On a Quarter-Symmetric Metric Connection in an LP-Sasakian Manifold

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Abstract: The object of the present paper is to study a quarter-symmetric metric connection in an LP-Sasakian manifold. We study some curvature properties of an LP-Sasakian manifold with respect to the quarter-symmetric metric connection.

Keywords: quarter-symmetric metric connection; LP-Sasakain manifold; locally φ-symmetric; φ-recurrent; locally projective φ-symmetric; φ-projectively flat; η-Einstein manifold.

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1 Introduction

The quarter-symmetric connection generalizes the semi-symmetric connection. The semi-symmetric metric connection is important in the geometry of Riemannian manifolds having also physical application; for instance, the displacement on the earth surface following a fixed point is metric and semi-symmetric ([1]).

In 1975, Golab ([2]) defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection $\nabla$ on an n-dimensional Riemannian manifold $(M^n, g)$ is said to be a quarter-symmetric connection ([2]) if its torsion tensor $\tilde{T}$ defined by

$\tilde{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$

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is of the form
\[ \tilde{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y, \]
where \( \eta \) is 1-form and \( \phi \) is a tensor of type (1, 1). In addition, a quarter-symmetric linear connection \( \tilde{\nabla} \) satisfies the condition
\[ (\tilde{\nabla}_X g)(Y, Z) = 0 \]
for all \( X, Y, Z \in \text{TM} \), where \( \text{TM} \) is the Lie algebra of vector fields of the manifold \( M^n \), then \( \tilde{\nabla} \) is said to be quarter-symmetric metric connection. In particular, if \( \phi X = X \) and \( \phi Y = Y \), then the quarter-symmetric connection reduces to a semi-symmetric connection \( (3) \).

After Golab (2), Rastogi (4-5) continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey (6) studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai (7) studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifolds. In 1991, Mukhopadhyay et al. (8) studied quarter-symmetric metric connection on a Riemannian manifold with an almost complex structure \( \phi \). Quarter-symmetric metric connection are also studied by De and Biswas (9), Singh (10), De and Mondal (11), De and De (12) and many others.

On the other hand, there is a class of almost paracontact metric manifolds, namely Lorentzian para-Sasakian manifolds. In 1989, Matsumoto (13) introduced the notion of LP-Sasakian manifolds. Then, Mihai and Rosca (14) introduced the same notion independently and obtained many interesting results. LP-Sasakian manifolds are also studied by De et al. (15), Mihai et al. (16), Saikh and Baishya (17), Singh et al. (18) and others. The paper is organized as follows.

In this paper, we study a quarter-symmetric metric connection in an Lorentzian para-Sasakian manifold. In Section 2, some preliminary results are recalled. In Section 3, we find the expression for curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric metric connection and investigate relations between curvature tensor (resp. Ricci tensor) with respect to quarter-symmetric metric connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. Section 4 deals with locally \( \phi \)-symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection. \( \phi \)-recurrant LP-Sasakian manifold admitting quarter-symmetric metric connection are studied in Section 5 and it is obtained that if an LP-Sasakian manifold is \( \phi \)-recurrant with respect to quarter-symmetric metric connection then \( (M^n, g) \) is an \( \eta \)-Einstein manifold with respect to Levi-Civita connection. Section 6 contains locally projective \( \phi \)-symmetric LP-Sasakian manifold with respect to quarter-symmetric metric connection. Section 7 is devoted to study of \( \phi \)-projectively flat LP-Sasakian manifold with respect to quarter-symmetric metric connection. In the last section, we study \( R, \tilde{R} = 0 \) and obtained \( (M^n, g) \) is an \( \eta \)-Einstein manifold.
2 Preliminaries

An n-dimensional, \((n = 2m+1)\), differentiable manifold \(M^n\) is called Lorentzian
para-Sasakian (briefly, LP-Sasakian) manifold \([13,19]\), if it admits a \((1, 1)\)-tensor
field \(\phi\), a contravariant vector field \(\xi\), a 1-form \(\eta\) and a Lorentzian metric \(g\) which
satisfy

\[
\eta(\xi) = -1, \quad (2.1)
\]

\[
\phi^2 X = X + \eta(X)\xi, \quad (2.2)
\]

\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)
\]

\[
g(X, \xi) = \eta(X), \quad (2.4)
\]

\[
\nabla_X \xi = \phi X, \quad (2.5)
\]

\[
(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.6)
\]

where \(\nabla\) denotes the covariant differentiation with respect to Lorentzian metric \(g\).

