Fixed Point Property of Real Unital Abelian Banach Algebras and Their Closed Subalgebras Generated by an Element with Infinite Spectrum

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Abstract A Banach space $X$ is said to have the fixed point property if for each nonexpansive mapping $T: E \rightarrow E$ on a bounded closed convex subset $E$ of $X$ has a fixed point. Let $X$ be an infinite dimensional unital Abelian real Banach algebra with $\Omega(X) \neq \emptyset$ satisfying: (i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$, (ii) $\inf \{r_X(x) : x \in X, \|x\| = 1\} > 0$. We prove that, for each element $x_0$ in $X$ with infinite spectrum, the Banach algebra $\langle x_0 \rangle = \{\sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R}\}$ generated by $x_0$ does not have the fixed point property.

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1. Introduction

A Banach space $X$ is said to have the fixed point property if for each nonexpansive mapping $T: E \rightarrow E$ on a bounded closed convex subset $E$ of $X$ has a fixed point, to have the weak fixed point property if for each nonexpansive mapping $T: E \rightarrow E$ on a weakly compact convex subset $E$ of $X$ has a fixed point.

In 1981, D. E. Alspach [1] proved that there exists an isometry $T: E \rightarrow E$ on a weakly compact convex subset $E$ of the Lebesgue space $L_1[0, 1]$ without a fixed point. Consequently, $L_1[0, 1]$ does not have the weak fixed point property.

In 1983, J. Elton, P. K. Lin, E. Odell, and S. Szarek [2] showed that $C_\mathbb{R}(\alpha)$ has the weak fixed point property, if $\alpha$ is a compact ordinal with $\alpha < \omega^\omega$.

Theorem 1.1. Let $X$ be a locally compact Hausdorff space. If $C_0(X)$ has the weak fixed point property, then $X$ is dispersed.

Moreover, by using Theorem 1.1, they proved the following corollaries.

Corollary 1.2. Let $G$ be a locally compact group. Then the $C^*$-algebra $C_0(G)$ has the weak fixed point property if and only if $G$ is discrete.

Corollary 1.3. A von Neumann algebra $M$ has the weak fixed point property if and only if $M$ is finite dimensional.

In 2005, Benavides and Pineda [4] proved the following results.

Theorem 1.4. Let $X$ be an $\omega$-almost weakly orthogonal closed subspace of $C(K)$ where $K$ is a metrizable compact space. Then $X$ has the weak fixed point property.

Theorem 1.5. Let $K$ be a metrizable compact space. Then, the following conditions are all equivalent:

1. $C(K)$ is $\omega$-almost weakly orthogonal,
2. $C(K)$ is $\omega$-weakly orthogonal,
3. $K(\omega) = \emptyset$.

Corollary 1.6. Let $K$ be a compact set with $K(\omega) = \emptyset$. Then $C(K)$ has the weak fixed point property.


Theorem 1.7. $A(G)$ has the fixed point property if and only if $G$ is finite.

As a consequence, they obtained the following corollary.

Corollary 1.8. $B(G)$ has the fixed point property if and only if $G$ is finite.

If $X$ is a complex Banach algebra, condition (A) is defined by:

(A) For each $x \in X$, there exists an element $y \in X$ such that $\tau(y) = \overline{\tau(x)}$, for each $\tau \in \Omega(X)$.

Note that each $C^*$-algebra satisfies condition (A).

In 2010, W. Fupinwong and S. Dhompongsa [6] proved that each infinite dimensional unital Abelian real Banach algebra $X$ with $\Omega(X) \neq \emptyset$ satisfying (i) if $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$, (ii) inf${r_X(x) : x \in X, \|x\| = 1}$ > 0, does not have the fixed point property. Moreover, they proved the following theorem.

Theorem 1.9. Let $X$ be an infinite dimensional unital Abelian complex Banach algebra satisfying condition (A) and each of the following statements:

(i) If $x, y \in X$ is such that $|\tau(x)| \leq |\tau(y)|$, for each $\tau \in \Omega(X)$, then $\|x\| \leq \|y\|$.
(ii) inf${r_X(x) : x \in X, \|x\| = 1}$ > 0.

Then $X$ does not have the fixed point property.

In 2010, by using Theorem 1.9, D. Alimohammadi and S. Moradi [7] obtain sufficient conditions to show that some unital uniformly closed subalgebras of $C(\Omega)$, where $\Omega$ is a compact space, do not have the fixed point property.

