On $\phi$–Quasiconformally Symmetric $N(k)$–Contact Metric Manifolds

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Abstract: The object of the present paper is to study locally and globally $\phi$–quasiconformally symmetric $N(k)$–metric manifolds. We prove that a globally $\phi$–quasiconformally $N(k)$–contact metric manifold $M^{2n+1}(n \geq 1)$ is Sasakian. Some observations for a 3-dimensional locally $\phi$–symmetric $N(k)$–contact metric manifold are given. We also give an example of a 3-dimensional locally $\phi$–quasiconformally symmetric $N(k)$–contact metric manifold.

Keywords: $N(k)$–contact manifold; quasiconformal curvature tensor; $\eta$–Einstein manifold.

2010 Mathematics Subject Classification: 53C25; 53C40.

1 Introduction

The notion of locally symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahasi $^\dagger$ introduced the notion of locally $\phi$–symmetry, De et al. $^\ddagger$ introduced the notion of $\phi$–recurrent Sasakian manifold. In the context of contact geometry the notion of $\phi$–symmetry is introduced and studied by Boeckx, Bueken and Vanhecke $^\ddagger$ with several examples. In a recent paper De and Gazi

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studied locally $\phi-$recurrent $N(k)-$contact metric manifolds. Also De, Özgür and Mondal [5] studied $\phi-$quasiconformally symmetric Sasakian manifolds. In the present paper we study $\phi-$quasiconformally symmetric $N(k)-$contact metric manifolds which generalizes the results of De, Özgür and Mondal [5] and also the result of Blair, Koufogiorgos and Sharma [6].

Let $(M, g)$ be a $(2n+1)$, $(n \geq 1)$-dimensional Riemannian manifold. The notion of the quasiconformal curvature tensor was introduced by Yano and Sawaki [7]. According to them a quasiconformal curvature tensor is defined by

$$C^*(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{2n+1} \left[ a \frac{a}{2n} + 2b \right] [g(Y,Z)X - g(X,Z)Y],$$

where $a, b$ are constants, $S$ is the Ricci tensor, $Q$ is the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and $r$ is the scalar curvature of the manifold $M$. If $a = 1$ and $b = -\frac{1}{2n-1}$, then (1.1) takes the form

$$C^*(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{2n} [g(Y,Z)X - g(X,Z)Y] = C(X,Y)Z,$$

where $C$ is the conformal curvature tensor. In [8], De and Matsuyama studied quasiconformally flat Riemannian manifolds satisfying certain condition on the Ricci tensor. From Theorem 5 of [8], it can be proved that a 4-dimensional quasiconformally flat semi-Riemannian manifold is the Robertson-Walker spacetime, Robertson-Walker spacetime is the warped product $I \times f M^*$, where $M^*$ is a space of constant curvature and $I$ is an open interval [9]. From (1.1), we obtain,

$$(\nabla W C^*)(X,Y)Z = a(\nabla W R)(X,Y)Z + b[(\nabla W S)(Y,Z)X - (\nabla W S)(X,Z)Y + g(Y,Z)(\nabla W Q)X - g(X,Z)(\nabla W Q)Y] - \frac{dr(W)}{2n+1} \left[ a \frac{a}{2n} + 2b \right] [g(Y,Z)X - g(X,Z)Y].$$

If the condition $\nabla C^* = 0$ holds on $M$, then $M$ is called quasiconformally symmetric, where $\nabla$ denotes the Levi-Civita connection on $M$. It is known [10] that a quasiconformally symmetric $N(k)-$contact metric manifold for $k \neq 0$ is a manifold of constant curvature $k$. This fact means that a quasiconformally symmetric condition is too strong for a $N(k)-$contact metric manifold. In [1], Takahashi introduced a weaker condition which is locally symmetry for a Sasakian manifold that satisfies the condition

$$\phi^2(\nabla X R)(Y,Z)W = 0,$$
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where $X,Y,Z,W$ are horizontal vector fields which means that it is horizontal with respect to the contact form $\eta$ of the local fibering, namely, a horizontal vector is nothing but a vector which is orthogonal to $\xi$. In [6], Blair, Koufogirgos and Sharma studied locally $\phi-$symmetric 3-dimensional $N(k)-$contact metric manifolds.