It can be easily seen that in an LP-Sasakian manifold the following relations
hold:

\[
\phi \xi = 0, \quad \eta(\phi) = 0, \quad (2.7)
\]

\[
\text{rank}(\phi) = n - 1. \quad (2.8)
\]

If we put

\[
\Phi(X, Y) = g(X, \phi Y) \quad (2.9)
\]

for any vector field \(X\) and \(Y\), then the tensor field \(\Phi(X, Y)\) is a symmetric \((0, 2)\)-
tensor field \([13]\). Also since the 1-form \(\eta\) is closed in an LP-Sasakian manifold,
we have \([13,15]\)

\[
(\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \xi) = 0, \quad (2.10)
\]

for all \(X, Y \in TM\).

Also in an LP-Sasakian manifold, the following relations hold \([15,19]\)

\[
g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.11)
\]

\[
R(\xi, X)Y = g(Y, \xi)X - \eta(Y)X, \quad (2.12)
\]

\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.13)
\]

\[
R(\xi, X) = X + \eta(X)\xi, \quad (2.14)
\]

\[
S(X, \xi) = (n - 1)\eta(X), \quad (2.15)
\]

\[
S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \quad (2.16)
\]

for any vector field \(X, Y\) and \(Z\), where \(R\) and \(S\) are the Riemannian curvature
tensor and Ricci tensor of the manifold respectively.
3 Curvature Tensor of an LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

Let \( \tilde{\nabla} \) be the linear connection and \( \nabla \) be Riemannian connection of an almost contact metric manifold such that

\[
\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y),
\]

where \( H \) is the tensor field of type (1, 1). For \( \tilde{\nabla} \) to be a quarter-symmetric metric connection in \( M^n \), we have

\[
H(X, Y) = \frac{1}{2}[\tilde{T}(X, Y) + \tilde{T}'(X, Y) + \tilde{T}'(Y, X)]
\]

and

\[
g(\tilde{T}'(X, Y), Z) = g(\tilde{T}(Z, X), Y).
\]

In view of equations (3.1) and (3.3), we have

\[
\tilde{T}'(X, Y) = \eta(Y)\phi X - g(\phi X, Y)\xi.
\]

Now, using equations (3.1) and (3.4) in equation (3.2), we get

\[
H(X, Y) = \eta(Y)\phi X - g(\phi X, Y)\xi.
\]

Hence a quarter-symmetric metric connection in an LP-Sasakian manifold is given by

\[
\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi.
\]

Thus the above equation is the relation between quarter-symmetric metric connection and the Levi-Civita connection.

The curvature tensor \( \tilde{\mathcal{R}} \) of \( M^n \) with respect to quarter-symmetric metric connection \( \tilde{\nabla} \) is defined by

\[
\tilde{\mathcal{R}}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z.
\]

In view of equation (3.7), above equation takes the form

\[
\tilde{\mathcal{R}}(X, Y)Z = R(X, Y)Z + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X
\]

\[
+ \eta(Z)(\eta(Y)X - \eta(X)Y) + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi,
\]

where \( \tilde{R} \) and \( R \) are the Riemannian curvature tensor with respect to \( \tilde{\nabla} \) and \( \nabla \) respectively.

From equation (3.8) it follows that

\[
\tilde{S}(Y, Z) = S(Y, Z) + (n - 1)\eta(Y)\eta(Z),
\]

where \( \tilde{S} \) and \( S \) are the Ricci tensor of the connection \( \tilde{\nabla} \) and \( \nabla \) respectively. Contracting above equation, we get

\[
\tilde{r} = r - (n - 1),
\]

where \( \tilde{r} \) and \( r \) are the scalar curvature tensor of the connection \( \tilde{\nabla} \) and \( \nabla \) respectively.
4 Locally $\phi$-Symmetric LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

Definition 4.1. An LP-Sasakian manifold $M^n$ is said to be locally $\phi$-symmetric if
\[ \phi^2(\nabla_W R)(X,Y)Z) = 0, \] (4.1)
for all vector fields $X, Y, Z, W$ orthogonal to $\xi$. This notion was introduced by Takahashi for Sasakian manifolds ([20]).

