Thai Journal of Mathematics
Volume 18 Number 3 (2020)
Pages 1433–1446

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Dedicated to Prof. Suthep Suantai on the occasion of his 60^{th} anniversary

The Stability of Dynamical System for the Quasi Mixed Equilibrium Problem in Hilbert Spaces

Narin Petrot[1] **and Jittiporn Tangkhawiwetkul**[2,*]

[1] *Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand*
e-mail : narinp@nu.ac.th (N. Petrot)

[2] *Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok, 65000, Thailand*
e-mail : j_suwannawit@psru.ac.th (J. Tangkhawiwetkul)

Abstract In this paper, we study the quasi mixed equilibrium problem in Hilbert spaces and consider the existence solution of such problem. The resolvent equation which is equivalent to the quasi mixed equilibrium problem is presented and the relation between the solution of the quasi mixed equilibrium problem and such equation is proposed. Using the previous relation, we can introduce a dynamical system associated with the quasi mixed equilibrium problem. Finally, the existence and stability of such dynamical system are proved.

MSC: 34A12; 37C75

Keywords: quasi mixed equilibrium problem; dynamical system; Wiener-Hopf equation; resolvent equation; Gronwall's inequality; Lyapunov function

Submission date: 13.01.2020 / Acceptance date: 24.07.2020

1. Introduction

Equilibrium problem theorem is significant role and interesting in mathematics. This problem is inspired to mathematicians for considering the applied math problems including economics, finance, transportation, mechanics, network analysis, optimization and operation research in general and unified way, see [1–4]. The equilibrium problem includes the variational inequality problem as special case which the variational inequality problem is widely useful problem and powerful tool in mathematics. By these reasons, the equilibrium problem has been extensively analyzed. A quasi mixed equilibrium problem is a problem which is developed from the equlibrium problem because this problem consist of the equilibrium problem and the variational inequality and, moreover, can be

*Corresponding author.

Published by The Mathematical Association of Thailand.

applied in various fields in nonlinear analysis including variational inequalities, complementarity problem, optimization problem, fixed point problem, saddle point problem and Nash equilibrium problem as special case, see [5, 6].

A dynamical system is a problem which relates to time and this problem is applied in many fields such as economics, physics, engineering, medicine and mathematics etc. Some problems in the previous fields can be written in the dynamical system model and then such model can be exchanged to the dynamical system equation that this equation is simple form to solve the results in the sense of mathematics such as to find equilibrium point, existence solution and stability solution, etc. see [7, 8]. In another way, in mathematics, some functions over time can be formulated in the dynamical system form and, using the results, applied in science and the real world problems. One of interesting aspects is the dynamical system of variational inequality problem that consider the dynamical system which associates with the variational inequality problem. This implies the variational inequality problem close to the real world problems and is simple for applications. For example, in paper of P. Dupuis and A. Nugurney [9] and M. A. Noor [10–12], authors studied the dynamical system associated with the variational inequality problem and obtained the following results. The set of stationary points of the dynamical system coincides with the set of the solutions of variational inequality problem. This concept is used for considering the solution, existence solution and the stability solution of the dynamical system and the variational inequality problem. By the previous article, researchers developed the dynamical system which relates with the variational inequality and this problem has received a lot of attention because of its application in financial equilibrium problem, optimization problem, complementarity problem and all problems in the framework of the variational inequality see [13–16].

The aim of this paper, we would like to develop the dynamical system associated with variational inequality, so we will consider a quasi-mixed equilibrium problem. The existence solution of such equilibrium problem is proved. The resolvent equation of quasi mixed equilibrium problem is presented and the relation of a solution of quasi mixed equilibrium problem and a solution of resolvent equation is considered. After that, using this relation, we can introduce the dynamical system associated with quasi mixed equilibrium problem. Finally, the existence solution and stability of such dynamical system is proved.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. The following basic concepts need for solving our results. Firstly, we will introduce the properties of mappings which are used in this paper.

Definition 2.1. A mapping $T : H \to H$ is said to be γ-*strongly monotone* if there exists a real number $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \gamma \|x - y\|^2,$$

for all $x, y \in H$.

Definition 2.2. A mapping $T : H \to H$ is said to be β-*Lipschitz continuous* if there exists a real number $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta \|x - y\|,$$

for all $x, y \in H$.

Definition 2.3 ([6]). A function $f : H \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semi-continuous at* x_0 if for all $\alpha < f(x_0)$, there exists a constant $\delta > 0$ such that

$$\alpha \leq f(x), \quad \forall x \in B(x_0, \delta),$$

where $B(x_0, \delta)$ denotes the ball with the center x_0 and the radius δ, i.e., $B(x_0, \delta) = \{y : \|y - x_0\| \leq \delta\}$. Furthermore, f is said to be lower semi-continuous if it is lower semi-continuous at every point of H.

Definition 2.4 ([6]). Let $F : H \times H \to \mathbb{R}$ be a real valued bifunction.

(1) F is said to be *monotone* if

$$F(x, y) + F(y, x) \leq 0,$$

for all $x, y \in H$.

