Strong convergence of modified Mann iteration method for an infinite family of nonexpansive mappings in a Banach space

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Abstract: In this paper we introduce a new modified Mann iteration for a $W$-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. The iteration is defined as follows:

$$
\begin{aligned}
x_1 &= x \in C, \text{ arbitrarily;} \\
y_n &= \alpha_n x_n + (1 - \alpha_n) W_n x_n, \quad n \geq 1 \\
x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n) y_n, \quad n \geq 1,
\end{aligned}
$$

where $W_n$ is a $W$-mapping, $C$ a nonempty closed convex subset of a Banach space $E$ with uniformly Gâteaux differentiable. Then we prove that under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ converges strongly to a common fixed point of $T_n, n \in \mathbb{N}$.

Keywords: Strong convergence: nonexpansive mappings: uniformly Gâteaux differentiable: Halpern type

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1 Introduction

Let $C$ be a closed convex subset of a Banach space $E$. Recall that a self-mapping $f : C \to C$ is a contraction on $C$ if there exists a constant $\alpha \in (0, 1)$ such that

$$
\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad x, y \in C.
$$

We use $\Pi_C$ to denote the collection of all contraction on $C$. That is

$$
\Pi_C = \{f : f : C \to C \text{ a contraction}\}.
$$

Note that each $f \in \Pi_C$ has a unique fixed point in $C$. Let now $T$ be a nonexpansive mapping of $C$ into itself, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Halpern [4] introduced the following iterative scheme for approximating a fixed point of $T$:

$$
x_{n+1} = \alpha_n x + (1 - \alpha_n) Tx_n
$$

(1.1)
for all \( n \in \mathbb{N} \), where \( x_1 = x \in C \) and \( \{\alpha_n\} \) is a sequence of \([0, 1]\). This iteration process is called a Halpern type iteration. Strong convergence of this type iterative sequence has been widely studied: Wittmann [17] discussed such a sequence in a Hilbert space. Shioji and Takahashi [14] extended Wittmann’s result; they prove strong converge of \( \{x_n\} \) defined by (1.1) in a Banach space; see also Kamimura and Takahashi [7] and Iiduka and Takahashi [5]. On the other hand, Bauschke [3] used a Halpern type iterative scheme to find a common fixed point of a finite family of nonexpansive mappings in a Hilbert space. Kimura et. al. [8] generalized the result of Shioji and Takahashi [14] and studied strong convergence to a common fixed point of a finite family of nonexpansive mappings in a Banach space; see also [13, 10]. In 2007, Aoyama, Kimura, Takahashi and Toyoda [1] introduce the following iterative sequence: Let \( x_1 = x \in C \) and

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)T_n x_n
\]

for all \( n \in \mathbb{N} \), where \( C \) is nonempty closed convex subset of a Banach space, \( \{\alpha_n\} \) is a sequence of \([0, 1]\), and \( \{T_n\} \) is a sequence of nonexpansive mappings. Then they prove that \( \{x_n\} \) defined by (1.2) converges strongly to a common fixed point

of \( \{T_n\} \).

In 2003, Kikkawa and Takahashi [9], introduce an iterative scheme for finding a common fixed point of infinite nonexpansive mappings in a Hilbert space by using the hybrid method:

\[
\begin{align*}
&y_n = W_n x_n, \\
&C_n = \{z \in C; \|y_n - z\| \leq \|x_n - z\|\}, \\
&Q_n = \{z \in C; (x_n - z, x_1 - x_n) \geq 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n}(x_1),
\end{align*}
\]

for every \( n \in \mathbb{N} \). Then we prove that \( \{x_n\} \) converges strongly to \( P_{F(U)}(x_1) \) where \( F(U) = \cap_{i=1}^\infty F(T_i) \).

Algorithm 1.1 Let \( T_1, T_2, \ldots \) be an infinite family of nonexpansive mappings of \( H \) into itself and let \( \lambda_1, \lambda_2, \ldots \) be real numbers such that \( 0 \leq \lambda_i \leq 1 \) for every \( i \in \mathbb{N} \), we define a mapping \( W_n \) of \( H \) into itself as follows:

\[
\begin{align*}
U_{n+1} &= I, \\
U_n &= \lambda_n T_n U_{n+1} + (1 - \lambda_n)I, \\
U_{n-1} &= \lambda_{n-1} T_{n-1} U_n + (1 - \lambda_{n-1})I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I,
\end{align*}
\]
such mapping $W_n$ is called the $W$-mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\lambda_n, \lambda_{n-1}, ..., \lambda_1$.

