Connections of \((m, n)\)-bi-quasi Hyperideals in Semihyperrings

Bundit Pibaljommee† and Warud Nakkhasen‡

†Department of Mathematics, Faculty of Science
Khon Kaen University, Khon Kaen 40002, Thailand
e-mail: banpib@kku.ac.th

‡Department of Mathematics, Faculty of Science
Mahasarakham University, Maha Sarakham 44150, Thailand
e-mail: warud.n@msu.ac.th

Abstract: We introduce the concept of left (resp. right) \((m, n)\)-bi-quasi hyperideals of semihyperrings as a generalization of \(n\)-bi-hyperideals, where \(m\) and \(n\) are positive integers. Then, we characterize regular semihyperrings using their left (resp. right) \((m, n)\)-bi-quasi hyperideals. Moreover, we study the connections between left (resp. right) \((m, n)\)-bi-quasi hyperideals and some many types of hyperideals in semihyperrings.

Keywords: semihyperring; \(m\)-bi-hyperideal; \((m, n)\)-quasi hyperideal; \((m, n)\)-bi-quasi hyperideal.

2010 Mathematics Subject Classification: 16Y60; 20M12; 20N20.

1 Introduction

In 1958, Iséki [1] introduced the notion of quasi-ideals for semirings without zero and proved results on semirings using their quasi-ideals. The concept of bi-ideals of associative rings was introduced by Lajos and Szász [2]. Any quasi-ideal is a generalization of a left and a right ideal, while every bi-ideal is a generalization of a quasi-ideal. Later, Chinram [3] introduced a generalization of quasi-ideals in semirings called \((m, n)\)-quasi-ideals and studied characterizations of regular semirings using their \((m, n)\)-quasi-ideals. Then, Munir and Shafiq [4] introduced and

1Corresponding author.

Copyright © 2020 by the Mathematical Association of Thailand. All rights reserved.
investigated some properties of the notion of $m$-bi-ideals in semirings as a generalization of bi-ideals. In 2018, Rao [5] introduced the concepts of left (resp. right) bi-quasi ideals of semirings which are generalizations of bi-ideals and quasi-ideals of semirings.

Algebraic hyperstructure was introduced in 1934 by Marty [6], at the 8th Congress of Scandinavian Mathematicians. In a classical algebraic structure, composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a nonempty set. This theory was studied in the following decades and nowadays by many mathematicians (see, e.g., [7], [8], [9], [10]).

The concept of semihyperrings, which both the sum and the product are hyperoperations, was defined by Vougiouklis [11] as a generalization of semirings. Omidi and Davvaz [12] generalized the notions of $m$-left and $n$-right hyperideals in ordered semihyperrings to be $(m,n)$-quasi-hyperideals. Afterword, Nakkhasen and Pibaljommee [13] introduced the concept of $m$-bi-hyperideals and characterized regular semihyperrings by their $m$-bi-hyperideals. In this paper, we introduce the concept of left and right $(m,n)$-bi-quasi hyperideals of semihyperrings which is a generalization of $n$-bi-hyperideals and $(m,n)$-quasi-hyperideals of semihyperrings and characterize regular semihyperrings using left (resp. right) $(m,n)$-bi-quasi hyperideals. In addition, we investigate the connections between left (resp. right) $(m,n)$-bi-quasi hyperideals and some many types of hyperideals in semihyperrings.