(2) F is said to be *strictly monotone* if

$$F(x, y) + F(y, x) < 0,$$

for all $x, y \in H$ with $x \neq y$.

(3) F is said to be *upper hemicontinuous* if for all $x, y, z \in H$

$$limsup_{t \to 0^+} F(tz + (1 - t)x, y) \leq F(x, y).$$

The following lemmas, we will recall the definition of $J^{\mu}_{F,K}$ and some properties of such mapping.

Lemma 2.5 ([6]). *Let K be a nonempty closed convex subset of H and F be a bifunction of $H \times H$ into $\mathbb{R}$ satisfying the following conditions:*

(1) F is monotone and upper hemicontinuous;

(2) $F(x, \cdot)$ is convex and lower semi-continuous for each $x \in K$.

Let $\mu > 0$ be fixed. Define a mapping $J^{\mu}_{F,K} : H \to K$ as follows:

$$J^{\mu}_{F,K}(x) = \{w \in K : \mu F(w, z) + \langle w - x, z - w\rangle \geq 0, \forall z \in K\},$$

for all $x \in H$. Then, $J^{\mu}_{F,K}$ is a single valued mapping.

Lemma 2.6 ([17]). *Let K be a nonempty closed convex subset of H. If $F : H \times H \to \mathbb{R}$ is a monotone function, then the operator $J^{\mu}_{F,K}$ is a nonexpansive mapping, that is,*

$$\|J^{\mu}_{F,K}(x) - J^{\mu}_{F,K}(y)\| \leq \|x - y\|,$$

for all $x, y \in H$.

Now, we will recall the following well known concepts of the dynamical system see [18].

A dynamical system is a mapping $\Phi : \mathbb{R} \times H \to H$ which is a C^1 mapping and writing $\Phi(t, x) := \Phi_t(x)$ and $f : H \to H$ is defined by

$$f(x) = \frac{d}{dt}\Phi_t(x)|_{t=0}. \tag{2.1}$$

Now, we may rewrite this in more conventional terms. If $\Phi_t : H \to H$ is a dynamical system and $x \in H$, let $x(t) = \Phi_t(x)$. Then, we rewrite (2.1) as

$$\dot{x} = f(x). \tag{2.2}$$

A solution of (2.2) is a differentiable function $x : I \to H$ where I is some intervals of R such that for all $t \in I$,

$$\dot{x}(t) = f(x(t)).$$

By the previous concept of dynamical system, we will propose the definition of equilibrium point and stability as follows.

Definition 2.7 ([19]). a) A point x^* is an *equilibrium point for (2.2)* if $f(x^*) = 0$;

b) An equilibrium point x^* of (2.2) is *stable* if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x_0 \in B(x^*, \delta)$, the solution $x(t)$ of the dynamical system with $x(0) = x_0$ exists and is contained in $B(x^*, \varepsilon)$ for all $t > 0$, where $B(x^*, r)$ denotes the open ball with center x^* and radius r;

c) A stable equilibrium point x^* of (2.2) is *asymptotically stable* if there exists $\delta > 0$ such that, for every solution $x(t)$ with $x(0) \in B(x^*, \delta)$, one has

$$\lim_{t\to\infty} x(t) = x^*.$$

Definition 2.8 ([20]). Let $x(t) = \Phi_t(x)$ in (2.1). For any $x^* \in K$, where K is a closed convex set, let L be a real continuous function defined on a neighborhood $N(x^*)$ of x^*, and differentiable everywhere on $N(x^*)$ except possibly at x^*. L is called a *Lyapunov function at* x^* if satisfies:

i) $L(x^*) = 0$ and $L(x) > 0$, for all $x \neq x^*$,

ii) $\dot{L}(x) \leq 0$ for all $x \neq x^*$ where

$$\dot{L}(x) = \frac{d}{dt} L(x(t))|_{t=0}. \tag{2.3}$$

Notice that, the equilibrium point x, which satisfies Definition 2.8 *ii*), is stable in the sense of Lyapunov.

Definition 2.9 ([10]). A dynamical system is said to be *globally convergent to the solution set* X of (2.2) if, irrespective of initial point, the trajectory of dynamical system satisfies

$$\lim_{t\to\infty} d(x(t), X) = 0. \tag{2.4}$$

If the set X has a unique point x^*, then (2.4) satisfies $\lim_{t\to\infty} x(t) = x^*$. If the dynamical system is still stable at x^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at x^* .

Definition 2.10 ([10]). The dynamical system is said to be *globally exponentially stable with degree* ω at x^* if, irrespective of the initial point, the trajectory of the dynamical system $x(t)$ satisfies

$$\|x(t) - x^*\| \leq c_0 \|x(t_0) - x^*\| \exp^{-\omega(t-t_0)},$$

for all $t \geq t_0$, where c_0 and ω are positive constants independent of initial point.

