Strong Convergence of the
Shrinking Projection Method for
the Split Equilibrium Problem and
an Infinite Family of Relatively
Nonexpansive Mappings in Banach spaces

Nutchari Niyamosot† and Warunun Inthakon‡

†PhD Degree Program in Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200 Thailand
e-mail : nudchareen@hotmail.com
‡Research Center in Mathematics and Applied Mathematic,
Department of Mathematics, Faculty of Science,
Chiang Mai University, Chiang Mai 50200 Thailand
e-mail : w_inthakon@hotmail.com

Abstract : In this paper, we use the shrinking projection method to prove a strong convergence theorem for finding a common solution of the split equilibrium problem and fixed point problem of a relatively quasi–nonexpansive mapping. Consequently, our main theorem can apply to find a common solution of the split equilibrium problem and common fixed point problem for an infinite family of relatively nonexpansive mappings in Banach spaces.

Keywords : split equilibrium problem; equilibrium problem; relatively quasi-nonexpansive; relatively nonexpansive; common fixed point; shrinking projection method; Banach space.

2010 Mathematics Subject Classification : 47H05; 47H09; 47H10; 47J25.

†Corresponding author.

Copyright © 2020 by the Mathematical Association of Thailand. All rights reserved.
1 Introduction

In 1994, Censor and Elfving [1] studied the split feasibility problem in two Hilbert spaces $H_1$ and $H_2$ which is to find $z \in H_1$ such that $z \in C \cap A^{-1}Q$, where $C$ and $Q$ are nonempty closed and convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively and $A : H_1 \to H_2$ is a bounded linear operator. Furthermore, if $C \cap A^{-1}Q$ is nonempty, then $z \in C \cap A^{-1}Q$ is equivalent to

$$z = P_C(I - \lambda A^* (I - P_Q)A)z,$$  \hspace{1cm} (1.1)

where $\lambda > 0$ and $P_C$ is the metric projection of $H_1$ onto $C$. Thus, many authors used such results to studied the split feasibility problem in Hilbert spaces; see, for instance [2, 3, 4, 5]. The result of (1.1) was extended to Banach spaces by Takahashi [6, 7]. Since then, many author have been investigating the split feasibility problem in Banach spaces (see [8, 9, 10, 11] and the reference therein). Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be any mappings, the split common fixed point problem [12, 13] is to find $z \in H_1$ such that $z \in F(S) \cap A^{-1}F(T)$, where $F(S)$ and $F(T)$ are the fixed point sets of $S$ and $T$, respectively. In 2016, Takahashi [14] studied the split common fixed point problem in two Banach spaces, see also [15, 16].

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for $F$ is to find $z \in C$ such that

$$F(z, y) \geq 0, \hspace{1cm} \forall y \in C.$$  \hspace{1cm} (1.2)

for all $y \in C$. The set of all solutions of the problem (1.2) is denoted by $EP(F)$.

In 1955, Nikaido and Isoda [17] first used the inequality in convex game models. In 1972, Fan [18] proved existence theorems for $EP(F)$. Moreover, many problems in physics, economics and others can be reduced to find a solution of the problem (1.2). After the works of [19, 20, 21, 22], the equilibrium problem has been investigated by many authors (see [23, 24, 25, 26, 27, 28, 29, 30] and the references therein).

In 2012, He [31] considered the split equilibrium problem in Hilbert spaces. Let $F_1 : C \times C \to \mathbb{R}, F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and $A : H_1 \to H_2$ be a bounded linear operator. The split equilibrium problem is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \forall x \in C \text{ and } y^* = Ax^* \in Q \text{ such that } F_2(y^*, y) \geq 0, \forall y \in Q.$$  \hspace{1cm} (1.3)

The authors also introduced an iterative algorithm to find a solution of the split equilibrium problem. Also, they introduced the following an iterative algorithm to find a solution of (1.3) involving $A^*$ is the adjoint of $A$. The split equilibrium problem and fixed point problems has been studied in Hilbert spaces by many authors; see [32, 33, 34] and the references therein.