Definition 4.2. An LP-Sasakian manifold $M^n$ is said to be $\phi$-symmetric if
\[ \phi^2(\nabla_W R)(X,Y)Z) = 0, \] (4.2)
for arbitrary vector fields $X, Y, Z, W$.

Analogous to the definition of locally $\phi$-symmetric LP-Sasakian manifolds with respect to Levi-Civita connection, we define a locally $\phi$-symmetric LP-Sasakian manifolds with respect to the quarter-symmetric metric connection by
\[ \phi^2(\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0, \] (4.3)
for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

In view of equations (3.6) and (3.8), we have
\[ (\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z + \eta(\tilde{R}(X,Y)Z)\phi W \]
\[ - g(\phi W, \tilde{R}(X,Y)Z)\xi. \] (4.4)

Now differentiating equation (3.8) covariantly with respect to $W$, we get
\[ (\nabla_W \tilde{R})(X,Y)Z \]
\[ = (\nabla_W R)(X,Y)Z + g((\nabla_W \phi)X,Z)\phi Y + g(\phi X,Z)(\nabla_W \phi)(Y) \]
\[ - g((\nabla_W \phi)Y,Z)\phi X - g(\phi Y,Z)(\nabla_W \phi)(X) + (\nabla_W \eta)(Z)\eta(Y)X \]
\[ - \eta(X)Y) + \eta(Z)(\nabla_W \eta)(X)X - (\nabla_W \eta)(X)Y + \eta(X) \]
\[ - g(X,Z)\eta(Y) \]
\[ (\tilde{\nabla}_W \xi) + \{g(Y,Z)(\nabla_W \eta)(X) - g(X,Z)(\nabla_W \eta)(Y)\} \xi, \] (4.5)

which on using equations (2.6), (2.9) and (2.10) reduces to
\[ (\nabla_W \tilde{R})(X,Y)Z \]
\[ = (\nabla_W R)(X,Y)Z - \{g(Y,W)\eta(Z) + g(Z,W)\eta(Y) \]
\[ + 2\eta(Y)\eta(W)\eta(Z)\phi X + \{g(X,W)\eta(Z) + g(Z,W)\eta(X) \]
\[ + 2\eta(X)\eta(W)\eta(Z)\phi Y + \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \phi W \]
\[ + \{g(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z)\} W + \{g(W,\phi Z)\eta(Y) \]
\[ + g(W,\phi Y)\eta(Z)\} X - \{g(W,\phi Z)\eta(X) + g(W,\phi X)\eta(Z)\} Y \]
\[ + \{g(\phi X,Z)g(Y,W) - g(\phi Y,Z)g(X,W) + 2\eta(Y)\eta(W)g(\phi X,Z) \]
\[ - 2\eta(X)\eta(W)g(\phi Y,Z) + g(\phi X,W)g(Y,Z) - g(\phi W)g(X,Z)\} \xi. \] (4.6)
Now taking the inner product of equation (3.8) with \( \xi \) and using equations (2.1), (2.7) and (2.11), we obtain

\[
\eta(\tilde{R}(X,Y)Z) = 0.
\] (4.7)

By virtue of equations (4.6), (4.7) and (2.7), equation (4.4) takes the form

\[
\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = \phi^2((\nabla W R)(X,Y)Z) + \eta(Y)g(W, \phi Z)\eta(Z) + \eta(Z)g(W, \phi Y)\eta(Y).
\] (4.8)

Consider \( X, Y, Z \) and \( W \) are orthogonal to \( \xi \), then equation (4.8) yields

\[
\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = \phi^2((\nabla W R)(X,Y)Z).
\] (4.9)

Hence we can state the following

**Theorem 4.3.** In an LP-Sasakian manifold the quarter-symmetric metric connection \( \tilde{\nabla} \) is locally \( \phi \)-symmetric iff the Levi-Civita connection is so.

