A Common Fixed Point Theorem in Fuzzy Metric Space Using the Property (CLRg)

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Abstract : In this paper, we generalize the results of Kumar and Fisher [S. Kumar, B. Fisher, A common fixed point theorem in fuzzy metric space using property (E.A.) and implicit relation, Thai J. Math. 8 (3) (2010) 439–446.] using weakly compatible mappings along with property (CLRg). We also provide an example in support our result.

Keywords : fuzzy metric space; common fixed point; weakly compatible maps; implicit relation; property (CLRg).

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1 Introduction

It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [1]. This notion laid the foundation of fuzzy mathematics. Kramosil and Michalek [2] introduced the notion of a fuzzy metric space by generalizing the concept of the probabilistic metric space to the fuzzy situation. George and Veeramani [3] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [2]. There are many view points of the notion of the metric space in fuzzy topology for instance one can refer to Kaleva and Seikkala [4], Kramosil and Michalek [2] and George and Veeramani [3]. This proved a milestone in fixed point point theory of fuzzy metric space and afterwards...
a flood of papers appeared for fixed point theorems in fuzzy metric space.

Mishra et al. [5] introduced the concept of compatible maps in FM-spaces which was further generalised by Singh and Jain [6] by introducing the notion of weak compatibility in FM-spaces. In 2002, Aamri and Moutawakil [7] introduced property (E.A.), which is a true generalization of non-compatible maps in metric spaces. Common fixed points for a pair of maps under the notion of property (E.A.) and non-compatible maps were studied by Pant and Pant [8]. Recently, Sintunavarat and Kumam [9] introduced a new concept of property (CLRg). The importance of property (CLRg) ensures that one does not require the closeness of range subspaces and hence, now a days, authors are giving much attention to this property for generalizing the results present in the literature. Works noted in the references [10–14] are some examples.

Popa [15, 16] introduced the idea of implicit function to prove a common fixed point theorem in metric spaces. Jain [17] further extended the result of Popa [15, 16] in fuzzy metric spaces. Afterwards, implicit relations are used as a tool for finding common fixed point of contraction maps (see, [18–23]). Altun and Turkoglu [24] proved two common fixed point theorems on complete FM-space with an implicit relation. In [24], common fixed point theorems have been proved for continuous compatible maps of type (α) or (β). Kumar and Fisher [25] generalized the results of Altun and Turkoglu [24] by removing the assumption of continuity, relaxing compatibility to weak compatibility and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation in FM-space. Our aim is to further generalize the result of Kumar and Fisher [25] by using the property (CLRg) and relaxing many conditions involved.

2 Preliminaries

Before we give our main result we need the following definitions:

Definition 2.1 ([1]). A fuzzy set $A$ in $X$ is a function with domain $X$ and values in $[0, 1]$.

Definition 2.2 ([26]). A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous $t$-norm if $([0, 1], *)$ is a topological abelian monoid with unit 1 s.t. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3 ([3]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) > 0$,

(FM-2) $M(x, y, t) = 1$ iff $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
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(FM-5) \( M(x, y, \cdot) : (0, \infty) \to [0, 1] \) is continuous, for all \( x, y, z \in X \) and \( s, t > 0 \).

Throughout this paper, we consider \( M \) to be a fuzzy metric space with condition:

(FM-6) \( \lim_{t \to \infty} M(x, y, t) = 1 \) for all \( x, y \in X \) and \( t > 0 \).

**Definition 2.4** ([3]). Let \((X, M, *)\) be fuzzy metric space. A sequence \( \{x_n\} \) in \( X \) is said to be

(i) **Convergent to a point** \( x \in X \), if \( \lim_{n \to \infty} M(x_n, x, t) = 1 \) for all \( t > 0 \);

(ii) **Cauchy sequence** if \( \lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \), for all \( t > 0 \) and \( p > 0 \).

**Definition 2.5** ([3]). A fuzzy metric space \((X, M, *)\) is said to be **complete** if and only if every Cauchy sequence in \( X \) is convergent.

**Lemma 2.6** ([27]). \( M(x, y, \cdot) \) is non-decreasing for all \( x, y \in X \).

**Lemma 2.7** ([27]). Let \( x_n \to x \) and \( y_n \to y \), then

(i) \( \lim_{n \to \infty} M(x_n, y_n, t) \geq M(x, y, t) \), for all \( t > 0 \),

(ii) \( \lim_{n \to \infty} (x_n, y_n, t) = M(x, y, t) \), for all \( t > 0 \), if \( M(x, y, t) \) is continuous.