In [1,4], if $X,Y,Z,W$ are not horizontal vectors, then we call the manifold globally $\phi-$symmetric.

In this paper we introduce a weaker condition than quasiconformally symmetry that satisfies

$$\phi^2(\nabla_W C^*)(X,Y)Z = 0,$$

which is called globally $\phi-$quasiconformally symmetric for arbitrary vector fields $X,Y,Z,W$ on $M$. If $X,Y,Z,W$ are horizontal vectors, then the manifold is called locally $\phi-$quasiconformally symmetric.

The paper is organized as follows: After preliminaries in Section 3, we consider globally $\phi-$quasiconformally symmetric $N(k)-$contact metric manifolds and prove that such a $N(k)-$contact metric manifold is Sasakian. Section 4 deals with 3-dimensional locally $\phi-$quasiconformally symmetric $N(k)-$contact metric manifold. We prove that a 3-dimensional $N(k)-$contact metric manifold is locally $\phi-$quasiconformally symmetric if and only if it is locally $\phi-$symmetric. Finally we construct an example of a 3-dimensional locally $\phi-$quasiconformally symmetric $N(k)-$contact metric manifold.

2 Preliminaries

A $(2n + 1)-$dimensional manifold $M$ is said to admit an almost contact metric structure if it admits a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$(a) \quad \phi^2 = -I + \eta \otimes \xi, \quad (b) \quad \eta(\xi) = 1, \quad (c) \quad \phi \xi = 0 \quad \text{and} \quad (d) \quad \eta \circ \phi = 0. \quad (2.1)$$

An almost contact metric structure is said to be normal if the induced almost complex structure $J$ on the product manifold $M \times R$ defined by

$$J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X), \frac{d}{dt})$$

is integrable, where $X$ is tangent to $M$, $t$ is the coordinate of $R$ and $f$ is a smooth function on $M \times R$. Let $g$ be a compatible Riemannian metric with almost contact structure $(\phi, \xi, \eta)$, that is

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2.2)$$

Then $M$ becomes an almost contact metric manifold equipped with an almost contact metric structure $(\phi, \xi, \eta, g)$. From (2.1) and (2.2) it can be easily seen that

$$(a) \quad g(X, \phi Y) = -g(\phi X, Y), \quad (b) \quad g(X, \xi) = \eta(X), \quad (2.3)$$
for all vector fields $X$, $Y$. An almost contact metric structure becomes a contact metric structure if
\[ g(X, \phi Y) = d\eta(X, Y), \] (2.4)
for all vector fields $X$, $Y$. The 1-form $\eta$ is then called a contact form and $\xi$ is its characteristic vector field. We define a $(1,1)$ tensor field $h$ by $h = \frac{1}{2} \mathcal{L}_{\xi} \phi$, where $\mathcal{L}$ denotes the Lie derivative. Then $h$ is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also
\[ \nabla_X \xi = -\phi X - \phi hX \] (2.5)
holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if
\[ (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM, \] (2.6)
where $\nabla$ is the Levi-Civita connection of the Riemannian metric $g$. A contact metric manifold $M(\phi, \xi, \eta, g)$ for which $\xi$ is a Killing vector field is said to be a $K$-contact manifold. A Sasakian manifold is $K$-contact but not conversely. However a 3-dimensional $K$-contact manifold is Sasakian [11]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [12]. On the other hand on a Sasakian manifold the following relation holds:
\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \] (2.7)
As a generalisation of both $R(X, Y)\xi = 0$ and the Sasakian case : D. E. Blair, T. Koufogiorgos and B. J. Papantoniou [13] introduced the $(k, \mu)$-nullity distribution on a contact metric manifold and gave several reasons for studying it. The $(k, \mu)$-nullity distribution $N(k, \mu)$ of a contact metric manifold $M$ is defined by
\[ N(k, \mu) : p \rightarrow N_p(k, \mu) = \{ W \in T_p M : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y) \}, \]
for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold $M$ with $\xi \in N(k, \mu)$ is called a $(k, \mu)$-contact manifold. In particular on a $(k, \mu)$-contact manifold, we have
\[ R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \] (2.8)
On a $(k, \mu)$-contact manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and $\mu$ is indeterminant) and if $k < 1$, then the $(k, \mu)$-nullity condition determines the curvature of $M$ completely [13]. In fact, for a $(k, \mu)$-contact manifold, the condition of being Sasakian, a $K$-contact manifold, $k = 1$ and $h = 0$ are all equivalent.