### 5 \( \phi \)-Recurrent LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

**Definition 5.1.** An n-dimensional LP-Sasakian manifold \( M^n \) is said to be \( \phi \)-recurrent if there exists a non-zero 1-form \( A \) such that

\[
\phi^2((\nabla W R)(X,Y)Z) = A(W)R(X,Y)Z,
\] (5.1)

for arbitrary vector fields \( X, Y, Z, W \).

If \( X, Y, Z, W \) are orthogonal to \( \xi \) then the manifold is called locally \( \phi \)-recurrent manifold.

If the 1-form \( A \) vanishes, then the manifold is reduces to \( \phi \)-symmetric manifold (20).

**Definition 5.2.** An n-dimensional LP-Sasakian manifold \( M^n \) is said to be \( \phi \)-recurrent with respect to quarter-symmetric metric connection if there exists a non-zero 1-form \( A \) such that

\[
\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = A(W)\tilde{R}(X,Y)Z,
\] (5.2)

for arbitrary vector fields \( X, Y, Z, W \).
Suppose $M^n$ is $\phi$-recurrent with respect to quarter-symmetric metric connection, then in view of equations (2.2) and (5.2), we can write
\[ g((\nabla_w R)(X, Y)Z, U) + \eta((\nabla_w R)(X, Y)Z)\eta(U) = A(W)g(\tilde{R}(X, Y)Z, U). \tag{5.3} \]

By virtue of equations (4.4) and (4.7) above equation reduces to
\[ g((\nabla_w R)(X, Y)Z, U) + \eta((\nabla_w R)(X, Y)Z)\eta(U) = A(W)g(\tilde{R}(X, Y)Z, U), \tag{5.4} \]
which on using equation (4.6) takes the form
\[
g((\nabla_w R)(X, Y)Z, U) + \{g(X, W)\eta(Z) + g(Z, W)\eta(X) \}
+ 2\eta(X)\eta(W)\eta(Z)g(\phi Y, U) - \{g(Y, W)\eta(Z) + g(Z, W)\eta(Y) \}
+ 2\eta(Y)\eta(W)\eta(Z)g(\phi X, U) + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \}g(\phi W, U)
+ \{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z) \}g(W, U) + \{g(W, \phi Z)\eta(Y) \}
+ g(W, \phi Y)\eta(Z)g(X, U) - \{g(W, \phi Z)\eta(X) + g(W, \phi X)\eta(Z) \}g(Y, U)
+ \eta((\nabla_w R)(X, Y)Z)\eta(U) + \{\eta(Y)\eta(W)\eta(U)g(\phi X, Z) \}
+ \{\eta(Z)\eta(Y)g(X, U) - \eta(X)g(Y, U) \}g(\phi Y, W) + \eta(\eta(Y)g(X, U))g(Y, Z)\eta(X)
- g(X, Z)\eta(Y)). \tag{5.5} \]

Putting $Z = \xi$ in above equation and using equations (2.1) and (2.7), we get
\[
g((\nabla_w R)(X, Y)\xi, U) - \{g(X, W)g(\phi Y, U) + \eta(X)\eta(W)g(\phi Y, U) \}
+ \{g(Y, W)g(\phi X, U) + \eta(Y)\eta(W)g(\phi X, U) \} + \{g(Y, Z)\eta(X) \}
- g(X, Z)\eta(Y)g(\phi W, U) + \{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z) \}g(W, U)
+ \{g(W, \phi X)g(Y, U) - g(W, \phi Y)g(X, U) \} + \eta((\nabla_w R)(X, Y)\xi)\eta(U)
+ \{\eta(Y)\eta(U)g(\phi X, W) - \eta(X)\eta(U)g(\phi Y, W) \}
= A(W)g(R(X, Y)\xi, U) - A(W)\{\eta(Y)g(X, U) - \eta(X)g(Y, U) \}. \tag{5.6} \]

Now, putting $X = U = e_i$ in above equation and taking summation over $i$, $1 \leq i \leq n$, we get
\[
(\nabla_w S)(Y, \xi) + \sum_{i=1}^{n} g((\nabla_w R)(e_i, Y)\xi, \xi)g(e_i, \xi) = A(W)S(Y, \xi) + ng(\phi Y, W) - (n - 1)A(W)\eta(Y). \tag{5.7} \]