**Lemma 2.8** ([5]). If for all \( x, y \in X \), \( t > 0 \) and for a number \( k \in (0, 1) \); 

\[ M(x, y, kt) \geq M(x, y, t), \quad \text{then } x = y. \]

**Definition 2.9** ([5]). Let \( A \) and \( B \) be maps from a FM-space \((X, M, \cdot)\) into itself. The maps \( A \) and \( B \) are said to be **compatible** (or asymptotically commuting), if for all \( t \),

\[ \lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1, \]

whenever \( \{x_n\} \) is a sequence in \( X \) such that

\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \quad \text{for some } z \in X. \]

From the above definition it is inferred that \( A \) and \( B \) are non-compatible maps from a FM-space \((X, M, \cdot)\) into itself if \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X \), but either \( \lim_{n \to \infty} M(ABx_n, BAx_n, t) \neq 1 \) or the limit does not exist.

**Definition 2.10** ([6]). Let \( A \) and \( B \) be maps from a FM-space \((X, M, \cdot)\) into itself. The maps are said to be **weakly compatible** if they commute at their coincidence points. Note that compatible mappings are weakly compatible but converse is not true in general.

**Definition 2.11** ([8]). Let \( A \) and \( B \) be two self-maps of a FM-space \((X, M, \cdot)\). We say that \( A \) and \( B \) satisfy the property (E.A.) if there exists a sequence \( \{x_n\} \) such that

\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \quad \text{for some } z \in X. \]
Note that weakly compatible and property (E.A.) are independent to each other (see [15], Example 2.2).

**Definition 2.12** ([9]). Let \((X, d)\) be a metric space. Two mappings \(f : X \to X\) and \(g : X \to X\) are said to satisfy property (CLRg) if there exists sequences \(\{x_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(p),
\]
for some \(p\) in \(X\).

Similarly, we can have the property (CLR\(T\)) and the property (CLR\(S\)) if in the Definition 2.12, the mapping \(g : X \to X\) has been replaced by the mapping \(T : X \to X\) and \(S : X \to X\) respectively.

Our result deal with the following implicit relation used by Altun and Turkoglu [24].

**Definition 2.13** ([24]). Let \(I = [0, 1]\), \(\ast\) be a continuous \(t\)-norm and \(F\) be the set of all real continuous functions \(F : I^6 \to R\) satisfying the following conditions:

(F-1) \(F\) is non-increasing in the fifth and sixth variables,

(F-2) if for some constant \(k \in (0, 1)\) we have
\[(F-a)\quad F\left(u(kt) \cdot v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \ast v\left(\frac{t}{2}\right)\right) \geq 1,
\]
or
\[(F-b)\quad F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) \ast v\left(\frac{t}{2}\right), 1\right) \geq 1,
\]
for any fixed \(t > 0\) and any non-decreasing functions \(u, v : (0, \infty) \to I\), then there exists \(h \in (0, 1)\) with \(u(ht) \geq v(t) \ast u(t)\),

(F-3) if for some constant \(k \in (0, 1)\), we have \(F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1\) for any fixed \(t > 0\) and any non-decreasing function \(u : (0, \infty) \to I\) then \(u(kt) \geq u(t)\).

### 3 Main Results

In [24], Altun and Turkoglu proved the following result:

**Theorem 3.1.** Let \((X, M, \ast)\) be a complete fuzzy metric space with \(a \ast b = \min\{a, b\}\). Let \(A, B, S, T\) be maps from \(X\) into itself satisfying the following conditions:

(3.1) \(A(X) \subseteq T(X), B(X) \subseteq S(X)\);

(3.2) one of the maps \(A, B, S, T\) is continuous;

(3.3) the pairs \((A, S)\) and \((B, T)\) are compatible of type (\(\alpha\)).
(3.4) there exists \( k \in (0, 1) \) and \( F \in \mathcal{F} \) such that

\[
F\{M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \geq 1
\]

for all \( x, y \in X \) and \( t > 0 \).

Then \( A, B, S, T \) have a unique common fixed point in \( X \).

In [25], Kumar and Fisher generalized Theorem 3.1 (which is Theorem 1 in [24]) as follows:

**Theorem 3.2.** Let \((X, M, *)\) be a fuzzy metric space with \( a * b = \min\{a, b\} \). Further, let \((A, S)\) and \((B, T)\) be weakly compatible pairs of self-maps of \( X \) satisfying (3.1), (3.4) with the following condition:

(3.5) one of the pairs \((A, S)\) or \((B, T)\) satisfies property \((E.A.)\).

If the range of one of the maps \( A, B, S \) or \( T \) is a complete subspace of \( X \), then \( A, B, S, T \) have a unique common fixed point in \( X \).