In 2011, S. Dhompongsa, W. Fupinwong, and W. Lawton [8] proved that a $C^*$-algebra has the fixed point property if and only if it is finite dimensional.

In 2012, W. Fupinwong [9] show that the unitality in Theorem 1.9 proved in [6] can be omitted.
In 2016, by using Urysohn’s lemma and Schauder-Tychonoff fixed point theorem, D. Alimohammadi [10] proved the following result.

**Theorem 1.10.** Let \( \Omega \) be a locally compact Hausdorff space. Then the following statements are equivalent:

(i) \( \Omega \) is infinite set.

(ii) \( C_0(\Omega) \) is infinite dimensional.

(iii) \( C_0(\Omega) \) does not have the fixed point property.


**Theorem 1.11.** Let \( X \) be an infinite dimensional real Abelian Banach algebra with \( \Omega(X) \neq \emptyset \) and satisfying each of the following:

(i) If \( x, y \in X \) is such that \( |\tau(x)| \leq |\tau(y)| \) for each \( \tau \in \Omega(X) \) then \( \|x\| \leq \|y\| \).

(ii) \( \inf \{r_X(x) : x \in X, \|x\| = 1\} > 0 \).

Then \( X \) does not have the fixed point property.

In 2018, P. Thongin and W. Fupinwong [12] proved that if \( X \) is an infinite dimensional complex unital Abelian Banach algebra satisfying condition (A) and satisfying (i) If \( x, y \in X \) is such that \( |\tau(x)| \leq |\tau(y)| \), for each \( \tau \in \Omega(X) \), then \( \|x\| \leq \|y\| \), (ii) \( \inf \{r_X(x) : x \in X, \|x\| = 1\} > 0 \), then there exists an element \( x_0 \in X \) such that

\[
\langle x_0 \rangle = \left\{ \sum_{i=1}^{k} \alpha_i x_i^0 : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}
\]

does not have the fixed point property. Our result is a generalization of Theorem 1.11.

2. Preliminaries

Let \( X \) be a Banach space. We say that a mapping \( T : E \to E \) is nonexpansive if

\[
\|Tx - Ty\| \leq \|x - y\|
\]

for each \( x, y \in E \), where \( E \) is a nonempty subset of \( X \). A Banach space \( X \) is said to have the fixed point property if for each nonexpansive mapping \( T : E \to E \) on a nonempty bounded closed convex subset \( E \) of \( X \) has a fixed point.

We define the spectrum of an element \( x \) of a real unital Banach algebra \( X \) to be the set

\[
\sigma_X(x) = \{ \lambda \in \mathbb{R} : \lambda 1 - x \notin Inv(X) \},
\]

where \( Inv(X) \) is the set of all invertible elements in \( X \).

The spectral radius of \( x \) is defined to be

\[
r_X(x) = \sup_{\lambda \in \sigma(x)} |\lambda|.
\]
We say that a mapping \( \tau : X \rightarrow \mathbb{R} \) is a character on a real algebra \( X \) if \( \tau \) is a non-zero homomorphism. We denote by \( \Omega(X) \) the set of all characters on \( X \). If \( X \) is a real unital Abelian Banach algebra with \( \Omega(X) \neq \emptyset \), it is known that \( \Omega(X) \) is compact.

If \( Y \) is a real unital subalgebra of a real unital Banach algebra \( X \) with \( \Omega(X) \neq \emptyset \), then \( \Omega(Y) \neq \emptyset \). In fact, for each \( \tau \in \Omega(X) \), the restriction \( \tau|Y \) of \( \tau \) on \( Y \) is in \( \Omega(Y) \). Note that \( \tau|Y \) is nonzero since \( Y \) is unital.

We denote by \( C_\mathbb{R}(S) \) the real unital Banach algebra of continuous functions from a topological space \( S \) to \( \mathbb{R} \) where the operations are defined pointwise and the norm is the sup-norm.

The following Theorem is known as the Stone-Weierstrass approximation theorem for \( C_\mathbb{R}(S) \).

**Theorem 2.1.** Let \( A \) be a subalgebra of \( C_\mathbb{R}(S) \) satisfying the following conditions:

(i) \( A \) separates the points of \( S \).

(ii) \( A \) annihilates no point of \( S \).

Then \( A \) is dense in \( C_\mathbb{R}(S) \).