The $k$-nullity distribution $N(k)$ of a Riemannian manifold $M$ is defined by [14]
\[ N(k) : p \rightarrow N_p(k) = \{ Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] \}, \]
Let \( k \) be a constant. If the characteristic vector field \( \xi \in N(k) \), then we call the manifold an \( N(k) \)-contact metric manifold \([14]\). If \( k = 1 \), then the manifold is Sasakian and if \( k = 0 \), then the manifold is locally isometric to the product \( E^{n+1}(0) \times S^n(4) \) for \( n > 1 \) and flat for \( n = 1 \) \([12]\). In a \((k,\mu)\)-contact manifold if \( \mu = 0 \), then the manifold becomes an \( N(k) \)-contact manifold.

In \([15]\), \( N(k) \)-contact metric manifold were studied in details. For more details we refer to \([6], [16]\).

In a \((2n+1)\)-dimensional \( N(k) \)-contact metric manifold \( M \), the following relations hold:

\[
\begin{align*}
\phi^2(Y) &= (k-1)\phi^2, \quad k \leq 1, \\
(\nabla_X \phi)(Y) &= g(X + hX, Y)\xi - \eta(Y)(X + hX), \\
R(\xi, X)Y &= k[g(X, Y)\xi - \eta(Y)X], \\
S(X, \xi) &= 2n\kappa\eta(X), \\
S(X, Y) &= 2(n-1)g(X, Y) + 2(n-1)g(hX, Y) \\
&\quad + [2(1-n) + 2nk]\eta(X)\eta(Y), \quad m \geq 1, \\
r &= 2n(2n - 2 + k),
\end{align*}
\]

\[
S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX, Y),
\]

\[
(\nabla_X \eta)(Y) = g(X + hX, \phi Y),
\]

\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],
\]

\[
\eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],
\]

for any vector fields \( X, Y, Z \) where \( R \) is the Riemannian curvature tensor and \( S \) is the Ricci tensor.

### 3 Globally \( \phi \)-Quasiconformally Symmetric \( N(k) \)-Contact Metric Manifolds

**Definition 3.1.** A \( N(k) \)-contact metric manifold \( M \) is said to be globally \( \phi \)-quasiconformally symmetric if the quasiconformal curvature tensor \( C^* \) satisfies

\[
\phi^2(\nabla_W C^*)(X, Y)Z = 0,
\]

for all vector fields \( X, Y, Z, W \in \chi(M) \).

A contact metric manifold is said to be an \( \eta \)-Einstein manifold if the Ricci tensor of the manifold is of the form

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]

where \( a, b \) are smooth functions on \( M \) and \( X, Y \in \chi(M) \).

Here we state the following Lemma due to Baikoussis and Koufogiorgos \([17]\):
Lemma 3.2. Let $M$ be an $\eta$-Einstein manifold of dimension $(2n + 1), (n \geq 1)$. If $\xi$ belongs to the $k$-nullity distribution, then $k = 1$ and the structure is Sasakian.