2 Preliminaries

Let $H$ be a nonempty set. A hyperoperation on $H$ is a mapping $\circ : H \times H \to P^\ast(H)$, where $P^\ast(H)$ denotes the set of all nonempty subsets of $H$ (see, e.g., [7], [8], [9], [10]). Then the structure $(H, \circ)$ is called a hypergroupoid. If $A,B \in P^\ast(H)$ and $x \in H$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$ 

A hypergroupoid $(H, \circ)$ is called a semihypergroup if for every $x,y,z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$ 

A hyperstructure $(S, +, \cdot)$ is called a semihyperring [11] if it satisfies the following conditions:

(i) $(S, +)$ is a semihypergroup;

(ii) $(S, \cdot)$ is a semihypergroup;

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$, for all $x, y, z \in S$. 
A nonempty subset $T$ of a semihyperring $(S,+,\cdot)$ is called a subsemihyperring of $S$ if for all $x, y \in T$, $x + y \subseteq T$ and $x \cdot y \subseteq T$. For more convenient, we write $S$ instead of a semihyperring $(S,+,\cdot)$ and $AB$ instead of $A \cdot B$, for any nonempty subsets $A$ and $B$ of $S$.

Next, we review some concepts in semihyperrings which will be used in later section. For a semihyperring $S$ and $m \in \mathbb{N}$, we denote $S^m = SS \cdots S$ ($m$ times), in addition, for every $m, n \in \mathbb{N}$ such that $m \geq n$, we conclude that $S^m \subseteq S^n$.

A subsemihyperring $A$ of a semihyperring $S$ is called an $m$-left (resp. $n$-right) hyperideal of $S$ if it satisfies $S^m A \subseteq A$ (resp. $A S^n \subseteq A$), where $m$ (resp. $n$) is a positive integer. A subsemihyperring $Q$ of a semihyperring $S$ is called an $(m,n)$-quasi-hyperideal of $S$ if it satisfies $(S^m Q) \cap (Q S^n) \subseteq Q$, where $m$ and $n$ are positive integers. A subsemihyperring $B$ of a semihyperring $S$ is called an $m$-bi-hyperideal of $S$ if it satisfies $BS^m B \subseteq B$, where $m$ is a positive integer.

### 3 Connections of $(m,n)$-bi-quasi hyperideals

In this section, we introduce the concept of left (resp. right) $(m,n)$-bi-quasi hyperideals of semihyperrings. Then, we characterize regular semihyperrings using their left (resp. right) $(m,n)$-bi-quasi hyperideals, and we present the connections between left (resp. right) $(m,n)$-bi-quasi hyperideals and some many types of hyperideals in semihyperrings.

**Definition 3.1.** A subsemihyperring $A$ of a semihyperring $S$ is called a left (resp. right) $(m,n)$-bi-quasi hyperideal of $S$ if it satisfies $(S^m A) \cap (A S^n) \subseteq A$ (resp. $(AS^n) \cap (AS^m) \subseteq A$), where $m$ and $n$ are positive integers.

If $A$ is both a left and a right $(m,n)$-bi-quasi hyperideal of a semihyperring $S$, then $A$ is called an $(m,n)$-bi-quasi hyperideal of $S$.

**Example 3.2.** Let $S = \left\{ \begin{bmatrix} 0 & u & v & w \\ 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid u, v, w, x, y, z \in \mathbb{N} \cup \{0\} \right\}$. Then $(S,+,\cdot)$ is a semiring under usual addition and multiplication of matrices, see [4]. For every $A, B \in S$, we define

$$A \leq B \text{ iff } a_{ij} \leq b_{ij},$$

where $i,j \in \{1,2,3,4\}$. Next, we define hyperoperations $\oplus$ and $\odot$ on $S$ by letting $A, B \in S$,