We now generalize Theorem 3.2 as follows:

**Theorem 3.3.** Let \((X, M, *)\) be a fuzzy metric space with \( a * b = \min\{a, b\} \). Let \( A, B, S, T \) be maps from \( X \) into itself satisfying (3.4) with the following conditions:

(3.6) \( B(X) \subseteq S(X) \) and the pair \((B, T)\) satisfies property \((CLR_T)\),

or

(3.7) the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

Then \( A, B, S, T \) have a unique common fixed point in \( X \).

**Proof.** Without loss of generality, assume that \( B(X) \subseteq S(X) \) and the pair \((B, T)\) satisfies property \((CLR_T)\), then there exists a sequence \( \{x_n\} \) in \( X \) such that \( Bx_n \) and \( Tx_n \) converges to \( Tx \) for some \( x \) in \( X \) as \( n \to \infty \). Since \( B(X) \not\subseteq S(X) \), so there exists a sequence \( \{y_n\} \) in \( X \) such that \( Bx_n = Sy_n \), hence \( Sy_n \to Tx \) as \( n \to \infty \).

We shall show that \( \lim_{n \to \infty} Ay_n = Tx \). Let \( \lim_{n \to \infty} Ay_n = z \). Taking \( x = y_n, y = x_n \) in (3.4),

\[
F\{M(Ay_n, Bx_n, kt), M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), M(Bx_n, Tx_n, t), M(Ay_n, Tx_n, t), M(Bx_n, Sy_n, t)\} \geq 1.
\]

Letting \( n \to \infty \), we have

\[
F\{M(z, Tx, kt), 1, M(z, Tx, t), 1, M(z, Tx, t), 1\} \geq 1.
\]
On the other hand, since

\[ M(z, Tx, t) \geq M\left(z, T x, \frac{t}{2}\right) = M\left(z, T x, \frac{t}{2}\right) \ast 1, \]

and \( F \) is non-increasing in the fifth variable, we have, for any \( t > 0 \)

\[
F\left\{ M(z, Tx, kt), 1, M(z, Tx, t), 1, M\left(z, T x, \frac{t}{2}\right), 1 \right\} \\
\geq F\{M(z, Tx, kt), 1, M(z, Tx, t), 1, M(z, Tx, t)\} \geq 1,
\]

which implies by \((F-2)\), that \( z = Tx \). Subsequently, we have \( Bx_n, Tx_n, Sy_n, Ay_n \)
converges to \( z \). We shall show that \( Bx = z \).

Taking \( x = y_n, y = x \) in (3.4),

\[
F\{M(Ay_n, Bx, kt), M(Sy_n, Tx, t), M(Ay_n, Sy_n, t), M(Bx, Tx, t), M(Ay_n, Tx, t), M(Bx, Sy_n, t)\} \geq 1.
\]

Letting \( n \to \infty \), we have

\[
F\{M(z, Bx, kt), 1, 1, M(z, Bx, t), 1, M(z, Bx, t)\} \geq 1.
\]

On the other hand, since

\[ M(z, Bx, t) \geq N\left(z, Bx, \frac{t}{2}\right) = M\left(z, Bx, \frac{t}{2}\right) \ast 1, \]

and \( F \) is non-increasing in the sixth variable, we have, for any \( t > 0 \)

\[
F\left\{ M(z, Bx, kt), 1, 1, M(z, Bx, t), 1, M\left(z, Bx, \frac{t}{2}\right) \ast 1 \right\} \\
\geq F\{M(z, Bx, kt), 1, 1, M(z, Bx, t), 1, M(z, Bx, t)\} \geq 1,
\]

which implies by \((F-2)\) that \( z = Bx = Tx \). Since, the pair \((B, T)\) is weak compatible, it follows that \( Bz = Tz \).

Also, since \( B(X) \subseteq S(X) \), there exists some \( y \) in \( X \) such that \( Bx = Sy(z) \).

We next show that \( Sy = Ay(z) \). Taking \( y = x_n, x = y \) in (3.4),

\[
F\{M(Ay, Bx_n, kt), M(Sy, Tx_n, t), M(Ay, Sy, t), M(Bx_n, Tx_n, t), M(Ay, Tx_n, t), M(Bx_n, Sy, t)\} \geq 1.
\]

Letting \( n \to \infty \), we have

\[
F\{M(Ay, z, kt), 1, M(Ay, z, t), 1, M(Ay, z, t)\} \geq 1.
\]

Other the other hand, since

\[ M(Ay, z, t) \geq M\left(Ay, z, \frac{t}{2}\right) = M\left(Ay, z, \frac{t}{2}\right) \ast 1, \]
and $F$ is non-increasing in the fifth variable, we have, for any $t > 0$

\[ F\left\{ M(Ay, z, kt), 1, M(Ay, z, t), 1, M(Ay, z, \frac{t}{2}) \ast 1, 1 \right\} \geq F\{M(Ay, z, kt), 1, M(Ay, z, t), 1, M(Ay, z, t), 1\} \geq 1, \]

which implies by (F-2) that $Ay = z = Sy$. But the pair $(A, S)$ is weakly compatible, it follows that $Az = Sz$.