Let \( X \) be a real Abelian Banach algebra with \( \Omega(X) \neq \emptyset \). The Gelfand representation \( \varphi : X \rightarrow C_\mathbb{R}(\Omega(X)) \) is defined by \( x \mapsto \hat{x} \), where \( \hat{x} \) is defined by

\[
\hat{x}(\tau) = \tau(x),
\]

for each \( \tau \in \Omega(X) \). If \( X \) is unital and Abelian, then \( \sigma(x) = \{ \tau(x) : \tau \in \Omega(X) \} \), for each \( x \in X \). It is known that \( r_X(x) = \| \hat{x} \|_{\infty,X} \) if \( X \) is Abelian, where

\[
\| \hat{x} \|_{\infty,X} = \sup_{\tau \in \Omega(X)} |\hat{x}(\tau)|.
\]

### 3. Lemmas

Some lemmas are proved in this section. In the next section, we will use them to prove our main theorem.

**Lemma 3.1.** Let \( X \) be a real unital Abelian Banach algebra satisfying \( \Omega(X) \neq \emptyset \) and

\[
\inf \{ r_X(x) : x \in X, \| x \| = 1 \} > 0.
\]

If \( Y \) is a unital subalgebra of \( X \), then

\[
\inf \{ r_Y(x) : x \in Y, \| x \| = 1 \} > 0.
\]

**Proof.** If \( x \in Y \), since \( \sigma_Y(x) \supseteq \sigma_X(x) \), then

\[
r_Y(x) = \sup_{\lambda \in \sigma_Y(x)} |\lambda| \geq \sup_{\lambda \in \sigma_X(x)} |\lambda| = r_X(x).
\]

So

\[
\inf \{ r_Y(x) : x \in Y, \| x \| = 1 \} \geq \inf \{ r_X(x) : x \in X, \| x \| = 1 \} > 0.
\]

Therefore, \( \inf \{ r_Y(x) : x \in Y, \| x \| = 1 \} \) is greater than zero.

**Lemma 3.2.** Let \( X \) be an infinite dimensional real unital Abelian Banach algebra with \( \Omega(X) \neq \emptyset \), and let \( x_0 \) be an element in \( X \) with infinite spectrum. Then \( \{ x_0^n : n \in \mathbb{N} \} \) is linearly independent. Consequently,

\[
\langle x_0 \rangle = \left\{ \sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}
\]
fixed point property of real unital abelian banach algebras}

Proof. Assume that

$$\sum_{i=1}^{k} \alpha_i x_0^i = 0,$$

where $\alpha_i \in \mathbb{R}$. Let $\{\lambda_1, \lambda_2, \lambda_3, \ldots \} \subseteq \sigma_X(x_0)$, with $\lambda_i \neq \lambda_j$ for each $i \neq j$. Since $\sigma_X(x_0) = \{\tau(x_0) : \tau \in \Omega(X)\}$, write $\lambda_j = \tau_j(x_0)$, where $\tau_j \in \Omega(X)$. Then

$$\sum_{i=1}^{k} \alpha_i (\tau_j(x_0))^i = \tau_j \left( \sum_{i=1}^{k} \alpha_i x_0^i \right) = 0,$$

therefore,

$$\sum_{i=1}^{k} \alpha_i \lambda_j^i = 0,$$

for each $j \in \mathbb{N}$. Hence $\alpha_i = 0$, for each $i \in \{1, 2, 3, \ldots, k\}$.

Lemma 3.3. Let $X$ be an infinite dimensional real unital Abelian Banach algebra with $\Omega(X) \neq \emptyset$, and let $x_0$ be an element in $X$ with infinite spectrum. Define

$$Z = \left\{ \sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}.$$

If $X$ satisfies

$$\inf \{ \|\hat{x}\|_{\infty, X} : x \in X, \|x\| = 1 \} > 0,$$

then $Z$ is a real unital Abelian Banach algebra satisfying the following conditions:

(i) The Gelfand representation $\varphi$ from $Z$ into $C_\mathbb{R}(\Omega(Z))$ is a bounded isomorphism.

(ii) The inverse $\varphi^{-1}$ is also a bounded isomorphism.