In this paper, we introduce the following iterative sequence as follows:

\[
\begin{align*}
    x_1 &= x \in C, \text{ arbitrarily;} \\
    y_n &= \alpha_n x_n + (1 - \alpha_n) W_n x_n, \\
    x_{n+1} &= \beta_n f(x_n) + (1 - \beta_n) y_n,
\end{align*}
\]

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences of $[0, 1]$, and $W_n$ is a $W$-nonexpansive mappings. Then we prove that $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $T_n, n \in \mathbb{N}$.

2 Preliminaries

Throughout this paper, we assume that $E$ is a reflexive Banach space, $C$ is a nonempty closed convex subset of $E$. $E^*$ is the dual space of $E$ and $J : E \to 2^{E^*}$ is the normalized mapping defined by

\[ J(x) = \{ f \in E^*, \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\| \}, \quad x \in E \]

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued normalized duality mapping $J$ by $j$.

Let $S = \{ x \in E : \|x\| = 1 \}$ denote the unit sphere of $E$. Recall that $E$ is said to have a Gâteaux differentiable norm if the limit

\[ \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \]

exists for each $x, y \in E$, and $E$ is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$, the limit is attained uniformly for $x \in S$.

Recall that a Banach space $E$ is said to be strictly convex if

\[ \|x\| = \|y\| = 1, \quad x \neq y \implies \frac{\|x + y\|}{2} < 1. \]

Lemma 2.1. [18] Let $E$ be a Banach space and $J$ the normalized duality mapping. Then for all $x, y \in E$

(i) $\|x + y\|^2 \leq \|x\|^2 + 2(y, j(x + y))$ for all $j(x + y) \in J(x + y)$;

(ii) $\|x + y\|^2 \geq \|x\|^2 + 2(y, j(x))$ for all $j(x) \in J(x)$.

Lemma 2.2. [18] Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,

\[ a_{n+1} \leq (1 - \gamma_n)a_n + b_n, \quad n \geq 0, \]

where $\{\gamma_n\} \subset (0, 1)$, and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n \to \infty} \frac{b_n}{\gamma_n} \leq 0$ or $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n \to \infty} a_n = 0$. 

Strong convergence of modified Mann iteration method ...
Lemma 2.3. [8] Let $E$ be a real reflexive and strictly convex Banach with uniformly Gâteaux differentiable norm. Suppose $C$ is a nonempty closed convex subset of $E$. Suppose that $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Then \{\(x_t\)\} defined by $x_t = tf(x_1) + (1 - t)Tx_1$ converges strongly to a fixed point of $T$ such that $p$ is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (f - I)p, j(x^* - p) \rangle \leq 0$$

for all $x^* \in F(T)$.

Let $\mu$ be a continuous linear functional on $l^\infty$ and $s = (a_0, a_1, \ldots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call $\mu$ a Banach limit if $\mu$ satisfies $||\mu|| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) \leq \mu_n(a_n)$ for all $(a_0, a_1, \ldots) \in l^\infty$. If $\mu$ is a Banach limit, then we have the following:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$,

(ii) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer $r$,

(iii) $\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$.

Remark 2.4. If $s = (a_0, a_1, \ldots) \in l^\infty$ with $a_n \to a$, then $\mu(s) = \mu_n(a_n) = a$ for any Banach limit $\mu$ by (iii). For more details on Banach limits, we refer readers to [10].

Lemma 2.5. [10] Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \subset l^\infty$ satisfying the condition $\mu_n(a_n) \leq a$ for all Banach limits. If $\limsup_{n \to \infty} (a_{n+1} - a_n) \leq 0$, then $\limsup_{n \to \infty} a_n \leq a$.