Let us denote the second term of left hand side of equation (5.7) by $E$. In this case $E$ vanishes. Namely, we have
\[
g((\nabla_w R)(e_i, Y)\xi, \xi) = g(\nabla_w R(e_i, Y)\xi, \xi) - g(R(\nabla_w e_i, Y)\xi, \xi)
- g(R(e_i, \nabla_w Y)\xi, \xi) - g(R(e_i, Y)\nabla_w \xi, \xi). \tag{5.8} \]
at \( p \in M \). In local co-ordinates \( \nabla_W e_i = W^j \Gamma^i_{jh} e_h \), where \( \Gamma^i_{jh} \) are the Christoffel symbols. Since \( \{ e_i \} \) is an orthonormal basis, the metric tensor \( g_{ij} = \delta_{ij} \), \( \delta_{ij} \) is the Kronecker delta and hence the Christoffel symbols are zero. Therefore \( \nabla_W e_i = 0 \). Since \( R \) is skew-symmetric, we have

\[
g(R(e_i, \nabla_W Y) \xi, \xi) = 0. \tag{5.9}
\]

Using equation \( (5.9) \) and \( \nabla_W e_i = 0 \) in equation \( (5.8) \), we get

\[
g((\nabla_W R)(e_i, Y) \xi, \xi) = g(\nabla_W R(e_i, Y) \xi, \xi) - g(R(e_i, Y) \nabla_W \xi, \xi). \tag{5.10}
\]

In view of \( g(R(e_i, Y) \xi, \xi) = -g(R(\xi, \xi)e_i, Y) = 0 \) and \( \nabla_W g = 0 \), we have

\[
g(\nabla_W R(e_i, Y) \xi, \xi) - g(R(e_i, Y) \xi, \nabla_W \xi) = 0, \tag{5.11}
\]

which implies

\[
g((\nabla_W R)(e_i, Y) \xi, \xi) = -g(R(e_i, Y) \xi, \nabla_W \xi) - g(R(e_i, Y) \nabla_W \xi, \xi).
\]

Since \( R \) is skew-symmetric, we have

\[
g((\nabla_W R)(e_i, Y) \xi, \xi) = 0. \tag{5.12}
\]

Using equation \( (5.12) \) in equation \( (5.7) \), we get

\[
(\nabla_W S)(Y, \xi) = A(W)S(Y, \xi) + ng(\phi Y, W) - (n - 1)\eta(Y)A(W). \tag{5.13}
\]

Now, we have

\[
(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi),
\]

which on using equations \( (2.5), (2.9), (2.10) \) and \( (2.15) \) takes the form

\[
(\nabla_W S)(Y, \xi) = (n - 1)g(W, \phi Y) - S(Y, \phi W). \tag{5.14}
\]

Form equations \( (5.13) \) and \( (5.14) \), we have

\[
S(Y, \phi W) + g(Y, \phi W) = 0. \tag{5.15}
\]

Replacing \( W \) by \( \phi W \) in above equation and using equation \( (2.2) \) we get

\[
S(Y, W) = -[g(Y, W) + n\eta(Y)\eta(W)], \tag{5.16}
\]

which shows that \( M^n \) is an \( \eta \)-Einstein manifold. Thus we state as follows:

**Theorem 5.3.** If an LP-Sasakian manifold is \( \phi \)-recurrent with respect to the quarter-symmetric metric connection then the manifold is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.
6 Locally Projective $\phi$-Symmetric LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection

**Definition 6.1.** An $n$-dimensional LP-Sasakian manifold $M^n$ is said to be locally projective $\phi$-symmetric if

$$\phi^2((\nabla_W P)(X,Y)Z) = 0,$$

(6.1)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$, where $P$ is the projective curvature tensor defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y].$$

(6.2)

Equivalently

**Definition 6.2.** An $n$-dimensional LP-Sasakian manifold $M^n$ is said to be locally projective $\phi$-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = 0,$$

(6.3)

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$, where $\tilde{P}$ is the projective curvature tensor with respect to the quarter-symmetric metric connection given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y],$$

(6.4)

where $\tilde{R}$ and $\tilde{S}$ are the Riemannian curvature tensor and Ricci tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$.