In 2017, Guo et al. [35] considered the split equilibrium problem in Banach spaces defined as : let $E_1, E_2$ be two Banach spaces and $C, Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Let $A : E_1 \to E_2$ be a bounded linear operator. Let $F : C \times C \to \mathbb{R}$ and $H : Q \times Q \to \mathbb{R}$ be two bifunctions. Let
Ω denote the set of solutions of the split equilibrium problem on \( F \) and \( H \), that is,

\[
\Omega = \{ z \in C : z \in EP(F), Az \in EP(H) \}.
\]

The authors proved a strong convergence theorem as follows:

Let \( E_1 \) be a uniformly smooth and uniformly convex Banach space and \( E_2 \) be a uniformly smooth, strictly convex and reflexive Banach space. Let \( A : E_1 \to E_2 \) be a linear and continuous operator. Let \( C \) and \( Q \) be nonempty closed and convex subsets of \( E_1 \) and \( E_2 \), respectively. Let \( S : C \to C \) be a relatively nonexpansive mapping and \( F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R} \) be two bifunctions satisfying the conditions (A1)-(A4) with \( \Omega \cap F(S) \neq \emptyset \). Define a sequence \( \{x_n\} \) by the following manner:

\[
\begin{align*}
take x_1 = x \in E, find v \in E_1 \text{ such that } Av \in Q, \\
V_n = \{ x \in E_1 : \|x-v\| \leq n \}, \\
U_n = \{ x \in V_n : Ax \in Q \}, \\
F(u_n, y) + \frac{1}{r_n} (y-u_n, J u_n - J x_n) \geq 0, \forall y \in C, \\
H(A z_n, y) + \frac{1}{s_n} (y-z_n, J z_n - J u_n) \geq 0, \forall y \in U_n, \\
y_n = J^{-1} (\alpha_n Ju_n + (1-\alpha_n) J S H z_n), \\
C_n = \{ z \in C : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
D_n = \bigcap_{i=1}^{n} C_i \\
x_{n+1} = \Pi_{D_n} x,
\end{align*}
\]

for each \( n \geq 1 \), where \( \{r_n\} \subset [r, \infty) \) with \( r > 0 \), \( \{s_n\} \subset [s, \infty) \) with \( s > 0 \) and \( \alpha_n \subset (0,1) \). Then the sequence \( \{x_n\} \) defined by (1.4) converges strongly to a point \( \Pi_{\Omega \cap F(S)} x \), where \( \Pi_{\Omega \cap F(S)} \) is the generalized projection of \( E_1 \) onto \( \Omega \cap F(S) \).

The algorithm (1.4) does not involve with the adjoint \( A^* \) of the operator \( A \) and the norm \( \|A\| \), which are quite difficult to compute, but involve only the operator \( A \). Furthermore, they also prove a weak convergence theorem for the set of solution of the split equilibrium problem and fixed point problem for a relatively nonexpansive mapping in Banach spaces. Using this idea, Inthakon and Niyamosot [36] also proved strong and weak convergence theorems for the split equilibrium problem and common fixed point problem for two relatively nonexpansive mappings in Banach spaces.

In 2008, Takahashi et al. [37] proved a strong convergence theorem for nonexpansive mapping by using the shrinking projection method as follows: Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( C \). Let \( T \) be a nonexpansive mapping of \( C \) into itself such that \( F(T) \neq \emptyset \) and let \( x_0 \in H \). For \( C_1 = C \) and \( u_1 = P_{C_1} x_0 \), define a sequence \( \{u_n\} \) of \( C \) as follows:

\[
\begin{align*}
y_n &= \alpha_n u_n + (1-\alpha_n) Tu_n, \\
C_{n+1} &= \{ z \in C_n : \|y_n - z\| \leq \|u_n - z\| \}, \\
u_{n+1} &= P_{C_{n+1}} u_0, n \in \mathbb{N},
\end{align*}
\]

where \( 0 \leq \alpha_n \leq a < 1 \) for all \( n \in \mathbb{N} \). Then, \( \{u_n\} \) converges strongly to \( z_0 = P_{F(T)} x_0 \).
Furthermore, studying strong convergence by the shrinking projection method has been used widely in Banach spaces; see for instance [38, 39] and the references therein.

In this paper, we focus on using the shrinking projection method to prove a strong convergence theorem for finding a common solution of the split equilibrium problem and fixed point problem of a relatively quasi-nonexpansive mapping. Consequently, our main theorem can apply to find a common solution of the split equilibrium problem and common fixed point problem for an infinite family of relatively nonexpansive mappings in Banach spaces.