Lemma 2.6. [15] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.7. [12] Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $T_1, T_2, \ldots$ be nonexpansive mappings of $C$ into itself such that $\cap_{n=1}^\infty F(T_n)$ is nonempty, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n \to \infty} U_{n,k}x$ exists.

Using Lemma 2.7, one can define mapping $W$ of $C$ into itself as follows:

$$Wx = \lim_{n \to \infty} W_nx = \lim_{n \to \infty} U_{n,1}x,$$

for every $x \in C$. Such a $W$ is called the $W$-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Throughout this paper we will assume that $0 < \lambda_n \leq b < 1$ for every $n \geq 1$.

Lemma 2.8. [12] Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $T_1, T_2, \ldots$ be nonexpansive mappings of $C$ into itself such that $\cap_{n=1}^\infty F(T_n)$ is nonempty, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then, $F(W) = \cap_{n=1}^\infty F(T_n)$. 

First we give our implicit iterative scheme as follows: For each \( k \geq 1 \) define a mapping \( S_k : H \to H \) by

\[ S_k(x) = \frac{1}{k} f(x) + (1 - \frac{1}{k})Wx, \quad \forall k \geq 1, x \in H. \]

It is easy to see that for each \( k \geq 1 \), \( S_k \) is a contraction on \( C \). Indeed, we note that

\[
\|S_k(x) - S_k(y)\| = \frac{1}{k}\|f(x) - f(y)\| + \left(1 - \frac{1}{k}\right)\|Wx - Wy\|
\leq \frac{1}{k}\|f(x) - f(y)\| + \left(1 - \frac{1}{k}\right)\|x - y\|
\leq \frac{1}{k}\|W\|\|x - y\| + \left(1 - \frac{1}{k}\right)\|x - y\|.
\]

By Banach contraction principle, there exist a unique fixed point \( u_k \in C \) of \( S_k \) such that

\[
u_k = \frac{1}{k} f(u_k) + (1 - \frac{1}{k})Wu_k, \quad \forall k \geq 1. \tag{2.1}
\]

3 Main Results

In this section, we to obtain our result, we need some Lemmas.

Lemma 3.1. Let \( E \) be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose \( C \) be a nonempty closed convex subset of \( E \). Let \( T_1, T_2, \ldots \) be nonexpansive mappings of \( C \) into itself such that \( \cap_{n=1}^{\infty} F(T_n) \) is nonempty, and let \( f \in \Pi_C \). Let \( \{x_n\} \) be a sequence in \( (1.3) \) with \( \lim_{n \to \infty} \beta_n = 0 \), then

\[
\mu_n(f(p) - p, j(x_n - p)) \leq 0,
\]

for \( p \in \cap_{n=1}^{\infty} F(T_n) \).

Proof. First we show that \( \{x_n\} \) is bounded. Let \( p \in \cap_{n=1}^{\infty} F(T_n) \). By the definition of \( y_n \) and \( x_n \), we have

\[
\|y_n - p\| = \|\alpha_n x_n + (1 - \alpha_n)W_n x_n - p\|
\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|W_n x_n - p\|
\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|x_n - p\|
= \|x_n - p\|
\]

and hence

\[
\|x_{n+1} - p\| = \|\beta_n f(x_n) + (1 - \beta_n) y_n - p\|
\leq \|\beta_n f(x_n) - p\| + (1 - \beta_n) \|y_n - p\|
\leq \beta_n \|f(x_n) - p\| + (1 - \beta_n) \|y_n - p\|
\leq \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + (1 - \beta_n) \|y_n - p\|
\leq \beta_n \|x_n - p\| + \beta_n \|f(p) - p\| + (1 - \beta_n) \|x_n - p\|
\leq (1 - \beta_n (1 - \alpha)) \|x_n - p\| + \beta_n (1 - \alpha) \|f(p) - p\|\|1 - \alpha\|
\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1 - \alpha}\}.\]
By induction on \( n \), we obtain
\[
||x_n - p|| \leq \max\{||x_1 - p||, ||f(p) - p||\}
\]
for every \( n \in \mathbb{N} \). Hence \( \{x_n\} \) is bounded. So are \( \{y_n\} \), \( \{f(x_n)\} \) and \( \{W_n x_n\} \).