Proof. (i) Since $\ker(\varphi) = \{0\}$, $\varphi$ is injective. We have $\varphi(Z)$ is a subalgebra of $C_\mathbb{R}(\Omega(Z))$ separating the points of $\Omega(Z)$, and annihilating no point of $\Omega(Z)$. Moreover, $\varphi(Z)$ is complete, so $\varphi(Z)$ is closed. Indeed, if $\{\hat{z}_n\}$ is a Cauchy sequence in $\varphi(Z)$, assume to the contrary that $\{z_n\}$ is not Cauchy, then there exists $\varepsilon_0 > 0$ and subsequences $\{z_n'\}$ and $\{z_n''\}$ of $\{z_n\}$ such that

$$\|z_n' - z_n''\| \geq \varepsilon_0,$$

for each $n \in \mathbb{N}$. Define $y_n = (z_n' - z_n'')/\varepsilon_0$. Thus $\|y_n\| \geq 1$, for each $n \in \mathbb{N}$. Since $\{\hat{z}_n\}$ is Cauchy, so $\lim \hat{y}_n = \hat{0}$. From Lemma 3.1, we have

$$\inf \{r_Z(x) : x \in Z, \|x\| = 1 \} > 0.$$

Hence

$$0 < \inf \{r_Z(x) : x \in Z, \|x\| = 1 \} = \inf \{ \|\hat{x}\|_{\infty, Z} : x \in Z, \|x\| = 1 \} \leq \inf_{n \in \mathbb{N}} \left\| \frac{y_n}{\|y_n\|} \right\|_{\infty, Z} \leq \inf_{n \in \mathbb{N}} \|\hat{y}_n\|_{\infty, Z} = 0,$$
which is a contradiction. Hence we conclude that \( \{z_n\} \) is a Cauchy sequence. Then \( \{z_n\} \) is a convergent sequence in \( Z \), say \( \lim z_n = z_0 \in Z \). From, for each \( n \in \mathbb{N} \),

\[
\|\tilde{z}_n - \tilde{z}_0\|_{\infty, Z} = \|\varphi(z_n - z_0)\|_{\infty, Z} \leq \|z_n - z_0\|,
\]

it follows that

\[
\lim \|\tilde{z}_n - \tilde{z}_0\|_{\infty, Z} = 0.
\]

So \( \varphi(Z) \) is complete. It follows from the Stone-Weierstrass theorem that \( \varphi \) is surjective.

(ii) From the open mapping theorem, \( \varphi^{-1} \) is a bounded isomorphism.

**Lemma 3.4.** Let \( X \) be an infinite dimensional real unital Abelian Banach algebra with \( \Omega(X) \neq \emptyset \). If there exists \( x_0 \) in \( X \) with infinite spectrum, then there exists \( y \in \langle x_0 \rangle \) satisfying the following conditions:

(i) \( 1 \in \sigma_X(y) \subset [0, 1] \).

(ii) There exists a strictly decreasing sequence in \( \sigma_X(y) \).

**Proof.** Note that \( Z = \left\{ \sum_{i=0}^{k} \alpha_i x^i_0 : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\} \) is unital and Abelian, so the spectrum of \( x_0 \) is \( \tilde{x}_0(\Omega(Z)) \). Hence

\[
\sigma_X \left( \frac{x^2_0}{r_X(x^2_0)} \right) = \left\{ \tau \left( \frac{x^2_0}{r_X(x^2_0)} \right) : \tau \in \Omega(Z) \right\} \subset [0, 1].
\]

Let \( \{a_n\} \) be an infinite sequence in \( \sigma_X \left( \frac{x^2_0}{r_X(x^2_0)} \right) \). We may assume that \( \{a_n\} \) is strictly increasing and \( a_1 > 0 \).

Define a continuous function \( g : [0, 1] \to [0, 1] \) by

\[
g(t) = \begin{cases} \frac{t}{a_1}, & \text{if } t \in [0, a_1), \\ \frac{1-t}{1-a_1}, & \text{if } t \in [a_1, 1]. \end{cases}
\]

So \( g \) is joining the points \((0, 0)\) and \((a_1, 1)\), and \( g(1) = 0 \). Let

\[
\hat{y} = g \circ \left( \frac{x^2_0}{r_X(x^2_0)} \right).
\]

It follows from Lemma 3.3 that \( y \in Z \). Since \( g(0) = 0 \), so \( y \in \langle x_0 \rangle \). We have \( \{g(a_n)\} \) is a strictly decreasing sequence in \( \sigma_X(y) \). Moreover, \( 1 = g(a_1) \in \sigma_X(y) \subset [0, 1] \).