Notice that, if it is a globally exponentially stability then it is a globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 2.11 ([21]). *(Gronwall's inequality) Let $\hat{u}$ and $\hat{v}$ be real valued nonnegative continuous functions with domain $\{t | t \geq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If for all $t \geq t_0$,*

$$\hat{u}(t) \leq \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)ds,$$

then,

$$\hat{u}(t) \leq \alpha(t) \exp^{\int_{t_0}^{t} \hat{v}(s)ds}.$$

3. Main Results

Throughout this paper, we let H be a real Hilbert space, K be a nonempty closed convex subset of H and $CC(H)$ be the family of all nonempty closed convex subsets of H. Firstly, we will propose the quasi mixed equilibrium problem (QMEP) in Hilbert spaces as follows. Let $F : H \times H \to \mathbb{R}$ be a given bifunction satisfying $F(x,x) = 0$ for all $x \in H, T : H \to H$ be a nonlinear operator and let $C : H \to CC(H)$ be a set valued mapping which associate a nonempty closed convex set $C(x)$ with any element x of H. To find $x^* \in C(x^*)$ such that

$$F(x^*, x) + \langle T(x^*), x - x^* \rangle \geq 0, \tag{3.1}$$

for all $x \in C(x^*)$. If $C(x) = m(x) + K$ for all $x \in H$ with a fixed closed convex set K and a single valued mapping m, then the problem (3.1) is equivalent to find $x^* - m(x^*) \in K$ such that

$$F(x^*, x) + \langle T(x^*), x - x^* \rangle \geq 0, \tag{3.2}$$

for all $x \in m(x^*) + K$.

Next, we will present the special cases of the problem (3.1) as follows:

(a) If we set $C(x) = K$ for all $x \in H$ then the problem (3.1) reduces to the mixed equilibrium problem (MEP), which was a case of the mixed equilibrium problems and was studied by A. Moudafi in [5], to find $x^* \in K$ such that

$$F(x^*, x) + \langle T(x^*), x - x^* \rangle \geq 0, \tag{3.3}$$

for all $x \in K$.

(b) If we set a mapping $T = 0$ then the problem (3.1) reduces to the quasi equilibrium problem (QEP), that is to find $x^* \in C(x^*)$ such that

$$F(x^*, x) \geq 0, \tag{3.4}$$

for all $x \in C(x^*)$ and, moreover, if we set $C(x) = K$ for all $x \in H$ then the problem (3.4) reduces to the equilibrium problem (EP), that is to find $x^* \in K$ such that

$$F(x^*, x) \geq 0, \tag{3.5}$$

for all $x \in K$.

(c) If we set $F(x,y) = \varphi(y) - \varphi(x)$ for all $x, y \in K$ where $\varphi : K \to R$ is a real valued function, a mapping $T = 0$ and $C(x) = K$ then the problem (3.1) reduces to the minimization problem (MP) subject to implicit constraints, that is to find $x^* \in K$ such that

$$\varphi(x^*) \leq \varphi(x), \tag{3.6}$$

for all $x \in K$.

(d) Furthermore, if $F(x,y) = \varphi(y) - \varphi(x)$ where $\partial\varphi$ is a subdifferential of proper, convex and lower-semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ then the problem (3.1) reduces to a case of quasi variational inequality problem (QVI), that is to find $x^* \in C(x^*)$ such that

$$\varphi(x) - \varphi(x^*) + \langle T(x^*), x - x^*\rangle \geq 0, \tag{3.7}$$

for all $x \in C(x^*)$ and if $C(x) = K$ for all $x \in H$ then the problem (3.7) reduces to the mixed variational inequality which was presented by M. A. Noor [11]. Moreover, if $\varphi = 0$ then the mixed variational inequality reduces to the classical Stampacchia's variational inequality problem (VI).

The following lemma is important for solving our results.

Lemma 3.1. *Let $F : H \times H \to \mathbb{R}$ be a monotone bifunction and $C : H \to CC(H)$ be a set valued mapping and $T : H \to H$ be a nonlinear operator.*

(i) If x^* is a solution of the problem (3.1) then for any $\mu > 0$,

$$x^* = J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)). \tag{3.8}$$

(ii) If there exists $\mu > 0$ such that

$$x^* = J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)), \tag{3.9}$$

then x^* is a solution of the problem (3.1).

Proof. (i) Assume that x^* is a solution of the problem (3.1), that is, $x^* \in C(x^*)$ such that

$$F(x^*, x) + \langle T(x^*), x - x^*\rangle \geq 0,$$

for all $x \in C(x^*)$. For any $\mu > 0$, we have

$$\mu F(x^*, x) + \langle x^* - (x^* - \mu T(x^*)), x - x^*\rangle \geq 0.$$

By the definition of $J^{\mu}_{F,C(x^*)}$, we have

$$x^* = J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)).$$

(ii) Assume that $x^* = J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*))$ for some $\mu > 0$. By the definition of $J^{\mu}_{F,C(x^*)}$, we have $x^* \in C(x^*)$ with

$$\mu F(x^*, x) + \langle x^* - (x^* - \mu T(x^*)), x - x^*\rangle \geq 0,$$

for all $x \in C(x^*)$. Since $\mu > 0$, we get $x^* \in C(x^*)$ such that

$$F(x^*, x) + \langle T(x^*)), x - x^*\rangle \geq 0,$$

for all $x \in C(x^*)$. We obtain that x^* is a solution of (3.1). This completes the proof. ∎

Next, the following theorem we will present the existence theorem of the problem (3.1) and the following condition is important to solve our results.

