the system (1.1) has a positive solution for $c_i \leq c_0$ ($1 \leq i \leq n$) and $\lambda \geq \lambda^*$.

Proof. To obtain the existence of positive weak solution to the system (1.1), we shall construct a positive subsolution $\psi = (\psi_1, \ldots, \psi_n)$ and a supersolution $z = (z_1, \ldots, z_n)$ of the system (1.1). First, we construct a positive subsolution.

For this, we shall verify that $(\psi_1, \ldots, \psi_n)$ with $\psi_i = \frac{2_i - 1}{p_i} \phi_i^{p_i}$ for $1 \leq i \leq n$ is a subsolution of the system (1.1). Let the test function $w = (w_1, \ldots, w_n) \in X$.

Thus $(\psi_1, \ldots, \psi_n)$ is a subsolution if

$$\lambda_{1,p_i} \phi_{1,p_i}^{p_i} - |\nabla \phi_{1,p_i} |^{p_i} \leq \lambda \alpha_i (x) \prod_{j=1}^{n} \psi_j^{\alpha_j^i} - c_i,$$

for $1 \leq i \leq n$.

This inequality holds, because we have from (2.2)

$$\lambda_{1,p_i} \phi_{1,p_i}^{p_i} - |\nabla \phi_{1,p_i} |^{p_i} \leq -k, \quad \text{in } \overline{\Omega},$$

for $1 \leq i \leq n$, and therefore if $c_i \leq c_0 := k$ for $1 \leq i \leq n$, then

$$\lambda_{1,p_i} \phi_{1,p_i}^{p_i} - |\nabla \phi_{1,p_i} |^{p_i} \leq -k = -c_0 \leq \lambda \alpha_i (x) \prod_{j=1}^{n} \psi_j^{\alpha_j^i} - c_i$$

for $1 \leq i \leq n$, since

$$\lambda \alpha_i (x) \prod_{j=1}^{n} \phi_j^{\alpha_j^i} \geq 0.$$

On the other hand, in $\Omega \backslash \overline{\Omega}$ we have $\phi_{1,p_i} \geq \mu > 0$ for $1 \leq i \leq n$. Thus in $\Omega \backslash \overline{\Omega}$ we have

$$\lambda_{1,p_i} \phi_{1,p_i}^{p_i} - |\nabla \phi_{1,p_i} |^{p_i} \leq \lambda_{1,p_i} \leq \lambda \alpha_i (x) \prod_{j=1}^{n} \psi_j^{\alpha_j^i} - c_i$$

for $1 \leq i \leq n$. Thus in $\Omega \backslash \overline{\Omega}$ we have

$$\lambda_{1,p_i} \phi_{1,p_i}^{p_i} - |\nabla \phi_{1,p_i} |^{p_i} \leq \lambda_{1,p_i} \leq \lambda \alpha_i (x) \prod_{j=1}^{n} \psi_j^{\alpha_j^i} - c_i$$

for $1 \leq i \leq n$. Thus in $\Omega \backslash \overline{\Omega}$ we have
Existence and Stability Properties of Positive Weak Solutions ...  

if
\[ \lambda \geq \hat{\lambda}_i := \frac{(\lambda_1 p_i + k) \prod_{j=1}^{n} (\frac{p_j}{p_j - 1})^{\alpha_j^i}}{a_i^0 \prod_{j=1}^{n} \mu_j^{\frac{\alpha_j^i}{p_j}}} \]  

for \( 1 \leq i \leq n \).

Therefore, \( \psi = (\psi_1, \ldots, \psi_n) \) is a subsolution of the system (1.1) for \( c_i \leq c_0 \) (1 \( \leq i \leq n \)) and
\[ \lambda \geq \lambda^* := \max\{\hat{\lambda}_1, \ldots, \hat{\lambda}_n\} \]

Next we construct a supersolution \( z = (z_1, \ldots, z_n) \) of the system (1.1) such that \( 0 < \psi_i(x) \leq z_i(x) \) for \( x \in \Omega \) and \( 1 \leq i \leq n \). We denote
\[ (z_1, \ldots, z_n) = (A_1 \zeta_1, \ldots, A_n \zeta_n), \]

where the constants \( A_1, \ldots, A_n > 0 \) are large and to be chosen later. We shall verify that \( z = (z_1, \ldots, z_n) \) is a supersolution of the system (1.1). To this end, letting \( w = (w_1, \ldots, w_n) \in X \), we have
\[
\int_{\Omega} \left| \nabla z_i(x) \right|^{p_i - 2} \nabla z_i(x) \nabla w_i(x) dx = A_i^{p_i - 1} \int_{\Omega} \left| \nabla \zeta_i(x) \right|^{p_i - 2} \nabla \zeta_i(x) \nabla w_i(x) dx
\]
\[
= A_i^{p_i - 1} \int_{\Omega} w_i(x) dx.
\]