2 Preliminaries

Let $E$ be a Banach space and let $E^*$ denote the dual of $E$. We denote the value of $x^*$ at $x$ by $\langle x, x^* \rangle$. Then the duality mapping $J$ on $E$ defined by

$$J(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}$$

for every $x \in E$. By the Hahn-Banach theorem, $J(x)$ is nonempty.

Let $S(E)$ be the unit sphere centered at the origin of $E$. A Banach space $E$ is said to be strictly convex if $\|(x + y)/2\| < 1$ wherever $x, y \in S(E)$ and $x \neq y$. The modulus $\delta$ of convexity of $E$ is defined by

$$\delta(\epsilon) = \inf \{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \}$$

for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. For $x \in E$ and $f \in E^*$ define $i(x)(f) = f(x)$. We know that $i(x) \in E^{**}$ and that the mapping $i : X \rightarrow E^{**}$ is an isometric isomorphism, called the canonical embedding of $E$ into $E^{**}$. If $i(E) = E^{**}$, then $E$ is said to be reflexive. A uniformly convex Banach space is strictly convex and reflexive. Then the space $E$ is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. The norm of $E$ is also said to be uniformly Gâteaux differentiable if for all $y \in S(E)$, the limit (2.1) attains uniformly for $x \in S(E)$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. The norm of $E$ is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit (2.1) is attained uniformly for $(x, y)$ in $S(E) \times S(E)$. We know that $E$ is smooth if and only if $J$ is a single-valued mapping of $E$ into $E^*$. We also know that $E$ is reflexive if and only if $J$ is surjective, and $E$ is strictly convex if and only if $J$ is one-to-one.

Let $E$ be a smooth, strictly convex and reflexive Banach space and $C$ be a nonempty closed and convex subset of $E$. Let $\phi$ be the function on $E \times E$ defined by

$$\phi(x, y) = \|y\|^2 - 2\langle x, Jy \rangle + \|x\|^2,$$
for all $x, y \in E$. From the definition of $\phi$, we have that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2,$$

for $x, y \in E$. In 1996, Alber [40] defined the generalized projection $\Pi_C$ from $E$ onto $C$ as $\Pi_C(x) = \text{arg min}_{y \in C} \phi(x, y)$, for all $x \in E$. If $E$ is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$ and $\Pi_C$ is the metric projection $P$ of $X^E$ onto $C$.

Let $C$ be a closed and convex subset of $E$ and let $T$ be a mapping from $C$ into itself. A point $p$ in $C$ is said to be an asymptotic fixed point [41] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of $T$ is denoted by $\hat{F}(T)$. We say that the mapping $T$ is called relatively nonexpansive [42, 43] if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$,
(R2) $\phi(p, Tx) \leq \phi(p, x)$, for each $x \in C, p \in F(T)$,
(R3) $F(T) = \hat{F}(T)$.

If $T$ satisfies (R1) and (R2), then $T$ is called relatively quasi-nonexpansive or quasi-$\phi$-nonexpansive. It is obvious that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings.

It is known from [43] that if $E$ be a strictly convex and smooth Banach space, let $C$ be a closed convex subset of $E$, and let $T$ be a relatively nonexpansive mapping from $C$ into itself. Then $F(T)$ is closed and convex. Furthermore, since the condition (R3) is not required in the proof of [43], we can concluded that the fixed point set of relatively quasi-nonexpansive mapping is closed and convex.

In 2008, Kohsaka and Takahashi [44] proved the following result for a countable family of relatively nonexpansive mappings.

**Lemma 2.1** ([44]). Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty} \subset (0, 1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $U : C \to E$ is defined by

$$Ux = J^{-1} \left( \sum_{i=1}^{\infty} \alpha_i (\beta_i Jx + (1 - \beta_i)JT_i x) \right) \text{ for each } x \in C.$$ 

Then $U$ is relatively nonexpansive and $F(U) = \bigcap_{i=1}^{\infty} F(T_i)$. 


In 2010, Nilsrakoo and SaJeung [45] also proved the following result.

**Lemma 2.2** ([45]). Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $\{T_i : C \to E\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\}_{i=1}^{\infty} \subset (0, 1)$ and $\{\beta_i\}_{i=1}^{\infty} \subset (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $S : C \to E$ is defined by

$$Sx = J^{-1} \left( \sum_{i=1}^{\infty} \alpha_i JT_i x \right) \text{ for each } x \in C.$$  

Then $S$ is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.