Let us suppose that the manifold $M$ is globally $\phi$–quasi conformally symmetric $N(k)$–contact metric manifold. Then by definition

$$\phi^2(\nabla^W C^*)(X, Y)Z = 0. \tag{3.3}$$

Using (2.1)(a), we have

$$-(\nabla^W C^*)(X, Y)Z + \eta((\nabla^W C^*)(X, Y)Z)\xi = 0. \tag{3.4}$$

Using (1.3) in (3.4), it follows that

$$-ag((\nabla^W R)(X, Y)Z, U) - bg(X, U)(\nabla^W S)(Y, Z) + bg(Y, U)(\nabla^W S)(X, Z) - bg(Y, Z)g((\nabla^W Q)X, U) + bg(X, Z)g((\nabla^W Q)Y, U) + \frac{1}{2n + 1} dr(W)[\frac{a}{2n + 2b}]$$

$$[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)] + an((\nabla^W R)(X, Y)Z)\eta(U) + b(\nabla^W S)(Y, Z)\eta(U)\eta(X) - b(\nabla^W S)(X, Z)\eta(Y)\eta(U) + bg(Y, Z)\eta((\nabla^W Q)X)\eta(U) - bg(X, Z)\eta((\nabla^W Q)Y)\eta(U) - \frac{1}{2n + 1} dr(W)[\frac{a}{2n + 2b}][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\eta(U) = 0. \tag{3.5}$$

Put $X = U = e_i$, in (3.5), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i$, we get

$$-a + (2n - 1)b[\nabla^W S](Y, Z) - \{bg((\nabla^W Q)e_i, e_i) - \frac{2n - 1}{2n + 1} dr(W)[\frac{a}{2n + 2b}] - b\eta((\nabla^W Q)\xi)\}g(Y, Z) + bg((\nabla^W Q)Y, Z) + an((\nabla^W R)(\xi, Y)Z) - b(\nabla^W S)(\xi, Z)\eta(Y) - b\eta((\nabla^W Q)Y)\eta(Z) + \frac{1}{2n + 1} dr(W)[\frac{a}{2n + 2b}][\eta(Y)]\eta(Z) = 0. \tag{3.6}$$

Putting $Z = \xi$ in (3.6) and using (2.1)(a) and (2.3)(b), we obtain

$$-a + (2n - 1)b[\nabla^W S](Y, \xi) - \{bdr(W) - \frac{2n - 1}{2n + 1} dr(W)[\frac{a}{2n + 2b}] - b\eta((\nabla^W Q)\xi)\}\eta(Y) + an((\nabla^W R)(\xi, Y)\xi) - b(\nabla^W S)(\xi, \xi)\eta(Y) + \frac{1}{2n + 1} dr(W)[\frac{a}{2n + 2b}]\eta(Y) = 0. \tag{3.7}$$