Condition ($\mathcal{A}$) There exists $\eta > 0$ such that

$$\|J^{\mu}_{F,C(x)}(z) - J^{\mu}_{F,C(y)}(z)\| \leq \eta\|x - y\|, \tag{3.10}$$

for all $x, y, z \in H$.

Remark 3.2. If K be a closed convex subset of H, then, we see that Condition $(\mathcal{A})$ is satistied for the case $C(x) = K$ for all $x \in H$ with $\eta = 0$. We can obtain that for the case $C(x) = m(x) + K$, the condition $(\mathcal{A})$ holds when m is a Lipschitz continuous and F satisfies $F(x - y, z) = F(x, z - y)$ for all $x, y, z \in H$ (see prove in [6]).

Theorem 3.3. *Let $F : H \times H \to \mathbb{R}$ be a monotone function, $C : H \to CC(H)$ be a setvalued mapping and $T : H \to H$ be a γ-strongly monotone mapping and β-Lipschitz continuous mapping. If the condition $(\mathcal{A})$ and the following conditions hold:*

(a) $|1 - \eta| < 1$;
(b) $\beta^2 < \gamma^2$;
(c) μ *satisfies*

$$\begin{aligned} \mu \quad \in \quad & \left(\frac{\gamma - \sqrt{\gamma^2 - \beta^2(1 - (1 - \eta)^2)}}{\beta^2}, \frac{\gamma - \sqrt{\gamma^2 - \beta^2}}{\beta^2} \right) \\ & \bigcup \left(\frac{\gamma + \sqrt{\gamma^2 - \beta^2}}{\beta^2}, \frac{\gamma + \sqrt{\gamma^2 - \beta^2(1 - (1 - \eta)^2)}}{\beta^2} \right). \end{aligned}$$

Then, the problem (3.1) has a unique solution.

Proof. Define the mapping $S : H \to H$ by

$$S(x) = J^{\mu}_{F,C(x)}(x - \mu T(x)), \tag{3.11}$$

for all $x \in H$ and μ satisfies (c). Next, we will show that S is a contraction mapping. Let $x, y \in H$. We see that

$$\begin{aligned} \|S(x) - S(y)\| &= \|J^{\mu}_{F,C(x)}(x - \mu T(x)) - J^{\mu}_{F,C(y)}(y - \mu T(y))\| \\ &\leq \|J^{\mu}_{F,C(x)}(x - \mu T(x)) - J^{\mu}_{F,C(x)}(y - \mu T(y))\| \\ &\quad + \|J^{\mu}_{F,C(x)}(y - \mu T(y)) - J^{\mu}_{F,C(y)}(y - \mu T(y))\| \\ &\leq \|x - \mu T(x) - y + \mu T(y)\| + \eta\|x - y\|. \end{aligned} \tag{3.12}$$

By the assumption of mapping T, we have

$$\begin{aligned} \|(x - y) - \mu(T(x) - T(y))\|^2 &= \|x - y\|^2 - 2\mu\langle T(x) - T(y), x - y\rangle + \mu^2\|T(x) - T(y)\|^2 \\ &\leq \|x - y\|^2 - 2\mu\gamma\|x - y\|^2 + \mu^2\beta^2\|x - y\|^2 \\ &= (1 - 2\mu\gamma + \mu^2\beta^2)\|x - y\|^2. \end{aligned}$$

That is,

$$\|x - \mu T(x) - y + \mu T(y)\| \leq \sqrt{1 - 2\mu\gamma + \mu^2\beta^2}\|x - y\|. \tag{3.13}$$

Replacing (3.13) in (3.12), we have

$$\begin{aligned} \|S(x) - S(y)\| &\leq \sqrt{1 - 2\mu\gamma + \mu^2\beta^2}\|x - y\| + \eta\|x - y\| \\ &= \left(\sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \eta\right)\|x - y\| \\ &= \theta\|x - y\|, \end{aligned}$$

where $\theta = \sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \eta$. By the assumption of μ, we obtain that $\theta < 1$. Thus, S has a unique fixed point in H. This implies that there exists $x \in H$ such that

$$x = J^{\mu}_{F,C(x)}(x - \mu T(x)), \quad \text{for some} \quad \mu > 0.$$

By Lemma 3.1, we obtain that the problem (3.1) has a unique solution. This completes the proof. ∎

Next, we will present the resolvent equation, which is equivalent to the quasi mixed equilibrium problem (QMEP). Starting with let x^* be fixed in H and μ be a fixed positive constant. Let $F : H \times H \to \mathbb{R}$ be bifunction, $T : H \to H$ be a nonlinear mapping and $C : H \to CC(H)$ be a set valued mapping. We consider to find $z^* := z^*(\mu, x^*) \in H$ such that

$$R^{\mu}_{F,C(x^*)}(z^*) + \mu T J^{\mu}_{F,C(x^*)}(z^*) = 0, \tag{3.14}$$

where $R^{\mu}_{F,C(x)} \equiv I - J^{\mu}_{F,C(x)}$ with $J^{\mu}_{F,C(x)}$ is a resolvent operator for all $x \in H$. Then, (3.14) is called the resolvent equation.