$$A \oplus B = \{X \in S \mid X \leq A + B\},$$

$$A \odot B = \{X \in S \mid X \leq A \cdot B\}.$$
We can show that \((S, \oplus, \odot)\) is a semihyperring. Now, let
\[
M = \left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ b & 0 & 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{N} \cup \{0\} \right\}.
\]
It is not difficult to check that \(M\) is a subsemihyperring of \(S\). We consider
\[
M \cdot S \cdot M = \left\{ \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid m \in \mathbb{N} \cup \{0\} \right\}.
\]
It follows that \(M \odot S \odot M = \{X \in S \mid X \leq M \cdot S \cdot M\} \nsubseteq M\). Next, we consider
\[
S^3 = \left\{ \begin{bmatrix} 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid n \in \mathbb{N} \cup \{0\} \right\}.
\]
This implies that
\[
S^3 \cdot M = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}.
\]
Thus,
\[
S^3 \odot M = \{X \in S \mid X \leq S^3 \cdot M\} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}.
\]
Hence, \((S^3 \odot M) \cap (M \odot S \odot M) \subseteq M\). Therefore, \(M\) is a left \((3, 1)\)-bi-quasi hyperideal of \(S\), but it is not a \(1\)-bi-hyperideal of \(S\).

Throughout this paper, we always assume that \(m\) and \(n\) are any positive integers.

**Theorem 3.3.** Every \(n\)-left hyperideal of a semihyperring \(S\) is an \((m, n)\)-bi-quasi hyperideal of \(S\).

**Proof.** Let \(A\) be an \(n\)-left hyperideal of a semihyperring \(S\). Then \(A\) is a subsemihyperring of \(S\) and \(S^n A \subseteq A\). Thus,
\[
(S^m A) \cap (A S^n A) \subseteq A S^n A \subseteq AA \subseteq A,
\]
\[
(A S^m) \cap (AS^n A) \subseteq AS^n A \subseteq AA \subseteq A.
\]
Hence, \(A\) is an \((m, n)\)-bi-quasi hyperideal of \(S\). \qed
Theorem 3.4. Every $n$-right hyperideal of a semihyperring $S$ is an $(m,n)$-bi-quasi hyperideal of $S$.

Remark 3.5. Let $S$ be a semihyperring. Then,

(i) every $m$-left hyperideal of $S$ is a left $(m,n)$-bi-quasi hyperideal of $S$;

(ii) every $m$-right hyperideal of $S$ is a right $(m,n)$-bi-quasi hyperideal of $S$.

Theorem 3.6. Every $(m,n)$-quasi-hyperideal of a semihyperring $S$ is a left $(m,n)$-bi-quasi hyperideal of $S$.

Proof. Let $Q$ be an $(m,n)$-quasi-hyperideal of a semihyperring $S$. Then $Q$ is a subsemihyperring of $S$ and $(S^m Q) \cap (QS^n) \subseteq Q$. So, $(S^m Q) \cap (QS^n) \subseteq (S^m Q) \cap (QS^n) \subseteq Q$. Hence, $Q$ is a left $(m,n)$-bi-quasi hyperideal of $S$. \qed

Theorem 3.7. Every $n$-bi-hyperideal of a semihyperring $S$ is an $(m,n)$-bi-quasi hyperideal of $S$.

Proof. Let $B$ be an $n$-bi-hyperideal of a semihyperring $S$. Then $B$ is a sub-semihyperring of $S$ and $BS^n B \subseteq B$. Consider $(S^m B) \cap (BS^n B) \subseteq BS^n B \subseteq B$ and $(BS^n B) \cap (BS^n B) \subseteq BS^n B \subseteq B$. Hence, $B$ is an $(m,n)$-bi-quasi hyperideal of $S$. \qed

We note that arbitrary intersection of left (resp. right) $(m,n)$-bi-quasi hyperideals of a semihyperring $S$ is not empty, then it is also a left (resp. right) $(m,n)$-bi-quasi hyperideal. If follows that arbitrary intersection of $(m,n)$-bi-quasi hyperideals of a semihyperring $S$ is not empty, then it is also an $(m,n)$-bi-quasi hyperideal.

A semihyperring $S$ is called regular (see, [14], [15]) if for each $a \in S$, there exists $x \in S$ such that $a \in axa$.

Theorem 3.8 ([14]). Let $S$ be a semihyperring. The following conditions are equivalent:

(i) $S$ is regular;

(ii) $a \in aSa$, for every $a \in S$;

(iii) $A \subseteq ASA$, for all $\emptyset \neq A \subseteq S$.