Let \( \ell_i = \|\zeta_i\|_{\infty} \) for \( 1 \leq i \leq n \). Bearing in mind that \( 0 < \alpha_j^i < 1 \) for \( 1 \leq i, j \leq n \), it is easy to prove that there exist positive large constants \( A_1, \ldots, A_n \) such that
\[ A_i \geq \left[ \lambda \|a_i\|_{\infty} \prod_{j=1}^{n} (A_j \ell_j)^{\alpha_j^i} \right]^{\frac{1}{p_i - 1}} \]

for \( 1 \leq i \leq n \), and then
\[
A_i^{p_i - 1} \geq \lambda \|a_i\|_{\infty} \prod_{j=1}^{n} (A_j \ell_j)^{\alpha_j^i} \geq \lambda a_i(x) \prod_{j=1}^{n} (A_j \ell_j)^{\alpha_j^i} - c_i
\]
\[
\geq \lambda a_i(x) \prod_{j=1}^{n} (A_j \zeta_j)^{\alpha_j^i} - c_i
\]
\[
= \lambda a_i(x) \prod_{j=1}^{n} z_j^{\alpha_j^i} - c_i
\]

for \( 1 \leq i \leq n \). Therefore
\[
\int_{\Omega} \left| \nabla z_i(x) \right|^{p_i - 2} \nabla z_i(x) \nabla w_i(x) dx = A_i^{p_i - 1} \int_{\Omega} w_i(x) dx
\]
\[
\geq \int_{\Omega} \left( \lambda a_i(x) \prod_{j=1}^{n} z_j^{\alpha_j^i} - c_i \right) w_i(x) dx,
\]
i.e., \( z = (z_1, \ldots, z_n) \) is a supersolution of the system (1.1) with \( z_i \geq \psi_i \) in \( \Omega \) for large \( A_i \), for \( 1 \leq i \leq n \). Then the system (1.1) has a positive solution \( u = (u_1, \ldots, u_n) \in X \) such that \( \psi_i \leq u_i \leq z_i \) for \( 1 \leq i \leq n \). Hence, Theorem 2.1 is proven. \( \square \)

3 Stability Results

Here, we would establish stability of positive solution \( u = (u_1, \ldots, u_n) \in X \) to the system (1.1) directly by showing that the principle eigenvalue \( \eta_1 \) of its linearization is positive.

We recall that, if \( u = (u_1, \ldots, u_n) \) be any positive solution to the system

\[
\begin{aligned}
-\Delta p_i u_i &= \lambda f^i (x, u_1, \ldots, u_n) \quad x \in \Omega, \\
u_i(x) &= 0 \quad x \in \partial \Omega
\end{aligned}
\]

for \( 1 \leq i \leq n \), then the linearized equation about \( u = (u_1, \ldots, u_n) \) is

\[
\begin{aligned}
-(p_i - 1) \text{div}(\nabla u_i |\nabla u_i|^{p_i-2}) - \lambda \sum_{j=1}^{n} f^i_{u_j} (x, u_1, \ldots, u_n) w_j &= \eta \omega_i \quad x \in \Omega, \\
w_i(x) &= 0 \quad x \in \partial \Omega
\end{aligned}
\]

for \( 1 \leq i \leq n \), where \( f^i_{u_j} (x, u_1, \ldots, u_n) \) denotes the partial derivative of \( f^i (x, u_1, \ldots, u_n) \) with respect to \( u_j \) for \( 1 \leq i \leq n \). Equation (3.1) obtained from the formal derivative of the operator \( \Delta_{p_i} \) (see [13]).

**Definition 3.1.** Let \( \eta_1 \) denote the first eigenvalue of (3.1). We say that \( u = (u_1, \ldots, u_n) \) is linearly stable, if all eigenvalues of (3.1) are strictly positive, which can be inferred if the principal eigenvalue \( \eta_1 > 0 \). Otherwise \( u = (u_1, \ldots, u_n) \) is linearly unstable.

Let \( u = (u_1, \ldots, u_n) \) be any positive solution of the system (1.1). Then from (3.1) the linearized equation about \( u = (u_1, \ldots, u_n) \) is

\[
\begin{aligned}
-(p_i - 1) \text{div}(\nabla u_i |\nabla u_i|^{p_i-2}) - \lambda a_i(x) &\alpha_i^1 u_1^{\alpha_i^1 - 1} u_2^{\alpha_2^i} \cdots u_n^{\alpha_n^i} w_1 \\
& \cdots \lambda a_i(x) \alpha_n^1 u_1^{\alpha_1^2} u_2^{\alpha_2^2} \cdots u_n^{\alpha_n^2} w_n &= \eta \omega_i \quad x \in \Omega, \\
w_i(x) &= 0 \quad x \in \partial \Omega
\end{aligned}
\]

for \( 1 \leq i \leq n \).

Let \( \eta_1 \) be the principal eigenvalue and \( \psi = (\psi_1, \ldots, \psi_n) \) be the corresponding eigenfunction. We make take \( \psi_i > 0 \) in \( \Omega \) and \( \| \psi_i \|_\infty = 1 \) for \( 1 \leq i \leq n \) (see [14]). Finally, we state our stability result as follows.