Using equation (6.3), we can write

$$(\tilde{\nabla}_W \tilde{P})(X,Y)Z = (\tilde{\nabla}_W \tilde{R})(X,Y)Z + \eta(\tilde{P}(X,Y)Z)\phi W - g(\phi W, \tilde{P}(X,Y)Z)\xi.$$ 

(6.5)

Now differentiating equation (6.4) with respect to $W$, we get

$$(\nabla_W \tilde{P})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z - \frac{1}{(n-1)}[\nabla_W \tilde{S}(Y,Z)X - (\nabla_W \tilde{S})(X,Z)Y].$$

(6.6)
In view of equations (4.6) and (3.9) above equation reduces to

\[
(\nabla_W \tilde{P})(X, Y)Z \\
= \left(\nabla_W R\right)(X, Y)Z - \{g(Y, W)\eta(Z) + g(Z, W)\eta(Y) + 2\eta(Y)\eta(W)\eta(Z)\} \phi X + \{g(X, W)\eta(Z) + g(Z, W)\eta(X) + 2\eta(X)\eta(W)\eta(Z)\} \phi Y + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \phi W \\
+ \{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)\} W + \{\eta(Y)g(\phi X, Z)\} W + \{\eta(X)g(\phi Y, Z)\} W + \{g(\phi X, Z)\} W \\
- g(\phi Y, Z)g(X, W) + 2\eta(Y)\eta(W)g(\phi X, Z) - 2\eta(X)\eta(W)g(\phi Y, Z) + g(\phi X, W)g(Y, Z) - g(\phi Y, W)g(X, Z) \xi \\
- \frac{1}{(n - 1)} [(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y]
\]

which on using equation (6.2) reduces to

\[
(\nabla_W \check{P})(X, Y)Z \\
= \left(\nabla_W P\right)(X, Y)Z - \{g(Y, W)\eta(Z) + g(Z, W)\eta(Y) + 2\eta(Y)\eta(W)\eta(Z)\} \phi X + \{g(X, W)\eta(Z) + g(Z, W)\eta(X) + 2\eta(X)\eta(W)\eta(Z)\} \phi Y + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \phi W \\
+ \{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)\} W + \{\eta(Y)g(\phi X, Z)\} W + \{\eta(X)g(\phi Y, Z)\} W + \{g(\phi X, Z)\} W \\
- g(\phi Y, Z)g(X, W) + 2\eta(Y)\eta(W)g(\phi X, Z) - 2\eta(X)\eta(W)g(\phi Y, Z) + g(\phi X, W)g(Y, Z) - g(\phi Y, W)g(X, Z) \xi.
\]

Now, using equations (3.8) and (3.9) in equation (6.4), we get

\[
\tilde{P}(X, Y)Z = R(X, Y)Z + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X + [g(Y, Z)\eta(X)] \eta(Y) \\
- g(\phi W, W)g(Y, Z) - g(\phi Y, W)g(X, Z) \xi - \frac{1}{(n - 1)} [S(Y, Z)X - S(X, Z)Y],
\]

which gives

\[
\check{P}(X, Y)Z = P(X, Y)Z + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X \\
+ [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \xi.
\]

Taking the inner product of equation (6.9) with \(\xi\) and using equations (2.1), (2.7) and (2.11), we get

\[
\eta(\check{P}(X, Y)Z) = -\frac{1}{(n - 1)} [S(Y, Z)X - S(X, Z)Y].
\]
Theorem 6.3. Hence we can state as follows:

Now applying equations (2.2), (6.8) and (6.11) in equation (6.5), we get

\[
\phi^2((\tilde{\nabla}_W \tilde{P})(X, Y)Z) = \phi^2((\nabla_W P)(X, Y)Z) - \{g(Y, W)\eta(Z) + g(Z, W)\eta(Y) + 2n(Y)\eta(W)\eta(Z)\phi^2(\phi X) + \{g(X, W)\eta(Z) + g(Z, W)\eta(X) + 2n(X)\eta(W)\eta(Z)\phi^2(\phi Y) + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\phi^2(\phi W) + \{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)\phi^2(W) - \frac{1}{n-1}\{S(Y, Z)\eta(X) + S(X, Z)\eta(Y)\}\phi^2(\phi W).}
\]

By assuming \(X, Y, Z, W\) orthogonal to \(\xi\), above equation reduces to

\[
\phi^2((\tilde{\nabla}_W \tilde{P})(X, Y)Z) = \phi^2((\nabla_W P)(X, Y)Z). \quad (6.13)
\]

Hence we can state as follows:

**Theorem 6.3.** An \(n\)-dimensional LP-Sasakian manifold is locally projective \(\phi\)-symmetric with respect to \(\tilde{\nabla}\) if and only if it is locally projective \(\phi\)-symmetric with respect to the Levi-Civita connection \(\nabla\).