For each \( k \in \mathbb{N} \), let \( u_k \) be a unique element of \( C \) such that
\[
u_k = \frac{1}{k}f(u_k) + (1 - \frac{1}{k})W u_k.
\]
(3.1)

From Lemma 2.3, and Lemma 2.8, we obtain that
\[
u_k \to p \in F(W) = \cap_{n=1}^{\infty} F(T_n) \text{ as } k \to \infty.
\]

For every \( n, k \in \mathbb{N} \), we have
\[
||x_{n+1} - W u_k|| = ||\beta_n f(x_n) + (1 - \beta_n)y_n - W u_k||
\]
\[
\leq \beta_n ||f(x_n) - W u_k|| + (1 - \beta_n)||y_n - W u_k||
\]
\[
\leq \beta_n ||f(x_n) - W u_k|| + (1 - \beta_n)||\alpha_n x_n + (1 - \alpha_n)W_n x_n - W u_k||
\]
\[
+ (1 - \beta_n)(1 - \alpha_n)||W_n x_n - W u_k||
\]
\[
\leq \beta_n ||f(x_n) - W u_k|| + (1 - \beta_n)(1 - \alpha_n)||W_n x_n - W u_k|| + (1 - \beta_n)(1 - \alpha_n)||W u_k - W u_k||
\]
\[
\leq \beta_n ||f(x_n) - W u_k|| + \alpha_n ||x_n - W u_k|| + (1 - \alpha_n)||x_n - u_k||
\]
\[
+ (1 - \beta_n)(1 - \alpha_n)||W u_k - W u_k||
\]
\[
= \alpha_n ||x_n - W u_k|| + (1 - \alpha_n)||x_n - u_k|| + b_n.
\]
(3.2)

where \( b_n = \beta_n ||f(x_n) - W u_k|| + (1 - \beta_n)(1 - \alpha_n)||W u_k - W u_k||. \) From \( \lim_{n \to \infty} \beta_n = 0 \) and Lemma 2.7, we have \( \lim_{n \to \infty} b_n = 0 \). From (3.2), we obtain
\[
||x_{n+1} - W u_k||^2 \leq \alpha_n ||x_n - W u_k||^2 + (1 - \alpha_n)||x_n - u_k||^2 + b_n^2
\]
\[
= \alpha_n ||x_n - W u_k||^2 + (1 - \alpha_n)||x_n - u_k||^2 + 2\alpha_n ||x_n - W u_k||
\]
\[
+ (1 - \alpha_n)||x_n - u_k||b_n + b_n^2
\]
\[
= \alpha_n ||x_n - W u_k||^2 + (1 - \alpha_n)^2||x_n - u_k||^2 + 2\alpha_n ||x_n - W u_k||
\]
\[
+ (1 - \alpha_n)||x_n - u_k||b_n + b_n^2
\]
\[
\leq \alpha_n^2 ||x_n - W u_k||^2 + (1 - \alpha_n)^2||x_n - u_k||^2 + (1 - \alpha_n)\alpha_n(||x_n - W u_k||^2
\]
\[
+ ||x_n - u_k||^2) + r_n
\]
\[
= \alpha_n ||x_n - W u_k||^2 + (1 - \alpha_n)||x_n - u_k||^2 + r_n
\]
(3.3)

where \( r_n = b_n(2\alpha_n ||x_n - W u_k|| + (1 - \alpha_n)||x_n - u_k||) + b_n \to 0 \) as \( n \to \infty \).

For any Banach limit \( \mu \), from (3.3), we obtain
\[
\mu_n ||x_n - W u_k||^2 = \mu_n ||x_{n+1} - W u_k||^2 \leq \mu_n ||x_n - u_k||^2.
\]
(3.4)

From (3.1), we have
\[
u_k - x_n = \frac{1}{k}(f(u_k) - x_n) + (1 - \frac{1}{k})(W u_k - x_n),
\]
that is

\[
(1 - \frac{1}{k})(x_n - W u_k) = (x_n - u_k) + \frac{1}{k}(f(u_k) - x_n).
\]