By the way, for solving the equilibrium problem, let us assume that a bifunction $F$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for all $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma is due to Takahashi and Zembayashi [30].

**Lemma 2.3** ([30]). Let $C$ be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$, and let $F$ be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1) - (A4). For $r > 0$ and $x \in E$, define a mapping $T_r^F : E \to C$ by

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}(y - z, Jz - Jy) \geq 0, \forall y \in C\},$$

for all $x \in E$. Then $T_r^F$ is well-defined and the followings hold:

(1) $T_r^F$ is single-valued;

(2) $T_r^F$ is firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,

$$(T_r^F x - T_r^F y, JT_r^F x - JT_r^F y) \leq (T_r^F x - T_r^F y, Jx - Jy);$$

(3) $F(T_r^F) = EP(F)$;

(4) $EP(F)$ is closed and convex.

The following results let us know more about the generalized projections.

**Lemma 2.4** ([40, 44]). Let $C$ be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y),$$

for all $x \in C$ and $y \in E$. 
Lemma 2.5 ([10, 14]). Let $C$ be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then, for any $x \in E$ and $z \in C$ we have
\[ z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0, \]
for all $y \in C$.

The following results also play the important role in our main theorems.

Lemma 2.6 ([14]). Let $E$ be a smooth and uniformly convex Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are the sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If \( \lim_{n \to \infty} \phi(x_n, y_n) = 0 \), then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

Lemma 2.7 ([30]). Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, $F$ be a bifunction from $C \times C \to \mathbb{R}$ satisfying the conditions (A1)-(A4) and let $r > 0$. Then, for any $x \in E$ and $q \in F(T_r^F)$,
\[ \phi(q, T_r^F x) + \phi(T_r^F x, x) \leq \phi(q, x). \]

3 Main Results

We use the shrinking projection method to prove strong convergence theorem as follows.

Theorem 3.1. Let $E_1$ be a uniformly smooth and uniformly convex Banach space and $E_2$ be a uniformly smooth, strictly convex and reflexive Banach space. Let $A : E_1 \to E_2$ be a linear and continuous operator. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Assume that $S : C \to C$ be a relatively quasi-nonexpansive mapping and $F : C \times C \to \mathbb{R}$, $H : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions (A1)-(A4) with $\Omega \cap F(S) \neq \emptyset$. Let $C_1 = C$ and define a sequence $\{x_n\}$ by the following manner:

\[
\begin{aligned}
&\text{take } x_1 = x \in E, \text{ find } v \in E_1 \text{ such that } Av \in Q, \\
&V_n = \{x \in E_1 : \|x - v\| \leq n\}, \\
&U_n = \{x \in V_n : Ax \in Q\}, \\
&F(u_n, y) + \frac{1}{r_n}(y - u_n, Ju_n - Jx_n) \geq 0, \forall y \in C, \\
&H(Az_n, Ay) + \frac{1}{r_n}(y - z_n, Jz_n - Ju_n) \geq 0, \forall y \in U_n, \\
&y_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JSL_C z_n), \\
&C_{n+1} = \{z : z \in C_n, \phi(z, y_n) \leq \phi(z, x_n)\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_n,
\end{aligned}
\]

for each $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$ and $\alpha_n \subset (0, 1)$. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $\Pi_{\Omega \cap F(S)} x$, where $\Pi_{\Omega \cap F(S)}$ is the generalized projection of $E_1$ onto $\Omega \cap F(S)$.
Proof. For each \( n \geq 1 \), we can see that \( v \) is contained in \( V_n \) and \( U_n \). Therefore \( V_n \) and \( U_n \) are nonempty. By the definition of \( V_n \), we have \( V_n \) is closed. Since \( A \) is linear and continuous, \( V_n \) is convex and \( U_n \) is closed and convex. It is obvious that \( C_1 = C \) is closed and convex. Suppose that \( C_k \) is closed and convex for some \( k \in \mathbb{N} \). For \( z \in C_k \), we see that

\[
\phi(z, y) \leq \phi(z, x_k) \iff \|y_k\|^2 - \|x_k\|^2 - 2\langle z, Jy_k - Jx_k \rangle \leq 0.
\]