The following lemma, we will show the relation between a solution of the problem (3.1) and a solution of the problem (3.14).

Lemma 3.4. *Let $F : H \times H \to \mathbb{R}$ be a bifunction, $C : H \to CC(H)$ be a set valued mapping and $T : H \to H$ be a single valued mapping. Then, the problem (3.1) has a solution x^* if and only if the problem (3.14) has a solution $z^* \in H$ where*

$$x^* = J^{\mu}_{F,C(x^*)}(z^*) \quad \text{and} \quad z^* = x^* - \mu T(x^*), \tag{3.15}$$

with μ is a positive constant.

Proof. ($\Rightarrow$) If x^* is a solution of the problem (3.1), then it follows from Lemma 3.1 that for any $\mu > 0$,

$$x^* = J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)).$$

Since $R^{\mu}_{F,C(x^*)} \equiv I - J^{\mu}_{F,C(x^*)}$, we see that

$$\begin{aligned} R^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) &= (I - J^{\mu}_{F,C(x^*)})(x^* - \mu T(x^*)) \\ &= x^* - \mu T(x^*) - J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) \\ &= -\mu T(x^*). \end{aligned}$$

This implies that $R^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) + \mu T(x^*) = 0$. Hence,

$$R^{\mu}_{F,C(x^*)}(z^*) + \mu T(J^{\mu}_{F,C(x^*)}(z^*)) = 0,$$

where $z^* = x^* - \mu T(x^*)$. Therefore, z^* is a solution of the problem (3.14).
($\Leftarrow$) Assume that $z^* \in H$ is a solution of (3.14) and satisfies (3.15), we have

$$x^* = J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) \quad \text{for some} \quad \mu > 0.$$

By Lemma 3.1, we have x^* is a solution of (3.1). This completes the proof. ∎

Notice that if x^* is a solution of the problem (3.1), by Lemma 3.4 we have $z^* = x^* - \mu T(x^*)$, for some $\mu > 0$, is a solution of (3.14). By the resolvent equation (3.14), this implies that

$$R^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) + \mu T J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) = 0.$$

Since $R^{\mu}_{F,C(x^*)} \equiv I - J^{\mu}_{F,C(x^*)}$, we obtain that

$$x^* - \mu T(x^*) - J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) + \mu T J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) = 0.$$

Now, we use the equivalent formulation to suggest a dynamical system associated with the quasi mixed equilibrium problem (DSQMEP). Let $F : H \times H \to \mathbb{R}$ be a bifunction, $C : H \to CC(H)$ be a set valued mapping and $T : C(x) \to C(x)$ be a nonlinear single valued mapping for all $x \in H$. Fixed $x^* \in C(x^*)$. Then, the problem (DSQMEP) as follows:

$$\frac{dx^*}{dt} = \lambda \left\{ J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) - \mu T J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) + \mu T(x^*) - x^* \right\}, \quad (3.16)$$

where $x(t_0) = x_0$ in H and λ is a positive constant with a positive real number t_0.

Notice that the right hand side is related to the resolvent and it is discontinuous on the boundary of $C(x^*)$. It is clear from the definition that the solution to (3.16) always stay in the constraint set $C(x^*)$ for $x^* \in C(x^*)$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data of (3.16) can be studied.

Theorem 3.5. *Let $T : C(x) \to C(x)$ be a nonlinear mapping for all $x \in H$. Assume that all of assumptions of Theorem 3.3 hold. Then, for each $x_0 \in H$, there exists the unique continuous solution $x(t)$ of the problem (3.16) with $x(t_0) = x_0$ over $[t_0, \infty)$.*

Proof. Let λ be a positive constant and define the mapping $A : H \to H$ by

$$A(x) = \lambda \left\{ J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T(x) - x \right\},$$

with $x \in C(x)$ for $x \in H$ and for some $\mu > 0$. Next, we will show that A is a Lipschitz continuous mapping. Let $x \in C(x)$ and $y \in C(y)$. By using the nonexpansive mapping of $J^{\mu}_{F,C(x)}$ and (3.13), we see that

$$\begin{aligned}
\|A(x) - A(y)\| &= \|\lambda \left\{ J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T(x) - x \right\} \\
&\quad - \lambda \left\{ J^{\mu}_{F,C(y)}(y - \mu T(y)) - \mu T J^{\mu}_{F,C(y)}(y - \mu T(y)) + \mu T(y) - y \right\} \| \\
&\leq \lambda\{\|J^{\mu}_{F,C(x)}(x - \mu T(x)) - J^{\mu}_{F,C(y)}(y - \mu T(y))\| \\
&\quad + \|\mu T J^{\mu}_{F,C(y)}(y - \mu T(y)) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x))\| + \|\mu T(x) - \mu T(y)\| \\
&\quad + \|y - x\|\} \\
&\leq \lambda\{(1+\mu\beta)\|J^{\mu}_{F,C(x)}(x - \mu T(x)) - J^{\mu}_{F,C(y)}(y - \mu T(y))\| + (1 + \mu\beta)\|x - y\|\} \\
&\leq \lambda(1 + \mu\beta)\{\|x - \mu T(x) - y + \mu T(y)\| + \eta\|x - y\| + \|x - y\|\} \\
&\leq \lambda(1 + \mu\beta)(\sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \eta + 1)\|x - y\|.
\end{aligned}$$

This implies that A is Lipschitz continuous. Hence, for each $x_0 \in H$, there exists a unique continuous solution $x(t) \in C(x(t))$ of the problem (3.16), defined in an interval $t_0 \leq t < \Gamma$ with the interval condition $x(t_0) = x_0$.