It follows from Lemma 2.1 (ii), that

\[
\|(1 - \frac{1}{k})(x_n - W u_k)\|^2 = \|(x_n - u_k) + \frac{1}{k}(f(u_k) - x_n)\|^2
\]

\[
\geq \|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - x_n, j(x_n - u_k) \rangle
\]

\[
= \|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - u_k - (x_n - u_k), j(x_n - u_k) \rangle
\]

\[
= \|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k) \rangle - \frac{2}{k}\|x_n - u_k\|^2
\]

\[
= (1 - \frac{2}{k})\|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k) \rangle.
\]

So by (3.4) and (3.5), we have

\[
(1 - \frac{1}{k})^2\|x_n - u_k\|^2 \geq (1 - \frac{1}{k})^2\|x_n - W u_k\|^2 \geq (1 - \frac{2}{k})\|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k) \rangle
\]

and hence

\[
\frac{1}{k^2}\|x_n - u_k\|^2 \geq \frac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k) \rangle.
\]

This implies that

\[
\frac{1}{2k}\mu_n\|x_n - u_k\|^2 \geq \mu_n\langle f(u_k) - u_k, j(x_n - u_k) \rangle.
\]

Since \( u_k \to p \in F(W) \) as \( k \to \infty \), we get

\[
\mu_n\langle f(p) - p, j(x_n - p) \rangle \leq 0.
\]

(3.6)

This completes the proof. \( \square \)

**Theorem 3.2.** Let \( E \) be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose \( C \) be a nonempty closed convex subset of \( E \). Let \( T_1, T_2, \ldots \) be nonexpansive mappings of \( C \) into itself such that \( \cap_{n=1}^\infty F(T_n) \) is nonempty. Suppose that the following conditions are satisfied:

\begin{enumerate}
  \item[i)] \( \lim_{n \to \infty} \beta_n = 0, \sum_{n=1}^\infty \beta_n = \infty \) and \( \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty \);
  \item[ii)] \( \sum_{n=1}^\infty \alpha_{n+1} - \alpha_n < \infty \);
  \item[iii)] \( \lim_{n \to \infty} \alpha_n = 0 \) or \( \alpha_n \in [0, a) \), \( \exists a \in (0, 1) \).
\end{enumerate}

Then \( \{x_n\} \) is defined by (1.5) converges strongly to a point in \( \cap_{n=1}^\infty F(T_n) \).

**Proof.** By the proved of Lemma 3.1 we have \( \{x_n\} \) is bounded. So are \( \{y_n\} \), \( \{f(x_n)\} \) and \( \{W_n x_n\} \). From (1.5), we have

\[
\|x_{n+1} - y_n\| = \beta_n \|f(x_n) - y_n\| \to 0 \text{ as } n \to \infty.
\]

(3.7)

Next, we show that

\[
\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.
\]

(3.8)
From \(\{x_n\}, \{y_n\}, \{f(x_n)\}\) and \(\{W_n x_n\}\) are bounded, we let

\[
M = \sup\{\|y_n - f(x_n)\| + \|W_{n+1} x_n - x_n\| + \|W_n x_n - W_{n+1} x_n\|\}.
\]

Moreover, we note that

\[
\|x_{n+2} - x_{n+1}\| = \|\beta_{n+1} f(x_{n+1}) + (1 - \beta_{n+1}) y_{n+1} - (\beta_n f(x_n) + (1 - \beta_n) y_n)\|
\]

\[
= \|\beta_{n+1} f(x_{n+1}) + (1 - \beta_{n+1}) y_{n+1} - (1 - \beta_{n+1}) y_n + (1 - \beta_{n+1}) y_n
- \beta_n f(x_n) - (1 - \beta_n) y_n - \beta_{n+1} f(x_n) + \beta_{n+1} f(x_n)\|
\]

\[
= \|(1 - \beta_{n+1})(y_{n+1} - y_n) + (\beta_n - \beta_{n+1}) y_n + \beta_{n+1}(f(x_{n+1}) - f(x_n))
+ (\beta_{n+1} - \beta_n) f(x_n)\|
\]

\[
\leq (1 - \beta_{n+1})\|y_{n+1} - y_n\| + |\beta_n - \beta_{n+1}|\|y_n - f(x_n)\| + \beta_{n+1}\|x_{n+1} - x_n\|
\]