This implies that \( C_{k+1} \) is closed and convex, and hence \( C_n \) is closed and convex for each \( n \geq 1 \). Next, we show that \( x_n \) is well defined. Let \( G(x, y) = H(Ax, Ay) \) for all \( x, y \in U_n \). Since \( A \) is linear and continuous, then \( G \) is a bifunction from \( U_n \times U_n \) into \( \mathbb{R} \) satisfying (A1)-(A4). Moreover, for each \( n \geq 1 \), we can rewrite

\[
H(Az_n, Ay) + \frac{1}{s_n} \langle y - z_n, Jz_n - Ju_n \rangle \geq 0,
\]

as

\[
G(x, y) + \frac{1}{s_n} \langle y - z_n, Jz_n - Ju_n \rangle \geq 0, \quad \text{for all } y \in U_n.
\]

Let \( p \in \Omega \cap F(S) \) so we have \( p \in EP(F) \) and \( Ap \in EP(H) \). By Lemma 2.3, we have \( p \in F(T_r^F) \) and hence \( p = T_r^F p \). Since \( Ap \in EP(H), H(Ap, z) \geq 0 \) for all \( z \in Q \). Since \( Az \in Q \) for all \( z \in U_n \), \( H(Ap, Az) \geq 0 \) for all \( z \in U_n \). It follows that \( G(p, z) \geq 0 \) for all \( z \in U_n \), which implies that \( p \in EP(G) \). By Lemma 2.3, we have \( p = T_r^G p \) and hence \( p \in C \). Let \( u_n = T_r^F x_n \) and \( z_n = T_{s_n}^G u_n \). By Lemma 2.7 and \( p \in F(T_r^F) \), we have

\[
\phi(p, T_r^F x_n) + \phi(T_r^F x_n, x_n) \leq \phi(p, x_n) \\
\phi(p, u_n) + \phi(u_n, x_n) \leq \phi(p, x_n).
\]

Thus,

\[
\phi(p, u_n) \leq \phi(p, x_n) - \phi(u_n, x_n),
\]

and hence

\[
\phi(p, u_n) \leq \phi(p, x_n).
\]

Since \( p \in F(S) \) and \( S \) is relatively quasi-nonexpansive mapping, we have

\[
\phi(p, S\Pi_C z_n) \leq \phi(p, \Pi_C z_n).
\]

Furthermore, we have from Lemma 2.4 that

\[
\phi(p, \Pi_C z_n) \leq \phi(p, u_n).
\]

On the other hand, since \( p \in EP(F) \), we can apply Lemma 2.3 and Lemma 2.7 to get that

\[
\phi(p, z_n) \leq \phi(p, u_n).
\]

It follows from (3.2) that

\[
\phi(p, S\Pi_C z_n) \leq \phi(p, x_n).
\]

By Lemma 2.7 and Lemma 2.3, we have

\[
\phi(p, u_n) \leq \phi(p, x_n).
\]

By Lemma 2.3, we have

\[
\phi(p, u_n) \leq \phi(p, x_n).
\]

By Lemma 2.7 and Lemma 2.3, we have

\[
\phi(p, u_n) \leq \phi(p, x_n).
\]

By Lemma 2.7 and Lemma 2.3, we have

\[
\phi(p, u_n) \leq \phi(p, x_n).
\]

By Lemma 2.7 and Lemma 2.3, we have

\[
\phi(p, u_n) \leq \phi(p, x_n).
\]
Thus,
\[
\phi(p, y_n) = \phi(p, J^{-1}(\alpha_n J u_n + (1 - \alpha_n) J y_n)) \\
= \|p\|^2 - 2\langle p, \alpha_n J u_n + (1 - \alpha_n) J y_n \rangle + \alpha_n u_n + (1 - \alpha_n) J y_n \|u_n\|^2 \\
\leq \|p\|^2 - 2\langle p, \alpha_n J u_n \rangle - 2\langle p, (1 - \alpha_n) J y_n \rangle + \alpha_n \|u_n\|^2 \\
+ (1 - \alpha_n) \|JS_{\Omega\cap F(S)} z_n\|^2 \\
= \alpha_n \phi(p, u_n) + (1 - \alpha_n) \phi(p, S_{\Omega\cap F(S)} z_n) \\
\leq \alpha_n \phi(p, x_n) + (1 - \alpha_n) \phi(p, x_n) \\
= \phi(p, x_n).
\]