Let $[t_0, \Gamma)$ be its maximal interval of existence, we will show that $\Gamma = \infty$. By the assumptions of Theorem 3.3 hold and Lemma 3.1, we have the problem (3.1) has a unique solution $x^* \in C(x^*)$ such that

$$x^* = J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)),$$

for some $\mu > 0$. Let $x \in C(x)$. By using (3.13), we obtain that

$$\begin{aligned}
\|Ax\| &= \left\| \lambda \left\{ J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T(x) - x \right\} \right\| \\
&\leq \lambda \left\{ \left\| J^{\mu}_{F,C(x)}(x - \mu T(x)) - x \right\| + \left\| \mu T(x) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) \right\| \right\} \\
&\leq \lambda(1 + \mu\beta) \left\| J^{\mu}_{F,C(x)}(x - \mu T(x)) - x \right\| \\
&= \lambda(1 + \mu\beta) \left\| J^{\mu}_{F,C(x)}(x - \mu T(x)) - x^* + x^* - x \right\| \\
&\leq \lambda(1 + \mu\beta) \left\{ \left\| J^{\mu}_{F,C(x)}(x - \mu T(x)) - J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*)) \right\| + \|x^* - x\| \right\} \\
&\leq \lambda(1 + \mu\beta)(\sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \eta + 1)\{\|x^*\| + \|x\|\} \\
&= \lambda(1 + \mu\beta)(\sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \eta + 1)\|x^*\| \\
&\quad + \lambda(1 + \mu\beta)(\sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \eta + 1)\|x\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x(t)\| &\leq \|x(t_0)\| + \int_{t_0}^{t} \|A(x(s))\| ds \\
&\leq \|x(t_0)\| + \int_{t_0}^{t} (\lambda(1 + \mu\beta)(1 + \eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2})\|x(s)\| \\
&\quad + \lambda(1 + \mu\beta)(1 + \eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2})\|x^*\|) ds \\
&= \|x(t_0)\| + k_1(t - t_0) + k_2 \int_{t_0}^{t} \|x(s)\| ds,
\end{aligned}$$

where $k_1 = \lambda(1 + \mu\beta)(1 + \eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2})\|x^*\|$ and $k_2 = \lambda(1 + \mu\beta)(1 + \eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2})$. By Gronwall's Lemma, we obtain that

$$\|x(t)\| \leq \{\|x(t_0)\| + k_1(t - t_0)\} e^{k_2(t - t_0)},$$

where $t \in [t_0, \Gamma)$. Therefore, the solution $x(t)$ is bounded on $[t_0, \Gamma)$, if Γ is finite. We conclude that $\Gamma = \infty$. This completes the proof. ∎

Theorem 3.6. *Assume that all of the assumptions of Theorem 3.5 hold and satisfy the following conditions:*

$$\eta < \frac{1 - \mu\beta}{1 + \mu\beta} - \sqrt{1 - 2\mu\gamma + \mu^2\beta^2} \tag{3.17}$$

$$\textit{where } \gamma > \frac{4\beta + \mu\beta^2(1 + \mu\beta)^2}{2(1 + \mu\beta)^2}. \tag{3.18}$$

Then the problem (3.16) converges a globally exponentially stable to the unique solution of the problem (3.1).

Proof. By Theorem 3.5, we known that the problem (3.16) has a unique continuous solution $x(t) \in C(x(t))$ over $[t_0, \Gamma)$ for any fixed $x_0 \in H$. Let $x_0(t) = x(t, t_0; x_0)$ be a solution of the initial value problem (3.16) and, by Theorem 3.3, there exists a solution of the problem (3.1), $x^* \in C(x^*)$. Now, we define Lyapunov function as follows: Let $L : H \to R$ by