(3.9)

for all \(n \in \mathbb{N}\). Observe that

\[
\|y_{n+1} - y_n\| = \|\alpha_n x_{n+1} + (1 - \alpha_n) W_{n+1} x_{n+1} - (\alpha_n x_n + (1 - \alpha_n) W_n x_n)\|
\]

\[
= \|\alpha_n x_{n+1} + (1 - \alpha_n) W_{n+1} x_{n+1} - (1 - \alpha_n) W_n x_n - (1 - \alpha_n) W_{n+1} x_n
- \alpha_n x_n - (1 - \alpha_n) W_n x_n - (1 - \alpha_n) W_{n+1} x_n + (1 - \alpha_n) W_n x_n - \alpha_n x_n + \alpha_n x_n\|
\]

\[
= \|(1 - \alpha_n)(W_{n+1} x_{n+1} - W_n x_n) + (\alpha_n - 1) W_n x_n
+ (\alpha_n - 1) y_n + (\alpha_n - 1) \|W_n x_n - W_{n+1} x_n\| + (\alpha_n - 1) \|W_n x_n - W_{n+1} x_n\|
+ (\alpha_n - 1) \|W_n x_n - W_{n+1} x_n\| + \alpha_n \|x_{n+1} - x_n\|
\]

\[
\leq \|x_{n+1} - x_n\| + \|W_n x_n - W_{n+1} x_n\| + |\alpha_n - \alpha_n| \|W_n x_n - W_{n+1} x_n\|
\]

(3.10)

for all \(n \in \mathbb{N}\). Substituting (3.10) in (3.9), we have

\[
\|x_{n+2} - x_{n+1}\| = (1 - \beta_{n+1})\|x_{n+1} - x_n\| + |\alpha_n - \alpha_n| \|W_n x_n - W_{n+1} x_n\|
+ \beta_{n+1}\|x_{n+1} - x_n\|
\]

\[
= (1 - \beta_{n+1})\|x_{n+1} - x_n\| + (1 - \beta_{n+1})\|x_{n+1} - x_n\| + |\alpha_n - \alpha_n| \|W_n x_n - W_{n+1} x_n\|
+ \beta_{n+1}\|x_{n+1} - x_n\|
\]

\[
\leq (1 - \beta_{n+1})\|x_{n+1} - x_n\| + |\alpha_n - \alpha_n| \|W_n x_n - W_{n+1} x_n\|
+ \beta_{n+1}\|x_{n+1} - x_n\|
\]

\[
= (1 - \beta_{n+1})(1 - \alpha)\|x_{n+1} - x_n\| + |\alpha_n - \alpha_n| \|W_n x_n - W_{n+1} x_n\|
+ \beta_{n+1}\|x_{n+1} - x_n\|
\]

\[
\leq (1 - \beta_{n+1})(1 - \alpha)\|x_{n+1} - x_n\| + 2|\alpha_n - \alpha_n|\|M + |\beta_n - \beta_{n+1}| M|
\]

Put \(b_n = 2|\alpha_n - \alpha_n|M + |\beta_n - \beta_{n+1}| M\). From (i) and (ii), we have