Now

$$\eta((\nabla^W Q)\xi) = g(\nabla^W Q\xi, \xi) - g(Q(\nabla^W \xi), \xi) = S(\phi X, \xi) + S(\phi h X, \xi) = 0, \tag{3.8}$$
Again
\[
g((\nabla_w R)(\xi, Y)\xi, \xi) = g(\nabla_w R(\xi, Y)\xi, \xi) - g(\nabla_w R(\xi, Y)\xi) - g(R(\xi, \nabla_w Y)\xi, \xi) - g(R(\xi, Y)\nabla_w \xi, \xi). \tag{3.9}
\]
From (2.11), we get by using (2.1)(b)
\[
g(R(\xi, Y)\xi, \xi) = 0.
\]
Since \(\nabla g = 0\), we obtain from above
\[
g(\nabla_w R(\xi, Y)\xi, \xi) + g(R(\xi, Y)\xi, \nabla_w \xi) = 0. \tag{3.10}
\]
Again using (2.11), we have
\[
g(R(\xi, \nabla_w Y)\xi, \xi) = kg(\eta(\nabla_w Y)\xi - \nabla_w Y, \xi) = k[\eta(\nabla_w Y) - \eta(\nabla_w Y)] = 0. \tag{3.11}
\]
By using (2.5), (2.11) and (2.1)(d), we have
\[
g(R(\xi, \nabla_w Y)\xi, \xi) = g(R(-\phi W - \phi h W, Y)\xi, \xi)
\[= -g(R(\phi W, Y)\xi, \xi) - g(R(\phi h W, Y)\xi, \xi)
\[= -kg(\eta(Y)\phi W - \eta(\phi W)Y, \xi) - kg(\eta(Y)\phi h W - \eta(\phi h W)Y, \xi)
\[= -k\eta(Y)g(\phi W, \xi) - k\eta(Y)g(\phi h W, \xi)
\[= 0 \quad \text{(since } \phi \text{ is skew symmetric and } \phi \xi = 0). \tag{3.12}
\]
Using (3.10), (3.11) and (3.12) in (3.9) yields
\[
g((\nabla_w R)(\xi, Y)\xi, \xi) = 0. \tag{3.13}
\]
From (2.11) by using (2.5) and \(\phi \xi = 0\), we get
\[
(\nabla_w S)(\xi, \xi) = \nabla_w S(\xi, \xi) = 2S(\nabla_w Y, \xi) = -2S(-\phi W - \phi h W, \xi) = 0. \tag{3.14}
\]
By the use of (3.8), (3.13) and (3.14), from (3.7), we obtain
\[
(\nabla_w S)(Y, \xi) = \frac{1}{2n + 1}dr(W)\eta(Y), \text{ if } a + (2n - 1)b \neq 0. \tag{3.15}
\]
Because \(a + (2n - 1)b = 0\) will imply \(C^* = aC\), from (1.1). So, we can not take \(a + (2n - 1)b = 0\). Putting \(Y = \xi\) in (3.15) we get \(dr(W) = 0\). This implies \(r\) is constant. So from (3.15), we have
\[
(\nabla_w S)(Y, \xi) = 0. \tag{3.16}
\]
Now we have
\[
(\nabla_w S)(Y, \xi) = \nabla_w S(Y, \xi) - S(\nabla_w Y, \xi) - S(Y, \nabla_w \xi).
\]
Using (2.12) and (2.5) in the above relation, it follows that
\[ (\nabla_W S)(Y, \xi) = 2nk(\nabla_W \eta)(Y) + S(Y, \phi W + \phi h W). \] (3.17)

In virtue of (3.17), (2.16) and (2.3)(a), we get
\[ (\nabla_W S)(Y, \xi) = -2nk g(\phi W + \phi h W, Y) + S(Y, \phi W + \phi h W). \] (3.18)

By (3.16) and (3.18), we have
\[ 2nk g(\phi W + \phi h W, Y) - S(Y, \phi W + \phi h W) = 0. \] (3.19)

Replacing \( Y \) by \( \phi Y \) in (3.19) and using (2.1)(d), (2.2) and (2.15), we get
\[ 2nk g(\phi W + \phi h W, \phi Y) - S(\phi Y, \phi W + \phi h W) = 0 \]
or,
\[ 2nk [g(W + h W, Y) - \eta(W + h W) \eta(Y)] - S(Y, W + h W)
+ 2nk \eta(W + h W) \eta(Y) + 4(n - 1) g(h Y, W + h W) = 0 \]
or,
\[ 2nk g(Y, W) + 2nk g(Y, h W) - S(Y, W) - S(Y, h W)
+ 4(n - 1) g(Y, h W) + 4(n - 1) g(Y, h^2 W) = 0 \]
since \( g(X, h Y) = g(h X, Y) \). Now by (2.9), (2.13) and (2.1)(a) this implies
\[ S(Y, W) + S(Y, h W) = 2nk g(Y, W) + [2nk + 4(n - 1)] g(Y, h W)
+ 4(n - 1)(k - 1) g(Y, -W + \eta(W) \xi) \]
or,
\[ S(Y, W) + 2(n - 1) g(Y, h W) - 2(n - 1)(k - 1) g(Y, W)
+ 2(n - 1)(k - 1) \eta(Y) \eta(W) = [2nk - 4(n - 1)(k - 1)] g(Y, W)
+ [2nk + 4(n - 1)] g(Y, h W) + 4(n - 1)(k - 1) \eta(Y) \eta(W), \]
which implies,
\[ S(Y, W) = 2(n + k - 1) g(Y, h W) + 2(nk + n - 1) g(Y, h W)
+ 2(n - 1)(k - 1) \eta(Y) \eta(W). \] (3.20)