**Lemma 3.5.** Let \( X \) be an infinite dimensional real unital Abelian Banach algebra satisfying \( \Omega(X) \neq \emptyset \) and

\[
\inf \{r_X(x) : x \in X, \|x\| = 1\} > 0,
\]

and let \( x_0 \) be an element in \( X \) with infinite spectrum and \( \tau(x_0) \in \mathbb{R} \), for each \( \tau \in \Omega(X) \). Then there exists a sequence \( \{z_n\} \) in \( \langle x_0 \rangle \) such that \( \{\tau(z_n) : \tau \in \Omega(Z)\} \subset [0, 1] \), for each \( n \in \mathbb{N} \), and \( \{(\tilde{z}_n)^{-1}\{1\}\} \) is a sequence of nonempty pairwise disjoint subsets of \( \Omega(Z) \), where

\[
Z = \left\{ \sum_{i=0}^{k} \alpha_i x^i_0 : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}.
\]
Proof. From Lemma 3.1, it follows that

$$\inf \{ r_Z(x) : z \in Z, \|z\| = 1 \} > 0.$$  

From Lemma 3.2, $Z$ is infinite dimensional, then, from Lemma 2.10 (iii) in [6], there exists $z_0 = \sum_{i=0}^{k} \alpha_i x_0^i \in Z$ such that $\{ \tau(z_0) : \tau \in \Omega(Z) \}$ is infinite. Let $z_1 = \sum_{i=1}^{k} \alpha_i x_0^i$. It can be seen that $z_1 \in \{ x_0 \}$ and $\sigma(z_1)$ is infinite. From Lemma 3.4, we may assume without generality that $z_1$ satisfies

$$1 \in \sigma_Z(z_1) \subset [0, 1]$$

and there exists a strictly decreasing sequence of real numbers in $\sigma_Z(z_1)$, say $\{a_n\}$. Moreover, we may assume that $a_1 < 1$.

Define a continuous function $g_1 : [0, 1] \to [0, 1]$ by

$$g_1(t) = \begin{cases} \frac{t}{a_1}, & \text{if } t \in [0, a_1], \\ 1 + \frac{(g_1(a_2) - 1)(t - a_1)}{2(1 - a_1)}, & \text{if } t \in [a_1, 1]. \end{cases}$$

So $g_1$ is joining the points $(0, 0)$ and $(a_1, 1)$, and $g_1(1) \in (g_1(a_2), 1)$.

Define $\hat{z}_2 = g_1 \circ \hat{z}_1 : \Omega(Z) \to \mathbb{R}$, and define a continuous function $g_2 : [0, 1] \to [0, 1]$ by

$$g_2(t) = \begin{cases} \frac{g_1(a_2)}{a_2}, & \text{if } t \in [0, g_1(a_2)], \\ 1 + \frac{(g_2(g_1(a_3)) - 1)(t - g_1(a_2))}{2(1 - g_1(a_2))}, & \text{if } t \in [g_1(a_2), 1]. \end{cases}$$

So $g_2$ is joining the points $(0, 0)$ and $(g_1(a_2), 1)$, and $g_2(1) \in (g_2(g_1(a_3)), 1)$.

Define $\hat{z}_3 = g_2 \circ \hat{z}_2 : \Omega(Z) \to \mathbb{R}$. Continuing in this process, we have a sequence $\{z_n\}$ in $Z$ with $1 \in \{ \tau(z_n) : \tau \in \Omega(Z) \} \subset [0, 1]$, for each $n \in \mathbb{N}$, and $\{(\hat{z}_n)^{-1}(1)\}$ is a sequence of nonempty pairwise disjoint subsets of $\Omega(Z)$. Note that, from Lemma 3.3, $\{z_n\}$ is in $Z$.

Since $g_n(0) = 0$, for each $n \in \mathbb{N}$, it follows that $z_n \in x_0$, for each $n \in \mathbb{N}$. □

Lemma 3.6. Let $X$ be an infinite dimensional real unital Abelian Banach algebra satisfying $\Omega(X) \neq \emptyset$ and

$$\inf \{ r(x) : x \in X, \|x\| = 1 \} > 0,$$

and let $x_0$ be an element in $X$ with infinite spectrum. Assume that there exists a bounded sequence $\{y_n\}$ in $\langle x_0 \rangle$ which contains no convergent subsequences and such that $\{ \tau(y_n) : \tau \in \Omega(Z) \}$ is finite, for each $n \in \mathbb{N}$, where

$$Z = \left\{ \sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}.$$

Then there exists an element $z_0 \in \langle x_0 \rangle$ such that $\{ \tau(z_0) : \tau \in \Omega(Z) \}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$.