\[
\Sigma_{n=1}^{\infty}|b_n| = 2\Sigma_{n=1}^{\infty}(|\alpha_n - \alpha_n| M) + \Sigma_{n=1}^{\infty}|\beta_n - \beta_{n+1}| M < \infty.
\]
Therefore, it follows from Lemma (2.2) that \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \). Next, we show that
\[
\limsup_{n \to \infty} (f(p) - p, j(x_n - p)) \leq 0,
\]
where \( p \in \cap_{n=1}^{\infty} F(T_n) \). Since \( \lim_{n \to \infty} \beta_n = 0 \), it follows from Lemma (3.1), we have
\[
\mu_n(f(p) - p, j(x_n - p)) \leq 0,
\]
where \( p \in \cap_{n=1}^{\infty} F(T_n) \). From \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \), thus
\[
\lim_{n \to \infty} (f(p) - p, j(x_{n+1} - p)) - (f(p) - p, j(x_n - p)) = 0,
\]
where \( p \in \cap_{n=1}^{\infty} F(T_n) \). From (3.11), (3.14) and Lemma (2.4) we have
\[
\limsup_{n \to \infty} (f(p) - p, j(x_n - p)) \leq 0,
\]
for \( p \in \cap_{n=1}^{\infty} F(T_n) \). Finally, we show that \( x_n \to p \) strongly and this concludes the proof. Indeed, using Lemma (2.1) we obtain
\[
\| x_{n+1} - p \|^2 = \| \beta_n f(x_n) + (1 - \beta_n) y_n - p \|^2
\]
\[
\leq (1 - \beta_n) \| y_n - p \|^2 + 2\beta_n (f(x_n) - p, j(x_n - p))
\]
\[
+ 2\beta_n (f(p) - p, j(x_n - p))
\]
\[
\leq (1 - \beta_n) \| y_n - p \|^2 + 2\beta_n (f(x_n) - p, j(x_n - p))
\]
\[
+ 2\beta_n (f(p) - p, j(x_n - p))
\]
\[
\leq (1 - \beta_n + \beta_n^2) \| x_n - p \|^2 + \beta_n \| x_n - x_{n+1} \|^2
\]
It follows that
\[
(1 - \beta_n \alpha) \| x_n - p \|^2 \leq (1 - \beta_n (2 - \alpha) + \beta_n^2) \| x_n - p \|^2 + 2\beta_n (f(p) - p, j(x_n - p)),
\]
that is
\[
\| x_{n+1} - p \|^2 \leq \left( \frac{1 - \beta_n (2 - \alpha)}{1 - \beta_n \alpha} \right) \| x_n - p \|^2 + \frac{\beta_n^2}{1 - \beta_n \alpha} \| x_n - p \|^2 + \frac{2\beta_n}{1 - \beta_n \alpha} (f(p) - p, j(x_{n+1} - p))
\]
\[
= \left[ 1 - \frac{2(1 - \alpha)\beta_n}{1 - \beta_n \alpha} \right] \| x_n - p \|^2 + \frac{2(1 - \alpha)\beta_n}{1 - \beta_n \alpha} \| x_n - p \|^2 + \frac{2\beta_n}{1 - \beta_n \alpha} (f(p) - p, j(x_{n+1} - p))
\]
for all \( n \in \mathbb{N} \), where \( M_1 \geq \| x_n - p \|^2 \geq 0, n \geq 1 \). Now, we apply Lemma (2.2) and use (3.13), we have \( \lim_{n \to \infty} \| x_n - p \|^2 = 0 \). Consequently, we deduce that \( \{ x_n \} \) converges strongly to fixed point \( p \in \cap_{n=1}^{\infty} F(T_n) \). This completes the proof. \( \square \)

**Theorem 3.3.** Let \( E \) be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose \( C \) be a nonempty closed convex subset of \( E \). Let \( T_1, T_2, \ldots \) be nonexpansive mappings of \( C \) into itself such that \( \cap_{n=1}^{\infty} F(T_n) \) is nonempty and \( f \in \Pi_C \) with \( \alpha \in (0, 1) \). Suppose that the following condition are satisfying

i) \( \lim_{n \to \infty} \beta_n = 0 \), and \( \sum_{n=1}^{\infty} \beta_n = \infty \);

ii) \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1 \).