Replacing \( W \) by \( h W \) and using (2.13), (2.9) and (2.1)(a), we get from (3.20)
\[ -2kg(Y, h W) = -2nk(k - 1) g(Y, W) + 2nk(k - 1) \eta(Y) \eta(W). \]

Since we may assume that \( k \neq 0 \), this implies
\[ g(Y, h W) = n(k - 1) g(Y, W) - n(k - 1) \eta(Y) \eta(W). \] (3.21)
From (3.20) and (3.21), we get
\[ S(Y, W) = A g(Y, W) + B \eta(Y) \eta(W), \] (3.22)
where \[ A = 2([n + k - 1] + n(k - 1)(nk + n - 1)] \] and \[ B = 2([n - 1](k - 1) - n(k - 1)(nk + n - 1)] \] are constants. So, the manifold is an \( \eta \)-Einstein manifold with constant coefficients.

Thus we state the following:

**Proposition 3.3.** A \((2n + 1)\)-dimensional globally \( \phi \)-quasiconformally symmetric \( N(k) \)-contact metric manifold is an \( \eta \)-Einstein manifold with constant coefficients.

In view of Lemma 3.2 and Proposition 3.3 we have the following:

**Theorem 3.4.** A \((2n + 1)\)-dimensional \((n \geq 1)\) globally \( \phi \)-quasiconformally symmetric \( N(k) \)-contact metric manifold is a Sasakian manifold.

If \( k = 1 \), then the manifold reduces to a Sasakian manifold. In this case from (3.22) it follows that the manifold is an Einstein manifold. Thus we obtain the following:

**Proposition 3.5.** A \((2n + 1)\)-dimensional globally \( \phi \)-quasiconformally symmetric Sasakian manifold is an Einstein manifold.

The above proposition have been proved by De, Özgür and Mondal [5].

### 4 3-Dimensional Locally \( \phi \)-Quasiconformally Symmetric \( N(k) \)-Contact Metric Manifolds

In a 3-dimensional Riemannian manifold, we have
\[ R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y + \frac{r}{2}[g(X, Z)Y - g(Y, Z)X], \] (4.1)
where \( Q \) is the Ricci-operator, that is, \( g(QX, Y) = S(X, Y) \) and \( r \) is the scalar curvature of the manifold. Now putting \( Z = \xi \) in (4.1) and using (2.1)\((a)\), (2.3)\((b)\) and (2.12) we get
\[ R(X, Y)\xi = \eta(Y)QX - \eta(X)QY + 2k[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(X)Y - \eta(Y)X], \] (4.2)
Using (2.17) in (4.2), we get
\[(k - \frac{r}{2})[\eta(Y)X - \eta(X)Y] = \eta(X)QY - \eta(Y)QX.\] (4.3)

Putting \(Y = \xi\) in (4.3) and using (2.12), we get
\[QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi.\] (4.4)

Therefore it follows from (4.4) that
\[S(X, Y) = (\frac{r}{2} - k)g(X, Y) + (3k - \frac{r}{2})\eta(X)\eta(Y).\] (4.5)