Next, we claim that $Az = Bz$. Taking $x = z, y = z$ in (3.4),

\[ F\{M(Az, Bz, kt), M(Az, Bz, t), 1, 1, M(Az, Bz, t), M(Az, Bz, t)\} \geq 1, \]

which implies by (F-3) that $Az = Bz$. Hence, $Az = Bz = Sz = Tz$.

We now show that $z = Az$. Taking $x = z, y = x$ in (3.4),

\[ F\{M(Az, Bx, kt), M(Sz, Tx, t), M(Az, Sz, t), M(Bx, Tx, t), M(Az, Tz, t), M(Bx, Sz, t)\} \geq 1, \]

that is,

\[ F\{M(Az, z, kt), M(Az, z, t), 1, 1, M(Az, z, t), M(Az, z, t)\} \geq 1. \]

Therefore, $z = Az = Bz = Sz = Tz$, that is $z$ is the common fixed point of the maps $A, B, S, T$. Uniqueness of $z$ follows immediately from (F-3) and (3.4).

**Example 3.4.** Let $(X, M, \ast)$ be a fuzzy metric space with $X = [0, 1]$, a $t$-norm $\ast$ be defined by $a \ast b = \min\{a, b\}$ for all $a, b$ in $[0,1]$ and $M$ be a fuzzy set on $X^2 \times (0, \infty)$ defined by

\[ M(x, y, t) = \left[ \exp\left(\frac{|x - y|}{t}\right) \right]^{-1} \]

for all $x, y$ in $X$ and $t > 0$.

Let $F : I^6 \to R$ be defined by $F(u_1, u_2, u_3, u_4, u_5, u_6) = \frac{u_1}{\min\{u_2, u_3, u_4, u_5, u_6\}}$. Let $t > 0, 0 < u(t), v(t) \leq 1, k \in (0, \frac{1}{2})$, where $u, v : [0, \infty) \to I$ are non-decreasing functions. Suppose that

\[ F\left(u(kt), v(t), v(t)u(t), 1, u\left(\frac{t}{2}\right) \ast v\left(\frac{t}{2}\right)\right) \geq 1, \]

that is,

\[ F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \ast v\left(\frac{t}{2}\right)\right) = \frac{u(kt)}{\min\{v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \ast v\left(\frac{t}{2}\right)\}} \geq 1. \]
Thus \( u(ht) \geq v(t) * u(t) \) if \( h = 2k \in (0, 1) \). A similar argument works if \((F_{k})\) is assumed. Finally, suppose that \( t > 0 \) is fixed, \( u : [0, \infty) \rightarrow I \) is a non-decreasing function and

\[
F(u(kt), u(t), 1, 1u(t), u(t)) = \frac{u(kt)}{u(t)} \geq 1, 
\]

for some \( k \in (0, 1) \). Then we have \( u(kt) \geq u(t) \) and thus \( F \in F \).

Define the mappings \( A, B, S, T : X \rightarrow X \) by

\[
Ax = \frac{x}{27}, \quad Bx = \frac{x}{9}, \quad Sx = \frac{x}{3}, \quad Tx = x, 
\]

respectively. Then, for some \( k \in \left(\frac{1}{9}, 1\right) \), we have

\[
M(Ax, By, kt) = \left[ \exp \left( \frac{|\frac{x}{27} - \frac{y}{9}|}{kt} \right) \right]^{-1} \geq \min \{ M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t) \}.
\]

Thus, the condition (3.4) of Theorem 3.3 is satisfied.

Further, the pairs \((A, S)\) and \((B, T)\) are weakly compatible. Also, \( B(X) = [0, \frac{1}{9}] \subset [0, \frac{1}{3}] = S(X) \). Considering the sequence \( \{x_n\} = \{\frac{1}{3}\} \) so that \( \lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = 0 = T(0) \), hence the pair \((B, T)\) satisfies property \((CLR_T)\).

Therefore, all the conditions of Theorem 3.3 are satisfied. Indeed 0 is the unique common fixed point of the mappings \( A, B, S, T \).

References


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