**Theorem 3.2.** Suppose \( c_i \leq c_0 \) for \( 1 \leq i \leq n \), and \( \lambda \geq \lambda^* \). Let \( u = (u_1, \ldots, u_n) \) be the solution of the system (1.1) obtained in Theorem 1. Moreover, let \( \alpha_j^i, \epsilon_i \)
for $1 \leq i, j \leq n$, and $\lambda$ be such that

$$\sum_{i=1}^{n} c_i (p_i - 1) \psi_i(x) + \lambda \sum_{i=1}^{n} a_i(x) u_i (\alpha_1^{i} u_1^{i - 1} u_2^{i} \cdots u_n^{i} \psi_1 + \cdots + \alpha_n^{i} u_1^{i} u_2^{i} \cdots u_n^{i - 1} \psi_n)$$

$$< \lambda \sum_{i=1}^{n} (p_i - 1) a_i(x) u_1^{i} u_2^{i} \cdots u_n^{i} \psi_i$$

for all $x \in \Omega$. Then $u = (u_1, \ldots, u_n)$ is linearly stable.

**Proof.** We give from equations (1.1) and (3.2) that

$$\int_{\Omega} \sum_{i=1}^{n} (p_i - 1) u_i(x) \text{div}(|\nabla u_i(x)|^{p_i - 2} \nabla \psi_i(x)) - \psi_i(x) \text{div}(|\nabla u_i(x)|^{p_i - 2} \nabla u_i(x)) \, dx$$

$$+ \lambda \int_{\Omega} \sum_{i=1}^{n} a_i(x) u_i(x) (\alpha_1^{i} u_1^{i - 1} u_2^{i} \cdots u_n^{i} \psi_1 + \cdots + \alpha_n^{i} u_1^{i} u_2^{i} \cdots u_n^{i - 1} \psi_n) \, dx$$

$$- \lambda \int_{\Omega} \sum_{i=1}^{n} (p_i - 1) \psi_i(x) (a_i(x) u_1^{i} \cdots u_n^{i}) \, dx + \int_{\Omega} \sum_{i=1}^{n} c_i (p_i - 1) \psi_i(x) \, dx$$

$$= - \eta \int_{\Omega} \sum_{i=1}^{n} u_i(x) \psi_i(x) \, dx. \tag{3.3}$$

But by using the Green’s first identity, for $1 \leq i \leq n$ we obtain

$$\int_{\Omega} u_i(x) \text{div}(|\nabla u_i(x)|^{p_i - 2} \nabla \psi_i(x)) \, dx = - \int_{\Omega} |\nabla u_i(x)|^{p_i - 2} \nabla u_i(x) \nabla \psi_i(x) \, dx$$

$$+ \int_{\partial \Omega} u_i(x) |\nabla u_i(x)|^{p_i - 2} \left( \frac{\partial \psi_i}{\partial n} \right) \, dS$$

$$= - \int_{\Omega} |\nabla u_i(x)|^{p_i - 2} \nabla u_i(x) \nabla \psi_i(x) \, dx \tag{3.4}$$

and

$$\int_{\Omega} \psi_i(x) \text{div}(|\nabla u_i(x)|^{p_i - 2} \nabla u_i(x)) \, dx = - \int_{\Omega} |\nabla u_i(x)|^{p_i - 2} \nabla u_i(x) \nabla \psi_i(x) \, dx$$

$$+ \int_{\partial \Omega} \psi_i(x) |\nabla u_i(x)|^{p_i - 2} \left( \frac{\partial u_i}{\partial n} \right) \, dS$$

$$= - \int_{\Omega} |\nabla u_i(x)|^{p_i - 2} \nabla u_i(x) \nabla \psi_i(x) \, dx. \tag{3.5}$$
By using (3.4) and (3.5) in (3.3) and from the hypotheses we get

\[- \eta_1 \int_{\Omega} \sum_{i=1}^{n} u_i(x) \psi_i(x) \]

\[= \lambda \int_{\Omega} \sum_{i=1}^{n} a_i(x) u_i(x) \left( \alpha_1^i u_1^{\alpha_1^i - 1} u_2^{\alpha_2^i} \cdots u_n^{\alpha_n^i} \psi_i + \cdots + \alpha_n^i u_1^{\alpha_1^i} u_2^{\alpha_2^i} \cdots u_n^{\alpha_n^i - 1} \psi_n \right) dx \]

\[- \lambda \int_{\Omega} \sum_{i=1}^{n} \left( (p_i - 1) \psi_i(x) (a_i(x) u_1^{\alpha_1^i} \cdots u_n^{\alpha_n^i}) \right) dx + \int_{\Omega} \sum_{i=1}^{n} c_i (p_i - 1) \psi_i(x) dx < 0.\]

But $\psi_1, \ldots, \psi_n > 0$ and $u_1, \ldots, u_n > 0$ in $\Omega$, and hence $\eta_1 > 0$. This completes the proof.

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