Using (4.1), (4.4) and (4.5) in (1.1) we get for \(n = 3\)
\[
C^*(X, Y)Z = a(\frac{r}{2} - 2k) + 2b(\frac{r}{2} - k) - \frac{r}{3}a(2 + 2b)[g(Y, Z)X - g(X, Z)Y] + [a(3k - \frac{r}{2}) + b(\frac{r}{2} - k)][g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] + \eta(X)\eta(Z)X - \eta(X)\eta(Y)Z
= (a + b)(r - 2k)[g(Y, Z)X - g(X, Z)Y] + [\frac{r}{2}(b - a) + k(3a - b)][g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] + \eta(Y)\eta(Z)X - \eta(X)\eta(Y)Z.\] (4.6)

Taking the covariant differentiation to the both sides of the equation (4.6), we have
\[
(\nabla_W C^*)(X, Y)Z = dr(W)(a + b)[g(Y, Z)X - g(X, Z)Y] + dr(W)\frac{b - a}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] + \eta(Y)\eta(Z)X - \eta(X)\eta(Y)Z + [\frac{r}{2}(b - a) + k(3a - b)][g(Y, Z)\nabla_W \eta(X)\xi - g(X, Z)(\nabla_W \eta)(Y)\xi] - g(X, Z)\eta(Y)\nabla_W \eta + (\nabla_W \eta)(Y)\eta(Z)X + \eta(Y)(\nabla_W \eta)(Z)X - (\nabla_W \eta)(X)\eta(Y)Z - \eta(X)(\nabla_W \eta)(Y)Z].\] (4.7)

Now, assume that \(X, Y\) and \(Z\) are horizontal vector fields. So, (4.7) becomes
\[
(\nabla_W C^*)(X, Y)Z = dr(W)(a + b)[g(Y, Z)X - g(X, Z)Y] + \frac{r}{2}(b - a) + k(3a - b)][g(Y, Z)(\nabla_W \eta)(X)\xi - g(X, Z)(\nabla_W \eta)(Y)\xi] - g(X, Z)(\nabla_W \eta)(Y)\xi].\] (4.8)
Using (2.1)(c) we obtain from (4.8)

\[ \phi^2 (\nabla_W C^*) (X,Y) Z = dr(W) (a + b) [g(X,Z)Y - g(Y,Z)X]. \] (4.9)

Assume \( \phi^2 (\nabla_W C^*) (X,Y) Z = 0 \). If \( a + b = 0 \), then putting \( a = -b \) into (1.1) we find for \( n = 3 \)

\[ C^* (X,Y) Z = a C(X,Y) Z, \]

where \( C \) is the weyl conformal curvature tensor. But for a 3-dimensional Riemannian manifold since \( C = 0 \), we obtain \( C^* = 0 \). Therefore \( a + b \neq 0 \). Then the equation (4.9) implies \( dr(W) = 0 \). Hence we conclude the following:

**Theorem 4.1.** A 3-dimensional \( N(k) \)-contact metric manifold is locally \( \phi \)-quasiconformally symmetric if and only if the scalar curvature \( r \) is constant.

In [6] Blair et al proved the following:
A 3-dimensional \( N(k) \)-contact metric manifold is locally \( \phi \)-symmetric if and only if the scalar curvature is constant.

Using the above result of Blair et al and Theorem 4.1, we state the following:

**Theorem 4.2.** A 3-dimensional \( N(k) \)-contact metric manifold is locally \( \phi \)-quasiconformally symmetric if and only if it is locally \( \phi \)-symmetric.