Therefore, \( p \in C_n \) for each \( n \geq 1 \) and hence \( C_n \) is nonempty. It follows that \( \Omega \cap F(S) \subset C_n \) for each \( n \geq 1 \) which implies that \( \{x_n\} \) is well-defined. For each \( n \geq 1 \), we have from Lemma 2.4 that
\[
\phi(x_{n+1}, x) = \phi(\Pi_{C_{n+1}} x, x) \\
\leq \phi(z, x) - \phi(z, \Pi_{C_{n+1}} x) \\
\leq \phi(z, x), \quad \forall z \in C_{n+1}.
\]

Since \( \Omega \) and \( F(S) \) are nonempty closed and convex, \( \Omega \cap F(S) \) is closed and convex. Let \( x^* = \Pi_{\Omega\cap F(S)} x \), one has \( x^* \in \Omega \cap F(S) \subset C_{n+1} \) and
\[
\phi(x_{n+1}, x) \leq \phi(x^*, x).
\]

Therefore \( \{\phi(x_n, x)\} \) is bounded which implies that \( \{x_n\} \) is bounded. It follows that \( \{u_n\} \) and \( \{z_n\} \) are also bounded. Since \( x_{n+2} = \Pi_{C_{n+2}} x \in C_{n+2} \subset C_{n+1} \), we have
\[
\phi(x_{n+1}, x) \leq \phi(x_{n+2}, x).
\]

Thus, we can conclude that the limit of \( \{\phi(x_n, x)\} \) exists.

For each \( m \geq 1 \), since \( x_{n+m} \in C_{n+m} \subset C_{n+m-1} \) and Lemma 2.4, we have
\[
\phi(x_{n+m}, x_n) = \phi(x_{n+m}, \Pi_{C_n} x) \\
\leq \phi(x_{n+m}, x) - \phi(\Pi_{C_n} x, x) \\
= \phi(x_{n+m}, x) - \phi(x_n, x).
\]

From the existence of \( \lim_{n \to \infty} \phi(x_n, x) \), we have
\[
\lim_{n \to \infty} \phi(x_{n+m}, x_n) = 0, \quad \text{for each} \quad m \geq 1. \tag{3.4}
\]

It follows from Lemma 2.6 that
\[
\lim_{n \to \infty} \|x_n - x_{n+m}\| = 0, \quad \text{for each} \quad m \geq 1. \tag{3.5}
\]

Thus, the sequence \( \{x_n\} \) is Cauchy. Therefore, there exists \( q \in C \) such that \( x_n \to q \) as \( n \to \infty \). Finally, we show that \( q = \Pi_{\Omega\cap F(S)} x \). Indeed, we have from
\[
x_{n+1} = \Pi_{C_{n+1}} x, \quad \Omega \cap F(S) \subset C_{n+1} \quad \text{and Lemma 2.5 that}
\]
\[
\langle y - x_{n+1}, Jx - Jx_{n+1} \rangle \leq 0, \quad \text{for all} \quad y \in \Omega \cap F(S). \tag{3.6}
\]
By letting \( n \to \infty \) in (3.6) and noting that \( x_n \to q \), we have
\[
\langle y - q, Jx - Jq \rangle \leq 0, \quad \text{for all } y \in \Omega \cap F(S).
\]
Therefore, we can conclude from Lemma 2.5 that
\[ q = \Pi_{\Omega \cap F(S)}x \]
and the proof is complete.

Since every relatively nonexpansive mapping is relatively quasi-nonexpansive mapping, Theorem [3.1] is also true when \( S \) is a relatively nonexpansive mapping and hence we can apply Lemma [2.1] and Theorem [3.1] to get a strong convergence theorem for finding an element in \( \Omega \cap \bigcap_{i=1}^{\infty} F(T_i) \), where \( T_i \) are relatively nonexpansive mappings in Banach spaces as follows.

**Theorem 3.2.** Let \( E_1 \) be a uniformly smooth and uniformly convex Banach space and \( E_2 \) be a uniformly smooth, strictly convex and reflexive Banach space. Let \( A : E_1 \to E_2 \) be a linear and continuous operator. Let \( C \) and \( Q \) be nonempty closed and convex subsets of \( E_1 \) and \( E_2 \), respectively. Assume that \( \{T_i : C \to C\}_{i=1}^{\infty} \) be a sequence of relatively nonexpansive mappings and \( F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R} \) be two bifunctions satisfying the conditions (A1)-(A4) with \( \Omega \cap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \).