Theorem 3.9 ([13]). Let $S$ be a semihyperring and $m_1, m_2 \in \mathbb{N}$ such that $m = \max\{m_1, m_2\}$. Then $S$ is regular if and only if $R \cap L = RS^m L$ for any $m_1$-left hyperideal $L$ and $m_2$-right hyperideal $R$ of $S$.

Theorem 3.10. Let $S$ be a semihyperring and $n \geq m$. Then $S$ is regular if and only if $A = (S^m A) \cap (AS^n A)$ for every $(m,n)$-bi-quasi hyperideal $A$ of $S$. 

Theorem 3.11. Let $S$ be a semihyperring and $n \geq m$. Then $S$ is regular if and only if $A = (AS^m) \cap (AS^n)A$ for every left $(m,n)$-bi-quasi hyperideal $A$ of $S$.

Theorem 3.12. Let $S$ be a semihyperring. Then $S$ is regular if and only if $A \cap L \subseteq AS^kA$ for every $m$-left hyperideal $L$ and $(m,n)$-bi-quasi hyperideal $A$ of $S$, where $k = \max\{m, n\}$.

Proof. Assume that $S$ is regular. Let $L$ be an $m$-left hyperideal, $A$ be an $(m,n)$-bi-quasi hyperideal of $S$ and $k = \max\{m, n\}$. Let $a \in A \cap L$. By Theorem 3.8, we have $a \in aSa \subseteq a(Sa)Sa \subseteq a(SaSa)Sa \subseteq \ldots \subseteq a(Sa \cdots Sa)Sa \subseteq aS^{k+1}a \subseteq AS^kL$. Hence, $A \cap L \subseteq AS^kL$. Conversely, let $L$ be an $m$-left hyperideal and $R$ be an $n$-right hyperideal of $S$. By Theorem 3.4, we have that $R$ is an $(m,n)$-bi-quasi hyperideal of $S$. By assumption, $R \cap L \subseteq RS^kL$, where $k = \max\{m, n\}$. It is easy to show that $RS^kL \subseteq R \cap L$. Thus, $R \cap L = RS^kL$. By Theorem 3.9, $S$ is regular.

Theorem 3.13. Let $S$ be a semihyperring. Then $S$ is regular if and only if $R \cap L \subseteq RS^kA$ for every $m$-right hyperideal $R$ and $(m,n)$-bi-quasi hyperideal $A$ of $S$, where $k = \max\{m, n\}$.

Theorem 3.14. Let $S$ be a regular semihyperring and $A$ be a nonempty set of $S$. Then the following statements hold:

(i) if $A$ is an $m$-left hyperideal of $S$, then $A$ is a right $(m,n)$-bi-quasi hyperideal of $S$;

(ii) if $A$ is an $m$-right hyperideal of $S$, then $A$ is a left $(m,n)$-bi-quasi hyperideal of $S$. 

(iii) if \( A \) is an \((m,n)\)-quasi-hyperideal of \( S \), then \( A \) is a right \((m,n)\)-bi-quasi hyperideal of \( S \).

**Proof.** (i) Assume that \( A \) is an \( m \)-left hyperideal of \( S \). By Theorem 3.8 we have \((AS^m) \cap (AS^n) A \subseteq AS^n A \subseteq (AS) A \subseteq (ASAS) A \subseteq \cdots \subseteq (AS \cdots AS) A \subseteq S^m A \subseteq A \). Hence, \( A \) is a right \((m,n)\)-bi-quasi hyperideal of \( S \).

(ii) The proof is similar to (i).