$$L(x) = \frac{1}{2}\|x - x^*\|^2,$$

with $x \in C(x)$ for any $x \in H$. We see that

$$\begin{aligned}
\frac{dL}{dt} &= \frac{dL}{dx} \cdot \frac{dx}{dt} \\
&= \left\langle x - x^*, \lambda \left\{ J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T(x) - x \right\} \right\rangle \\
&= -\lambda \left\langle x - x^*, x - J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T(x) \right\rangle \\
&= -\lambda \left\langle x - x^*, x - x^* + x^* - J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T(x) \right\rangle \\
&= -\lambda\|x - x^*\|^2 + \lambda \left\langle x - x^*, J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T(x) - x^* \right\rangle \\
&\leq -\lambda\|x - x^*\|^2 + \lambda\|x - x^*\|\|J^{\mu}_{F,C(x)}(x - \mu T(x)) - \mu T J^{\mu}_{F,C(x)}(x - \mu T(x)) + \mu T(x) - x^*\| \\
&\leq -\lambda\|x - x^*\|^2 \\
&\quad + \lambda\|x - x^*\|\{\|J^{\mu}_{F,C(x)}(x - \mu T(x)) - J^{\mu}_{F,C(x^*)}(x^* - \mu T(x^*))\| \\
&\quad + \mu\beta\|x - x^* + x^* - J^{\mu}_{F,C(x)}(x - \mu T(x))\|\} \\
&\leq -\lambda\|x - x^*\|^2 \\
&\quad + \lambda\|x - x^*\|\{(\eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2})\|x - x^*\| + \mu\beta\|x - x^*\| \\
&\quad + \mu\beta(\eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2})\|x^* - x\|\} \\
&\leq -\lambda\|x - x^*\|^2 + \lambda(\eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \mu\beta + \mu\beta(\eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2}))\|x - x^*\|^2 \\
&= \lambda(\eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \mu\beta + \mu\beta(\eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2}) - 1)\|x - x^*\|^2 \\
&\leq \lambda\omega\|x - x^*\|,
\end{aligned}$$

where $\omega = \eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2} + \mu\beta + \mu\beta(\eta + \sqrt{1 - 2\mu\gamma + \mu^2\beta^2}) - 1$. By the assumption of (3.17) and (3.18), we obtain that $\omega < 0$. Thus,

$$\begin{aligned}
\|x(t) - x^*\| &\leq \|x_0 - x^*\| + \int_{t_0}^{t} \|L(x(s))\|ds \\
&\leq \|x_0 - x^*\| + \omega \int_{t_0}^{t} \|x(s) - x^*\|ds \\
&\leq \|x_0 - x^*\|e^{\omega(t - t_0)}.
\end{aligned}$$

Since $\omega < 0$, we have the problem (3.16) is a globally exponentially stable with degree $-\omega$ at x^*. We conclude that the solutions of the problem (3.16) converges globally exponentially to the unique solution of the problem (3.1). This completes the proof. ■

Remark 3.7. In our results, for any $x, y \in H$, if we let $F(x, y) = \varphi(y) - \varphi(x)$ where $\partial\varphi$ is a subdifferential of proper, convex and lower-semicontinuous function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ and $C(x) = K$ for all $x \in H$ and K is a closed and convex subset of H, then the problem (3.1) reduces to the mixed variational inequality problem which was studied by M. A.

Noor [11], that is, to find $x^* \in K$ such that

$$\varphi(x) - \varphi(x^*) + \langle T(x^*), x - x^* \rangle \geq 0, \tag{3.19}$$

for all $x \in K$. By the lemma of H. Brezis [22] that for a give $z \in H, u \in H$ satisfies the inequality

$$\langle u - z, v - u \rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \forall v \in H,$$

if and only if

$$u = J_\varphi(z),$$

where $J_\varphi(u) := (I + \rho\partial\varphi)^{-1}(u)$ for all $u \in H$ with $\partial\varphi$ is a subdifferential of a proper, convex and lower-semicontinuous function and $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a maximal monotone operators and $\rho > 0$ is a constant. Thus, in this case, $J^\mu_{F,K}(x)$ which is defined in Definition 2.5 is $J_\varphi(x)$. We can be obtain the same as results of M. A. Noor [11] that is

$$\frac{dx}{dt} = \lambda \left\{ J_\varphi(x - \mu T(x)) - \mu T J_\varphi(x - \mu T(x)) + \mu T(x) - x \right\}. \tag{3.20}$$

So the following corollaries are obtained which is the same results of M. A. Noor [11].

Corollary 3.8. *Let $T : H \to H$ be a γ-strongly monotone mapping and a β-Lipschitz continuous mapping and the following conditions satisfy:*

(a) $0 < 1 - 2\mu\gamma + \mu^2\beta^2 < 1$;

(b) $\beta^2 < \gamma^2$.

Then, the problem (3.20) has a unique solution.

Corollary 3.9. *Assume that all of assumptions of Corollary 3.8 hold. Then, for each $x_0 \in H$, there exists a unique continuous solution $x(t)$ of the problem (3.20) with $x(t_0) = x_0$ over $[t_0, \infty)$.*

Corollary 3.10. *Assume that all of the assumptions of Corollary 3.8 hold and satisfy the condition (3.18). Then the problem (3.20) converges a globally exponentially stable to the unique solution of the problem (3.19).*

Remark 3.11. Furthermore, if we let $F = 0$ then the problem (3.16) reduces to the problem (3.7) and, moreover, if $C(x) = K$ for all $x \in H$ then the problem (3.16) reduces to the Stampacchia's variational inequality problem, that is, find $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \tag{3.21}$$

for all $x \in K$. By the well known the projection property, for given $x \in H$ and $z \in K$ satisfy

$$\langle z - x, y - z \rangle \geq 0, \forall y \in K,$$

if and only if

$$z = Proj_K(x),$$

where $Proj_K$ is the projection of H onto K. Hence, in this case, the $J^\mu_{F,K}(x)$ which is defined in Definition 2.5 is $Proj_K(x)$. Hence, the problem (3.16) reduced to

$$\frac{dx}{dt} = \lambda \left\{ Proj_K(x - \mu T(x)) - \mu T Proj_K(x - \mu T(x)) + \mu T(x) - x \right\}, \tag{3.22}$$

which this problem was studied by M. A. Noor [12]. In the same assumptions of Corollary 3.8 to Corollary 3.10, we can obtain the same results in [12], that is, there exists a unique continuous solution and the problem (3.22) converges a globally exponential stable to the unique solution of the problem (3.21).