Again from equations (2.2), (6.7) and (6.11), we have

\[
\phi^2((\tilde{\nabla}_W \tilde{P})(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z) - \{g(Y, W)\eta(Z) + g(Z, W)\eta(Y) + 2n(Y)\eta(W)\eta(Z)\phi^2(\phi X) + \{g(X, W)\eta(Z) + g(Z, W)\eta(X) + 2n(X)\eta(W)\eta(Z)\phi^2(\phi Y) + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\phi^2(\phi W) + \{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)\phi^2(W) - \frac{1}{n-1}\{S(Y, Z)\eta(X) - (\nabla_W S)(X, Z)\eta(Y)\} - \frac{1}{n-1}\{S(Y, Z)\eta(X) - S(X, Z)\eta(Y)\}\phi^2(\phi W).}
\]

Taking \(X, Y, Z\) and \(W\) orthogonal to \(\xi\) in equation (6.14), we obtain by simple calculation

\[
\phi^2((\tilde{\nabla}_W \tilde{P})(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z). \quad (6.15)
\]

Thus we can state as follows:

**Theorem 6.4.** A \(\phi\)-symmetric LP-Sasakian manifold admitting the quarter-symmetric metric connection \(\tilde{\nabla}\) is locally projective \(\phi\)-symmetric with respect to the quarter-symmetric metric connection \(\tilde{\nabla}\) if and only if it is locally \(\phi\)-symmetric with respect to the Levi-Civita connection \(\nabla\).
\section{\textbf{\textphi-projectively Flat LP-Sasakian Manifold with respect to Quarter-Symmetric Metric Connection}}

\textbf{Definition 7.1.} An n-dimensional differentiable manifold \((M^n, g)\) satisfying the equation

\[ \phi^2(P(\phi X, \phi Y)\phi Z) = 0 \]  

(7.1)

is called \(\phi\)-projectively flat. Analogous to the equation (7.1) we define an n-dimensional LP-Sasakian manifold is said to be \(\phi\)-projectively flat with respect to quarter-symmetric metric connection if it satisfies

\[ \phi^2(\tilde{P}(\phi X, \phi Y)\phi Z) = 0, \]  

(7.2)

where \(\tilde{P}\) is the projective curvature tensor of the manifold with respect to quarter-symmetric metric connection.

Suppose \(M^n\) is \(\phi\)-projectively flat LP-Sasakian manifold with respect to quarter-symmetric metric connection. It is easy to see that \(\phi^2(\tilde{P}(\phi X, \phi Y)\phi Z) = 0\) holds if and only if

\[ g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W) = 0, \]  

(7.3)

for any \(X, Y, Z, W \in TM\). So by the use of equation (6.4) \(\phi\)-projectively flat means

\[ g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi W) + g(X, \phi Z)g(Y, \phi e_i) - g(Y, \phi Z)g(X, \phi e_i) \]

(7.5)

which on using equations (3.8) and (3.9) reduces to

\[ n-1 \sum_{i=1}^{n-1} [S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - S(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \]  

(7.6)

Let \(\{e_1, e_2, \ldots, e_{n-1}, \xi\}\) be a local orthonormal basis of the vector fields in \(M^n\). Using the fact that \(\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}\) is also local orthonormal basis. Putting \(X = W = \phi_i\) in equation (7.5) and summing over \(i\), we get

\[ \sum_{i=1}^{n-1} [g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + g(\phi e_i, \phi Z)g(Y, \phi e_i) - g(Y, \phi Z)g(\phi e_i, \phi e_i)] \]

(7.7)

Also, it can be seen that (21)

\[ \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \]  

(7.7)
On a Quarter-Symmetric Metric Connection in an LP-Sasakian Manifold

\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \quad (7.8) \]

\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1 \quad (7.9) \]

and

\[ \sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \quad (7.10) \]