(iii) Assume that \( A \) is an \((m,n)\)-quasi-hyperideal of \( S \). By Theorem 3.8 we have

\[
AS^m \subseteq AS \subseteq AS(AS) \subseteq AS(ASAS) \subseteq \cdots \subseteq AS(AS \cdots AS) \subseteq AS^{n+1} \subseteq AS^n,
\]

\[
AS^n A \subseteq (AS) A \subseteq (ASAS) A \subseteq \cdots \subseteq (AS \cdots AS) A \subseteq S^n A.
\]

This implies that \((AS^m) \cap (AS^n) A \subseteq (S^m A) \cap (AS^n) A \subseteq A \). Therefore, \( A \) is a right \((m,n)\)-bi-quasi hyperideal of \( S \).

**Theorem 3.15.** Let \( S \) be a regular semihyperring and \( A \) be a nonempty subset of \( S \). Then the following statements are equivalent:

(i) \( A \) is an \((m,n)\)-bi-quasi hyperideal of \( S \);

(ii) \( A \) is an \( m \)-bi-hyperideal of \( S \);

(iii) \( A \) is an \( n \)-bi-hyperideal of \( S \);

(iv) \( A \) is an \((m,n)\)-quasi-hyperideal of \( S \).

**Proof.** (i) \( \Rightarrow \) (ii): Assume that \( A \) is an \((m,n)\)-bi-quasi hyperideal of \( S \). Clearly, \( AS^m S \subseteq S^m A \). By Theorem 3.8 we have \( AS^m A \subseteq ASA \subseteq A(SA)SA \subseteq A(SAS)SA \subseteq \cdots \subseteq A(SA \cdots SA)SA \subseteq AS^{n+1} A \subseteq AS^n A \). It follows that

\[
AS^m A \subseteq (S^m A) \cap (AS^n A) \subseteq A \). Therefore, \( A \) is an \( m \)-bi-hyperideal of \( S \).

(ii) \( \Rightarrow \) (iii): Assume that \( A \) is an \( m \)-bi-hyperideal of \( S \). By Theorem 3.8 we have \( AS^n A \subseteq ASA \subseteq A(SA)SA \subseteq A(SAS)SA \subseteq \cdots \subseteq A(SA \cdots SA)SA \subseteq \)

\( AS^{m+1} A \subseteq AS^n A \subseteq A \). Hence, \( A \) is an \( n \)-bi-hyperideal of \( S \).

(iii) \( \Rightarrow \) (iv): Assume that \( A \) is an \( n \)-bi-hyperideal of \( S \). Let \( a \in (S^m A) \cap (AS^n) \). By Theorem 3.8 we have \( a \in aSa \subseteq (AS^n) S(S^m A) \subseteq AS^n A \subseteq A \). That is, \( (S^m A) \cap (AS^n) \subseteq A \). Hence, \( A \) is an \((m,n)\)-quasi-hyperideal of \( S \).

(iv) \( \Rightarrow \) (i): Assume that \( A \) is an \((m,n)\)-quasi-hyperideal of \( S \). By Theorem 3.8 we have that \( A \) is a left \((m,n)\)-bi-quasi-hyperideal of \( S \). By Theorem 3.8 we have

\[
AS^m \subseteq AS \subseteq AS(AS) \subseteq AS(ASAS) \subseteq \cdots \subseteq AS(AS \cdots AS) \subseteq AS^{n+1} \subseteq AS^n,
\]

\[
AS^n A \subseteq (AS) A \subseteq (ASAS) A \subseteq \cdots \subseteq (AS \cdots AS) A \subseteq S^n A.
\]
Thus, \((AS^m) \cap (AS^n \cdot A) \subseteq (S^m \cdot A) \cap (AS^n) \subseteq A\). Hence, \(A\) is a right \((m,n)\)-bi-quasi hyperideal of \(S\). Therefore, \(A\) is an \((m,n)\)-bi-quasi hyperideal of \(S\).

The following example, we show that any \((m,n)\)-quasi-hyperideal in a regular semihyperring is not necessary an \(m\)-left hyperideal and an \(n\)-right hyperideal.