Define \( S : C \to C \) by \( Sx = J^{-1} \left( \sum_{i=1}^{\infty} \alpha_i (\beta_i Jx + (1 - \beta_i)JT_i x) \right) \) for each \( x \in C \), where \( \{\alpha_i\}_{i=1}^{\infty} \subset (0,1) \) and \( \{\beta_i\}_{i=1}^{\infty} \subset (0,1) \) are sequences such that \( \sum_{i=1}^{\infty} \alpha_i = 1 \).

Let \( C_1 = C \) and define a sequence \( \{x_n\} \) by the following manner:

\[
\begin{align*}
\text{take } x_1 = x & \in E, \text{ find } v \in E_1 \text{ such that } Av \in Q, \\
V_n & = \{ x \in E_1 : \|x - v\| \leq n \}, \\
U_n & = \{ x \in V_n : Ax \in Q \}, \\
F(u_n, y) + \frac{1}{n} \langle y - u_n, Ju_n - Jx_n \rangle & \geq 0, \forall y \in C, \\
H(Az_n, Ay) + \frac{1}{s_n} \langle y - z_n, Jz_n - Ju_n \rangle & \geq 0, \forall y \in U_n, \\
y_n & = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JT_n z_n) \\
C_{n+1} & = \{ z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
x_{n+1} & = \Pi_{C_{n+1}} x,
\end{align*}
\]

for each \( n \geq 1 \), where \( \{r_n\} \subset [r, \infty) \) with \( r > 0 \), \( \{s_n\} \subset [s, \infty) \) with \( s > 0 \). Then the sequence \( \{x_n\} \) defined by (3.7) converges strongly to a point \( \Pi_{\Omega \cap F(S)}x \), where \( \Pi_{\Omega \cap F(S)} \) is the generalized projection of \( E_1 \) onto \( \Omega \cap F(S) \) and \( F(S) = \bigcap_{i=1}^{\infty} F(T_i) \).
Furthermore, Lemma 2.2 and Theorem 3.1 also allow us to get the following result.

**Theorem 3.3.** Let $E_1$ be a uniformly smooth and uniformly convex Banach space and $E_2$ be a uniformly smooth, strictly convex and reflexive Banach space. Let $A : E_1 \to E_2$ be a linear and continuous operator. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Assume that $\{T_i : C \to C\}_{i=1}^{\infty}$ be a sequence of relatively nonexpansive mappings and $F : C \times C \to \mathbb{R}, H : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions (A1)-(A4) with $\Omega \cap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Define $S : C \to C$ by

$$Sx = J^{-1} \left( \sum_{i=1}^{\infty} \alpha_i JT_ix \right)$$

for each $x \in C$, where $\{\alpha_i\}_{i=1}^{\infty} \subset (0,1)$ is a sequence such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Let $C_1 = C$ and define a sequence $\{x_n\}$ by the following manner:

$$\begin{align*}
take x_1 & = x \in E, find v \in E_1 such that Av \in Q, \\
V_n & = \{x \in E_1 : \|x - v\| \leq n\}, \\
U_n & = \{x \in V_n : Ax \in Q\}, \\
F(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jx_n \rangle & \geq 0, \forall y \in C, \\
H(Az_n, Ay) + \frac{1}{s_n} \langle y - z_n, Jz_n - Ju_n \rangle & \geq 0, \forall y \in U_n, \\
y_n & = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JS\Pi_C z_n) \\
C_{n+1} & = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
x_{n+1} & = \Pi_{C_{n+1}}x, \quad (3.8)
\end{align*}$$

for each $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$. Then the sequence $\{x_n\}$ defined by (3.8) converges strongly to a point $\Pi_{\Omega \cap F(S)}x$, where $\Pi_{\Omega \cap F(S)}$ is the generalized projection of $E_1$ onto $\Omega \cap F(S)$ and $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.

**Acknowledgements:** We would like to thank the referee(s) for comments and suggestions on the manuscript. This work was supported by Chiang Mai University.

**References**


(Received 17 June 2019)
(Accepted 24 December 2019)