Proof. It follows form Lemma 3.2 and Lemma 3.3 that $Z$ is an infinite dimensional real unital Abelian Banach algebra with $\Omega(Z) \neq \emptyset$ and homeomorphic to $C_Z(\Omega(Z))$. Assume that there exists a bounded sequence $\{y_n\}$ in $\langle x_0 \rangle$ which contains no convergent subsequences and such that $\{ \tau(y_n) : \tau \in \Omega(Z) \}$ is finite, for each $n \in \mathbb{N}$. From the proof of Lemma 2.10 (ii) in [6], we have

$$\Omega(Z) = (\cup_{n \in \mathbb{N}} G_n) \cup F,$$
where \( F \) is a closed set in \( \Omega(Z) \), \( G_n \) is closed and open for each \( n \in \mathbb{N} \), and \( \{ F, G_1, G_2, \ldots \} \) is a partition of \( \Omega(Z) \). Define \( \tau_Z : Z \to \mathbb{R} \) by

\[ \tau_Z \left( \sum_{i=0}^{k} \alpha_i x_0^i \right) = \alpha_0, \]

for each \( \sum_{i=0}^{k} \alpha_i x_0^i \in Z \). So \( \tau_Z \in \Omega(Z) \). There are two cases to be considered. If \( \tau_Z \) is in \( F \), define \( \psi : \Omega(Z) \to \mathbb{R} \) by

\[ \psi(\tau) = \begin{cases} 1, & \text{if } \tau \in G_1, \\ \frac{n}{n+1}, & \text{if } \tau \in G_n, n \geq 2, \\ 0, & \text{if } \tau \in F. \end{cases} \]

If \( \tau_Z \) is in \( G_{n_0} \), for some \( n_0 \in \mathbb{N} \), without loss of generality, we may assume that \( n_0 = 1 \), define \( \psi : \Omega(Z) \to \mathbb{R} \) by

\[ \psi(\tau) = \begin{cases} 0, & \text{if } \tau \in G_1, \\ \frac{n-1}{n}, & \text{if } \tau \in G_n, n \geq 2, \\ 1, & \text{if } \tau \in F. \end{cases} \]

For each case, the inverse image of each closed set in \( \psi(\Omega(Z)) \) is closed, so \( \psi \in C_{\mathbb{R}}(\Omega(Z)) \).

Let \( \varphi : Z \to C_{\mathbb{R}}(\Omega(Z)) \) be the Gelfand representation. Therefore, \( \varphi^{-1}(\psi) \) is an element in \( Z \). Write \( z_0 = \varphi^{-1}(\psi) \). Then \( \{ \tau(z_0) : \tau \in \Omega(Z) \} \) is equal to \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \} or \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}. Moreover, \( z_0 \in \langle x_0 \rangle \) since \( \tau_Z(z_0) = \psi(\tau_Z) = 0 \).

**Lemma 3.7.** Let \( X \) be an infinite dimensional real unital Abelian Banach algebra satisfying \( \Omega(X) \neq \emptyset \) and the following conditions:

(i) If \( x, y \in X \) is such that \( |\tau(x)| \leq |\tau(y)| \), for each \( \tau \in \Omega(X) \), then \( \|x\| \leq \|y\| \),

(ii) \( \inf \{ r_X(x) : x \in X, \|x\| = 1 \} > 0 \).

Let \( x_0 \) be an element in \( X \) with infinite spectrum, and let \( x \in \langle x_0 \rangle \) with \( (\bar{x})^{-1}\{1\} \neq \emptyset \), and \( 0 \leq \tau(x) \leq 1 \), for each \( \tau \in \Omega(Z) \), where

\[ Z = \left\{ \sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}. \]

Define

\[ E = \{ z \in \langle x_0 \rangle : 0 \leq \tau(z) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(z) = 1 \text{ if } \tau \in A \}, \]

where \( A = (\bar{x})^{-1}\{1\} \), and define \( T : E \to E \) by

\[ z \mapsto xz. \]

Then \( E \) is a nonempty bounded closed convex subset of \( \langle x_0 \rangle \) and \( T : E \to E \) is a nonexpansive mapping.