4. Conclusion

In this work, we showed the existence of the quasi mixed equilibrium problem. To present the dynamical system associated with the quasi mixed equilibrium problem, we considered the resolvent equation which is equivalent to the quasi mixed equilibrium problem, so we obtained that if a solution of the quasi mixed equilibrium problem exists then there exists a solution of such resolvent equation and conversely; obvious. Using the previous relation, we introduced the dynamical system associated with the quasi mixed equilibrium problem. Furthermore, the existence and the convergent globally exponential of such dynamical system was presented.

Acknowledgements

This work was supported by Thailand Research Fund (Project No. MRG6280117) and Pibulsongkram Rajabhat University.

References

[1] O. Chadli, N.C. Wong, J.C. Yao, Equilibrium problems with apllications to eigenvalue problems, J. Optim. Theo. Appl. 117 (2) (2003) 245–266.

[2] A. Moudafi, M. Thera, Proximal and dynamical approaches to equilibrium problems, Ill-Posed Variational Problems and Regularization Techniques, Springer. Berlin, Heidelberg (1999), 187–201 .

[3] L. Scrimali, The financial equlibrium problem with implicit budget constraints, Cent. Eur. J. Oper. Res. 16 (2008) 191–203.

[4] L. Scrimali, Pollution control quasi-equilibrium problems with joint implementation of environmental projects, Appl. Math. Lett. 25 (3) (2012) 385–392.

[5] A. Moudafi, Mixed equilibrium problems: sensitivity analysis and algorithmic aspect, Computers & Mathematics with Application. 44 (2002) 1099–1108.

[6] S. Suantai, N. Petrot, Existence and stability of iterative algorithms for the system of nonlinear quasi-mixed equilibrium problems, Appl. Math. Lett. 24 (2011) 308–313.

[7] S.Kr. Biswas, S. Chakraborty, Interacting dark energy in $f(T)$ cosmology: A dynamical system analysis, International Journal of Modern Physics D. 24 (7) (2015).

[8] M. Bahiana, Y. Oono, Cell dynamical system approach to block copolymers, Physical Review A. 41 (12) (1990) 6763–6771.

[9] P. Dupuis, A. Nagurney, Dynamical systems and variational inequalities, Ann. Oper. Res. 44 (1993) 9–42.

[10] M.A. Noor, K.I. Noor, A.G. Khan, Dynamical systems for quasi variational inequalities, Ann. Funct. Anal. 6 (2015) doi: 10.15352/afa/06-1-14.

[11] M.A. Noor, Resolvent dynamical systems for mixed variational inequalities, Korean J. Comput. & Appl. Math. 9 (1) (2002) 15–26.

[12] M.A. Noor, Weiner-Hopf dynamical system for variational inequality, New Zealand J. Math. 31 (2002) 173–182.

[13] Q.H. Ansari, J. Balooee, J.C. Yao, Extended general nonlinear quasi variational inequalities and projection dynamical systems, Taiwanese J. Math. 17 (4) (2013) 1321–1352.

[14] J. Balooee, Y.J. Cho, Algorithms for solutions of extended general mixed variational inequalities and fixed points, Optim. Lett. 7 (2013) 1929–1955.

[15] M.G. Cojocaru, Monotonicity and existence of periodic orbits for projected dynamical systems on Hilbert spaces, Proceeding of the American Mathematical Society 134 (3) (2005) 793–804.

[16] J. Dong, D. Zhang, A. Nagurney, A projected dynamical systems model of general financial equilibrium with stability analysis, Mathl. Comput. Modelling 24 (2) (1996).

[17] A.M. Harder, T.L. Hick, Stability results for fixed-point iteration procedures, Math. Japonica. 33 (5) (1998) 693–706.

[18] M.W. Hirsch, S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic press, New York, 1974.

[19] N.T.T. Ha, J.J. Strodiot, P.T. Vuong, On the global exponential stability of a projected dynamical system for strongly pseudomonotone variational inequlities, Optim. Lett. 12 (2018) 1–14.

[20] A. Nagurney, A.D. Zhang, Projected Dynamical Systems and Variational Inequalities with Applications, Kluwer Academic, Boston, 1996.

[21] R.K. Miller, A.N. Michel, Ordinary Differential Equations, Academic Press, New York, 1982.

[22] H. Brezis, Operators Maximaux Monotone et Semigroupes de Contractions dan les Espaces de Hilbert, Nortth-Holland, Amsterdam, 1973.