Hence by virtue of equations (7.7), (7.8), (7.9) and (7.10) equation (7.6) takes the form

\[ S(\phi Y, \phi Z) = -2(n - 1)g(\phi Y, \phi Z). \quad (7.11) \]

Now, using equations (2.3) and (2.16) in above equation, we get

\[ S(Y, Z) = -(n - 1)[g(Y, Z) + 3\eta(X)\eta(Y)]. \]

Thus we can state as follows:

**Theorem 7.2.** An n-dimensional \( \phi \)-projectively flat LP-Sasakian manifold admitting the quarter-symmetric metric connection is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

### 8 An LP-Sasakian Manifold with Quarter-Symmetric Metric Connection satisfying \( R.\tilde{R} = 0 \)

Suppose \( (R(\xi, X)\tilde{R})(Y, Z)U = 0 \) on \( M^n \). Then it can be written as

\[ R(\xi, X)\tilde{R}(Y, Z)U - \tilde{R}(R(\xi, X)Y, Z)U - \tilde{R}(Y, R(\xi, X)Z)U \]
\[ - \tilde{R}(Y, Z)R(\xi, X)U = 0. \quad (8.1) \]

In view of equation (2.12) above equation reduces to

\[ ^t\tilde{R}(Y, Z, U, X)\xi - \eta(\tilde{R}(Y, Z)U)X - g(X, Y)\tilde{R}(\xi, Z)U \]
\[ + \eta(Y)\tilde{R}(X, Z)U - g(X, Z)\tilde{R}(Y, \xi)U + \eta(Z)\tilde{R}(X, Y)U \]
\[ - g(X, U)\tilde{R}(Y, Z)\xi + \eta(U)\tilde{R}(Y, Z)X = 0. \quad (8.2) \]
By virtue of equation (3.8) above equation takes the form

\[ \begin{align*}
& \rho'(Y, Z, U, X) + \{ g(\phi Y, U)g(\phi Z, X) - g(\phi Z, U)g(\phi Y, X) \} \xi \\
& + \eta(U)\{ g(X, Y)\eta(Z) - g(X, Z)\eta(Y) \} \xi + \eta(X)\{ g(Z, U)\eta(Y) - g(Y, U)\eta(Z) \} \xi \\
& - g(Y, U)\eta(Z) \xi + \eta(Y)\{ g(\phi X, U)\phi Z - g(\phi Z, U)\phi X \} \\
& + \eta(U)\{ g(\phi Z, U)\phi X - g(\phi X, U)\phi Y + \eta(U)\eta(Y) \} \\
& + \eta(X)\{ g(\phi Y, U)\phi Z - g(\phi Z, U)\phi Y \} \\
& + \eta(Z)\{ g(\phi Y, U)\phi X - g(\phi X, U)\phi Y \} = 0.
\end{align*} \tag{8.3} \]

Now, taking the inner product of above equation with \( \xi \) and using equations (2.1) and (2.4), we get

\[ \begin{align*}
& \rho'(Y, Z, U, X) = g(\phi Z, U)g(\phi Y, X) - g(\phi Z, X)g(\phi Y, U) - \eta(U)\{ g(X, Y)\eta(Z) - g(X, Z)\eta(Y) \} \\
& - g(Y, U)\eta(Z) \xi + \eta(Y)\{ g(\phi X, U)\phi Z - g(\phi Z, U)\phi X \} \\
& + \eta(U)\{ g(\phi Z, U)\phi X - g(\phi X, U)\phi Y + \eta(U)\eta(Y) \} \\
& + \eta(X)\{ g(\phi Y, U)\phi Z - g(\phi Z, U)\phi Y \} \\
& + \eta(Z)\{ g(\phi Y, U)\phi X - g(\phi X, U)\phi Y \} = 0.
\end{align*} \tag{8.4} \]

Putting \( Y = X = e_i \) in above equation and taking summation over \( i \), we get

\[ S(Z, U) = g(Z, U) + (-n)\eta(Z)\eta(U). \]

Thus we can state as follows:

**Theorem 8.1.** An n-dimensional LP-Sasakian manifold admitting quarter-symmetric metric connection satisfying \( R(\xi, X)\tilde{R} = 0 \) is an \( \eta \)-Einstein manifold.

**References**


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