**Proof.** Obviously, \( E \) is closed and convex. \( E \) is nonempty since \( x \in E \). From Lemma 3.1, we have

\[ \inf \{ r_Z(x) : x \in Z, \|x\| = 1 \} > 0. \]
If \( z \in E \), then
\[
\inf \{ r_Z(x) : x \in Z, \|x\| = 1 \} = \inf \{ \|\hat{x}\|_{\infty,Z} : x \in Z, \|x\| = 1 \}
\]
\[
\leq \left\| \frac{z}{\|z\|} \right\|_{\infty,Z} \leq \frac{1}{\|z\|}.
\]
Then, for each \( z \in E \),
\[
\|z\| \leq \frac{1}{\inf \{ r_Z(x) : x \in Z, \|x\| = 1 \}}.
\]
Therefore, \( E \) is bounded.

Let \( \omega \in \Omega(X) \), and let \( z, z' \in E \). Note that the restriction \( \omega|_Z \) of \( \omega \) on \( Z \) is in \( \Omega(Z) \).

Then
\[
|\omega(Tz - Tz')| = |\omega|_Z(Tz - Tz'),
\]
\[
= |\omega|_Z(xz - xz'),
\]
\[
= |\omega|_Z(x)|\omega|_Z(z - z'),
\]
\[
\leq |\omega|_Z(z - z'),
\]
\[
= |\omega(z - z'|.
\]

From (i), we have
\[
\|Tz - Tz'\| \leq \|z - z'|.
\]

So \( T \) is nonexpansive.

4. MAIN RESULTS

We now present the main theorem.

**Theorem 4.1.** Let \( X \) be an infinite dimensional real unital Abelian Banach algebra with \( \Omega(X) \neq \emptyset \), and let \( X \) satisfy the following conditions:
(i) If \( x, y \in X \) is such that \( |\tau(x)| \leq |\tau(y)| \), for each \( \tau \in \Omega(X) \), then \( \|x\| \leq \|y\| \);
(ii) \( \inf \{ r_X(x) : x \in X, \|x\| = 1 \} > 0 \).

If \( x_0 \) is an element in \( X \) with infinite spectrum, then the closed subalgebra

\[
\langle x_0 \rangle = \left\{ \sum_{i=1}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}
\]
does not have the fixed point property.

**Proof.** From Lemma 3.2, \( \langle x_0 \rangle \) and \( Z \) are infinite dimensional real Abelian Banach algebras, where
\[
Z = \left\{ \sum_{i=0}^{k} \alpha_i x_0^i : k \in \mathbb{N}, \alpha_i \in \mathbb{R} \right\}.
\]

Note that \( \Omega(Z) \neq \emptyset \) since \( \Omega(X) \neq \emptyset \).

From Lemma 3.5, there exists a sequence \( \{z_n\} \) in \( \langle x_0 \rangle \) such that, for each \( n \in \mathbb{N} \),
\[
\{\tau(z_n) : \tau \in \Omega(Z)\} \subset [0, 1],
\]
and \( \{(\epsilon_n)^{-1}\{1\}\} \) is a sequence of nonempty pairwise disjoint subsets of \( \Omega(Z) \).
Let $A_n = (\hat{z}_n)^{-1}\{1\}$, and define $T_n : E_n \to E_n$ by
\[ x \mapsto z_nx, \]
where
\[ E_n = \{x \in \langle x_0 \rangle : 0 \leq \tau(x) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(x) = 1 \text{ if } \tau \in A_n\}. \]

From Lemma 3.7, $T_n : E_n \to E_n$ is a nonexpansive mapping on the bounded closed convex set $E_n$, for each $n \in \mathbb{N}$.

Assume to the contrary that $\langle x_0 \rangle$ has fixed point property. So $T_n$ has a fixed point, for each $n \in \mathbb{N}$. Let $y_n$ be a fixed point of $T_n$, for each $n \in \mathbb{N}$. We have $y_n = z_ny_n$, hence $\hat{y}_n = \hat{z}_n\hat{y}_n$, and then
\[ \hat{y}_n(\tau) = \begin{cases} 0, & \text{if } \tau \text{ is not in } A_n, \\ 1, & \text{if } \tau \text{ is in } A_n, \end{cases} \]
for each $n \in \mathbb{N}$. It follows that $\|\hat{y}_n - \hat{y}_n\|_{\infty, Z} = 1$, if $m \neq n$, since $A_1, A_2, A_3, \ldots$ are pairwise disjoint. Then $\{\hat{y}_n\}$ has no convergent subsequences. From Lemma 3.3, $Z$ and $C_\mathbb{R}(\Omega(Z))$ are homeomorphic, so $\{y_n\}$ has no convergent subsequences. Note that $\{y_n\}$ is in $\langle x_0 \rangle$. It follows from Lemma 3.6 that there exists an element $z_0$ in $\langle x_0 \rangle_\mathbb{R}$ such that $\{\tau(z_0) : \tau \in \Omega(Z)\}$ is equal to $\{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ or $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