Example 3.16. Let \(S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{N} \cup \{0\} \right\} \). Then \((S, +, \cdot)\) is a semiring under usual the matrix addition and the matrix multiplication. For any \(A, B \in S\), we define \(A \leq B\) iff \(a_{ij} \leq b_{ij}\), where \(i, j = \{1, 2\}\). Next, we define hyperoperations \(\oplus\) and \(\odot\) on \(S\) by letting \(A, B \in S\),

\[
A \oplus B = \{ X \in S \mid X \leq A + B \},
\]

\[
A \odot B = \{ X \in S \mid X \leq A \cdot B \}.
\]

We can show that \((S, \oplus, \odot)\) is a semihyperring. Let \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S\), where \(a, b, c, d \in \mathbb{N} \cup \{0\}\). Choose \(X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in S\). Then

\[
A \cdot X \cdot A = \begin{bmatrix} a^2 + ab + ac + bc & ab + b^2 + ad + bd \\
ac + ad + c^2 + cd & bc + bd + cd + d^2 \end{bmatrix}.
\]

Consider \(a \leq a^2 + ab + ac + bc, b \leq ab + b^2 + ad + bd, c \leq ac + ad + c^2 + cd, \) and \(d \leq bc + bd + cd + d^2\). This implies that \(A \leq A \cdot X \cdot A\). Thus, \(A \in A \odot X \odot A\).

Hence, \(S\) is a regular semihyperring. Now, let

\[
Q = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \mid x \in \mathbb{N} \cup \{0\} \right\}.
\]

It is easy to show that \(Q\) is a subsemihyperring of \(S\). Consider

\[
S^3 Q = \left\{ \begin{bmatrix} 0 & m \\ 0 & n \end{bmatrix} \mid m, n \in \mathbb{N} \cup \{0\} \right\},
\]

\[
Q S^2 = \left\{ \begin{bmatrix} 0 & k \\ k & l \end{bmatrix} \mid k, l \in \mathbb{N} \cup \{0\} \right\}.
\]

Hence,

\[
S^3 \odot Q = \{ X \in S \mid X \leq S^3 \cdot Q \} \not\subseteq Q,
\]

\[
Q \odot S^2 = \{ X \in S \mid X \leq Q \cdot S^2 \} \not\subseteq Q.
\]

On the other hand, \((S^3 \odot Q) \cap (Q \odot S^2) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \mid y \in \mathbb{N} \cup \{0\} \right\} = Q\). Therefore, \(Q\) is a \((3, 2)\)-quasi-hyperideal of \(S\), but it is not a 3-left hyperideal and is not a 2-right hyperideal of \(S\).

Finally, we conclude the connections between left (resp. right) \((m,n)\)-bi-quasi hyperideals and many types of hyperideals in semihyperrings as the following figure:
Connections of \((m, n)\)-bi-quasi Hyperideals in Semihyperrings

where:

\[ LBQ_{(m,n)} \] denotes the set of all left \((m, n)\)-bi-quasi hyperideals;

\[ RBQ_{(m,n)} \] denotes the set of all right \((m, n)\)-bi-quasi hyperideals;

\[ B_m \] denotes the set of all \(m\)-bi-hyperideals;

\[ B_n \] denotes the set of all \(n\)-bi-hyperideals;

\[ Q_{(m,n)} \] denotes the set of all \((m, n)\)-quasi-hyperideals;

\[ L_m \] denotes the set of all \(m\)-left hyperideals;

\[ L_n \] denotes the set of all \(n\)-left hyperideals;

\[ R_m \] denotes the set of all \(m\)-right hyperideals;

\[ R_n \] denotes the set of all \(n\)-right hyperideals;

\( \rightarrow \) denotes the normal implication;

\( \rightarrow \rightarrow \rightarrow \) denotes the implication with regular semihyperring as an assumption.

References


(Received 23 May 2019)
(Accepted 24 December 2019)