**5 Example**

In this section, we construct an example of a locally \( \phi \)-quasiconformally symmetric 3-dimensional \( N(k) \)-contact manifold. We consider 3-dimensional manifold \( M = \{(x,y,z) \in R^3 \} \), where \( (x,y,z) \) are the standard coordinate in \( R^3 \). Let \( \{e_1, e_2, e_3\} \) be linearly independent global frame on \( M \) given by

\[ [e_2,e_3] = 2e_1, \quad [e_3,e_1] = \frac{3}{2} e_2, \quad [e_1,e_2] = \frac{1}{2} e_3. \]

Let \( g \) be the Riemannian metric defined by

\[ g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0, \quad g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1. \]

Let \( \eta \) be the 1-form defined by

\[ \eta(U) = g(U,e_1) \]

for any \( U \in \chi(M) \). Let \( \phi \) be the \((1,1)\)-tensor field defined by

\[ \phi e_1 = 0, \quad \phi e_2 = e_3, \quad \phi e_3 = -e_2. \]

Using the linearity of \( \phi \) and \( g \) we have

\[ \eta(e_1) = 1, \]
\[ \phi^2(U) = -U + \eta(U)e_1 \]

and

\[ g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W) \]

for any \( U, W \in \chi(M) \). Moreover

\[ he_1 = 0, \quad he_2 = -\frac{1}{2}e_2 \text{ and } he_3 = \frac{1}{2}e_3. \]

The Riemannian connection \( \nabla \) of the metric tensor \( g \) is given by the Koszul’s formulae as

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]

We have

\[
2g(\nabla_{e_2} e_3, e_1) = e_2 g(e_3, e_1) + e_3 g(e_1, e_2) - e_1 g(e_2, e_3) \\
- g(e_2, [e_3, e_1]) - g(e_3, [e_2, e_1]) + g(e_1, [e_2, e_3])
\]

\[ = 1 = 2g(\frac{1}{2}e_1, e_1). \]

Similarly, we have

\[ 2g(\nabla_{e_2} e_3, e_2) = 0 = 2g(\frac{1}{2}e_1, e_2) \]

and

\[ 2g(\nabla_{e_2} e_3, e_3) = 0 = 2g(\frac{1}{2}e_1, e_3). \]

Therefore, we have \( \nabla_{e_2} e_3 = \frac{1}{2}e_1 \).

Similarly, we have

\[ \nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \]

\[ \nabla_{e_2} e_1 = -\frac{1}{2}e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = \frac{1}{2}e_1, \]

\[ \nabla_{e_3} e_1 = \frac{3}{2}e_2, \quad \nabla_{e_3} e_2 = -\frac{3}{2}e_1, \quad \nabla_{e_3} e_3 = 0. \]

Therefore, the manifold satisfies the relation

\[ \nabla_{e_2} e_1 = -\phi e_2 - \phi(he_2) \]

and

\[ \nabla_{e_3} e_1 = -\phi e_3 - \phi(he_3). \]

Hence we have

\[ \nabla_X \xi = -\phi X - \phi hX, \]
for any vector field $X$. Hence the manifold is a contact metric manifold for $e_1 = \xi$.

Now, we find the curvature tensors as

\[
R(e_2, e_1)e_1 = \frac{3}{4}e_2, \quad R(e_3, e_1)e_1 = \frac{3}{4}e_3, \quad R(e_2, e_3)e_1 = 0,
\]

\[
R(e_2, e_3)e_2 = \frac{3}{4}e_3, \quad R(e_2, e_1)e_3 = 0, \quad R(e_2, e_3)e_3 = -\frac{3}{4}e_2,
\]

\[
R(e_1, e_2)e_2 = \frac{3}{4}e_1, \quad R(e_1, e_3)e_3 = \frac{3}{4}e_1, \quad R(e_1, e_3)e_2 = 0.
\]

From the expressions of $R(e_2, e_1)e_1$ and $R(e_3, e_1)e_1$ we conclude the manifold is a $N(\frac{3}{4})$-contact metric manifold.

The Ricci tensors of this manifold are given as follows:

\[
S(e_1, e_1) = \frac{3}{2}, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = 0.
\]

Hence the scalar curvature is

\[
r = \frac{3}{2} = \text{constant}.
\]

Therefore, in view of the Theorem 4.1, we can say that the manifold is locally $\phi$-quasiconformally symmetric.

References


(Received 14 December 2015)
(Accepted 20 October 2017)