Write $A_0 = (\hat{z}_0)^{-1}\{1\}$, define $T_0 : E_0 \to E_0$ by
\[ x \mapsto z_0x, \]
where
\[ E_0 = \{x \in \langle x_0 \rangle : 0 \leq \tau(x) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(x) = 1 \text{ if } \tau \in A_0\}. \]
It follows from Lemma 3.7 that $T_0$ is a nonexpansive mapping on a nonempty bounded closed convex subset $E_0$ in $\langle x_0 \rangle_\mathbb{R}$. So $T_0$ has a fixed point in $E_0$, say $y_0$. There are two cases to be considered.

Case(1) $\{\tau(z_0) : \tau \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ :

We have $\hat{y}_0 = \hat{z}_0\hat{y}_0$ since $y_0$ is a fixed point of $T_0$. Then
\[ \hat{y}_0(\tau) = \begin{cases} 0, & \text{if } \tau \text{ is not in } A_0, \\ 1, & \text{if } \tau \text{ is in } A_0. \end{cases} \]
Therefore,
\[ (\hat{y}_0)^{-1}\{1\} = (\hat{z}_0)^{-1}\{1\} = A_0 \]
and
\[ \Omega(Z) \setminus A_0 = (\hat{y}_0)^{-1}\{0\} = \bigcup_{n=0}^{\infty} (\hat{z}_0)^{-1}\{\frac{n}{n+1}\}. \]

It follows from
\[ \{\tau(z_0) : \tau \in \Omega(Z)\} = \{0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} \]
that $\left\{(\hat{z}_0)^{-1}\left\{\frac{n}{n+1}\right\} : n \in \mathbb{N}\right\} \cup \{\hat{z}_0)^{-1}\{0\}\}$ is a nonempty pairwise disjoint open covering of the compact set $\Omega(Z) \setminus A_0$, which is a contradiction.
Case (2) \( \{ \tau(z_0) : \tau \in \Omega(Z) \} = \{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \} : \)

Define \( T : E \to E \) by
\[
1 + x \mapsto (1 - z_0)(1 + x),
\]
for each \( 1 + x \in E \), where
\[
E = \{ 1 + x \in 1 + \langle x_0 \rangle : 0 \leq \tau(1 + x) \leq 1 \text{ for each } \tau \in \Omega(Z), \text{ and } \tau(1 + x) = 1 \text{ if } \tau \in A \},
\]
where \( A = (1 - z_0)^{-1}\{1\} \). Then
\[
\{ \tau(1 - z_0) : \tau \in \Omega(Z) \} = \{ 0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ... \}.
\]

Define \( S : Z \to Z \) by
\[
\lambda + x \mapsto (-\lambda + 1) + x.
\]
Note that \( S^2 \) is the identity mapping. It follows that \( STS : S(E) \to S(E) \) is a nonexpansive mapping on a nonempty bounded closed convex subset \( S(E) \) of \( \langle x_0 \rangle \).

Since \( \langle x_0 \rangle \) has the fixed point property, so \( STS \) has a fixed point. Then \( T \) has a fixed point, say \( 1 + y_0 \). Therefore,
\[
(1 + y_0)(\tau) = \begin{cases} 
0, & \text{if } \tau \text{ is not in } A, \\
1, & \text{if } \tau \text{ is in } A.
\end{cases}
\]

Then
\[
(1 + y_0)^{-1}\{1\} = (1 - z_0)^{-1}\{1\} = A
\]
and
\[
\Omega(Z) \setminus A = (1 + y_0)^{-1}\{0\} = \bigcup_{n=0}^{\infty} \left( (1 - z_0)^{-1}\left\{ \frac{n}{n+1} \right\} \right).
\]

From
\[
\{ \tau(1 - z_0) : \tau \in \Omega(Z) \} = \{ 0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ... \},
\]
we have \( \left\{ (1 - z_0)^{-1}\left\{ \frac{n}{n+1} \right\} : n \in \mathbb{N} \right\} \cup \left\{ (1 - z_0)^{-1}\{0\} \right\} \) is a nonempty pairwise disjoint open covering of the compact set \( \Omega(Z) \setminus A \), which is a contradiction.

So we conclude that \( \langle x_0 \rangle \) does not have the fixed point property. \( \blacksquare \)

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**References**