Then \( \{ x_n \} \) is defined by (1.5) converges strongly to a point in \( \cap_{n=1}^{\infty} F(T_n) \).
Proof. By using the same arguments and techniques as those of Lemma 3.1, we note that \( \{x_n\} \) is bounded, and so are the \( \{W_n x_n\}, \{y_n\} \) and \( \{f(x_n)\} \). Setting 
\[
\gamma_n = (1 - \beta_n)\alpha_n, \forall n \geq 1,
\]
it follows from \( \lim_{n \to \infty} \beta_n = 0 \) and (ii) that 
\[
0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1. \tag{3.14}
\]
Define 
\[
x_{n+1} = \gamma_n x_n + (1 - \gamma_n)z_n. \tag{3.15}
\]
We observe that 
\[
z_{n+1} - z_n = \frac{x_{n+2} - \gamma_{n+2}x_{n+1}}{1 - \gamma_{n+1}} - \frac{x_{n+1} - \gamma_{n}x_n}{1 - \gamma_n}
= \frac{\beta_{n+1}f(x_{n+1}) + (1 - \beta_{n+1})y_{n+1} - \gamma_{n+1}x_{n+1}}{1 - \gamma_{n+1}} - \frac{\beta_n f(x_n) + (1 - \beta_n)y_n - \gamma_n x_n}{1 - \gamma_n}
+ \frac{(1 - \beta_{n+1})\alpha_n x_n + (1 - \alpha_n)W_n x_n - \gamma_{n+1}x_{n+1}}{1 - \gamma_n}.
\]
Thus, we have 
\[
\|z_{n+1} - z_n\| \leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}}\left(\|f(x_{n+1})\| + \|W_{n+1} x_{n+1}\|\right) + \frac{\beta_n}{1 - \gamma_n}\left(\|f(x_n)\| + \|W_n x_n\|\right)
+ \|W_{n+1} x_{n+1} - W_n x_n\|
\leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}}\left(\|f(x_{n+1})\| + \|W_{n+1} x_{n+1}\|\right) + \frac{\beta_n}{1 - \gamma_n}\left(\|f(x_n)\| + \|W_n x_n\|\right)
+ \|W_{n+1} x_{n+1} - W_{n+1} x_n\| + \|W_{n+1} x_n - W_n x_n\|
\leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}}\left(\|f(x_{n+1})\| + \|W_{n+1} x_{n+1}\|\right) + \frac{\beta_n}{1 - \gamma_n}\left(\|f(x_n)\| + \|W_n x_n\|\right)
+ \|x_{n+1} - x_n\| + \|W_{n+1} x_n - W_n x_n\|. \tag{3.16}
\]
It follows that 
\[
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}}\left(\|f(x_{n+1})\| + \|W_{n+1} x_{n+1}\|\right)
+ \frac{\beta_n}{1 - \gamma_n}\left(\|f(x_n)\| + \|W_n x_n\|\right) + \|W_{n+1} x_n - W_n x_n\|. \tag{3.16}
\]
From (14), since \( T_i \) and \( U_{n,i} \) are nonexpansive, we have 
\[
\|W_{n+1} x_n - W_n x_n\| = \|\lambda_1 T_1 U_{n+1,2} x_n - \lambda_1 T_1 U_{n,2} x_n\|
\leq \lambda_1 \|U_{n+1,2} x_n - U_{n,2} x_n\|
\leq \lambda_2 \|U_{n+1,3} x_n - U_{n,3} x_n\|
\leq \cdots
\leq \lambda_1 \lambda_2 \cdots \lambda_n \|U_{n+1,n+1} x_n - U_{n,n+1} x_n\|
\leq M \prod_{i=1}^{n} \lambda_i, \tag{3.17}
\]
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where $M \geq 0$ is constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M$ for all $n \geq 0$. Substituting (3.17) into (3.16), we have

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \frac{\beta_{n+1}}{1 - \gamma_{n+1}}(\|f(x_{n+1})\| + \|W_{n+1}x_{n+1}\| + \|W_nx_n\|) + \frac{\beta_n}{1 - \gamma_n}(\|f(x_n)\| + \|W_nx_n\|) + M \prod_{i=1}^{n} \lambda_i,$$

which implies that (noting that (i) and $0 < \lambda_i \leq b < 1, \forall i \geq 1$)

$$\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.6 we have

$$\lim_{n \to \infty} \|z_n - x_n\| = 0.$$

Consequently

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = (1 - \gamma_n) \lim_{n \to \infty} \|z_n - x_n\| = 0.$$

Argument of the proved in Theorem 3.2 we have

$$\limsup_{n \to \infty} (f(p) - p, j(x_n - p)) \leq 0,$$

for $p \in \cap_{n=1}^{\infty} F(T_n)$. By using the same arguments and techniques as those Theorem 3.2, we have \(\{x_n\}\) converges strongly to a point $p \in \cap_{n=1}^{\infty} F(T_n)$. This completes the proof. \(\square\)

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References


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