On some Algebraic Structures of AG*-groupoids

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Abstract: An AG*-groupoid is an AG-groupoid $S$ satisfying the identity $(ab)c = b(ac)$ for all $a, b, c \in S$. In this paper, we study some properties of AG*-groupoids. Moreover, we construct a congruence relation on a cancellative AG*-groupoid.

Keywords: AG-groupoid; AG*-groupoid; cancellative; regular; congruence.

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1 Introduction and preliminaries

By a groupoid $(S, \cdot)$ we mean a nonempty set $S$ on which a binary operation $\cdot$ is defined. We say that $S$ is an AG-groupoid (Abel-Grassmann’s groupoid) if $\cdot$ is left invertive, that is, $(ab)c = (cb)a$ for all $a, b, c \in S$. The notion of an AG-groupoid was first introduced by Kazim and Naseeruddin in 1977 and they have called it a left almost semigroup (LA-semigroup) [1]. Such a groupoid satisfies the medial law: $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$ [2]. In fact, if $S$ is an AG-groupoid with left identity, then $S$ satisfies the paramedical law: $(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in S$ [3]. If an AG-groupoid satisfies the identity: $(ab)c = b(ac)$ for all $a, b, c \in S$, then it is called AG*-groupoid. It is well known that every AG*-groupoid satisfies the paramedical law. Both AG-groupoids and AG*-groupoids have been wildly studied. Some properties of AG*-groupoids were investigated in [4].
In [5], a description of a fully regular AG*-groupoid was presented. Other algebraic properties of AG-groupoids and AG*-groupoids can be found in [1, 3, 6, 7]. In this paper, we study some algebraic structures of AG*-groupoids. Furthermore, we define a commutative congruence on a cancellative AG*-groupoid.

An element $a$ of a groupoid $S$ is called left (right) cancellative if for every $x, y \in S$, $ax = ay$ ($xa = ya$) implies $x = y$. An element $a$ of a groupoid $S$ is called cancellative if it is both left and right cancellative. An AG-groupoid $S$ is called a left cancellative (right cancellative, cancellative) AG-groupoid if every element of $S$ is left cancellative (right cancellative, cancellative).

We first present some propositions on AG-groupoids most of which will be used later.

**Theorem 1.1** ([3]). Let $S$ be an AG-groupoid. If $a$ is right cancellative of $S$, then $a$ is left cancellative. Hence every right cancellative element of $S$ is cancellative.

**Theorem 1.2** ([3]). Let $S$ be an AG-groupoid with left identity. Every left cancellative element of $S$ is also right cancellative.

**Theorem 1.3** ([7]). Let $S$ be an AG-groupoid with left identity. Then $S$ is commutative if and only if $S$ is associative.

An element $a$ of an AG-groupoid $S$ is called 3-band if $a = (aa)a$.

The following results show that a left cancellative element of an AG-groupoid is right cancellative if it is 3-band.

**Proposition 1.4.** Let $S$ be an AG-groupoid. If $a$ is left cancellative and 3-band, then $a$ is right cancellative.

**Proof.** Suppose that $a$ is a left cancellative element of $S$ and $a = (aa)a$. Let $x, y \in S$ be such that $xa = ya$. Then

$$ax = ((aa)a)x = (xa)(aa) = (ya)(aa) = ((aa)a)y = ay.$$ 

Since $a$ is left cancellative, we have $x = y$. Therefore $a$ is a right cancellative element of $S$. \hfill \Box

The following is an immediate consequence of Proposition 1.4.

**Corollary 1.5.** Let $a$ be an element of an AG-groupoid $S$ such that $a^2 = a$. If $a$ is left cancellative, then $a$ is also right cancellative.

**Proposition 1.6.** Let $S$ be an AG-groupoid. If $a$ is a left cancellative element of $S$ and $a = bc$ for some $b, c \in S$, then $b$ and $c$ are left and right cancellative elements of $S$, respectively.
Proof. Suppose that \(a\) is a left cancellative element of \(S\) and \(a = bc\) for some \(b, c \in S\). Let \(x, y \in S\) be such that \(xc = yc\). Then 

\[
ax = (bc)x = (xc)b = (yc)b = (bc)y = ay.
\]

Since \(a\) is left cancellative, we have \(x = y\). This shows that \(c\) is right cancellative. Let \(x, y \in S\) be such that \(bx = by\). Then 

\[
a(xc) = (bc)(xc) = (bx)(cc) = (by)(cc) = (bc)(yc) = a(yc).
\]

Since \(a\) is left cancellative and \(c\) is right cancellative, we deduce that \(x = y\). Hence \(b\) is left cancellative.

An element \(a\) of AG-groupoid \(S\) is called a regular element of \(S\) if \(a = (ax)a\) for some \(x \in S\).

**Proposition 1.7.** Let \(S\) be an AG-groupoid. If \(a\) and \(b\) are regular elements of \(S\), then \(ab\) is a regular element of \(S\). In particular, the set of all regular elements of \(S\) becomes an AG-subgroupoid of \(S\) if it is nonempty.

Proof. Suppose that \(a\) and \(b\) are regular elements of \(S\). Then \(a = (ax)a\) and \(b = (by)b\) for some \(x, y \in S\). Since 

\[
ab = ((ax)a)((by)b) = ((ax)(by))(ab) = ((ab)(xy))(ab),
\]

it follows that \(ab\) is a regular element of \(S\). \(\square\)

2 Main Results

We first study some properties of cancellative elements of an AG*-groupoid.

**Theorem 2.1.** Let \(a\) be an element of an AG*-groupoid \(S\). Then the following statements are equivalent.

(i) \(a\) is a left cancellative element of \(S\).

(ii) \(a\) is a right cancellative element of \(S\).

(iii) \(a\) is a cancellative element of \(S\).

Proof. (i) \(\Rightarrow\) (ii) Suppose that \(a\) is a left cancellative element of \(S\). Let \(x, y \in S\) be such that \(xa = ya\). Since 

\[
a(a(ay)) = a((aa)y) = a((ya)a) = a((xa)a) = a((aa)x) = a(a(ax))
\]

and by assumption, we deduce that \(x = y\). Therefore \(a\) is right cancellative.

(ii) \(\Rightarrow\) (iii) It is clearly by Theorem 1.1.

(iii) \(\Rightarrow\) (i) Obvious. \(\square\)
Theorem 2.2. Let $a$ and $b$ be elements of an AG*-groupoid $S$. Then $a$ and $b$ are cancellative if and only if $ab$ is cancellative. In particular, the set of all cancellative elements of $S$ is an AG*-subgroupoid of $S$ if it is nonempty.

Proof. Suppose that $a$ and $b$ are cancellative elements of $S$. Let $x, y \in S$ be such that $x(ab) = y(ab)$. This implies that $(ax)b = x(ab) = y(ab) = (ay)b$. By cancellativity of $b$ and $a$, we deduce that $x = y$. Therefore $ab$ is right cancellative. By Theorem 2.1, $ab$ is cancellative of $S$.

Conversely, it follows directly from Proposition 1.6 and Theorem 2.1.

Immediately we adapt the statement by using Theorem 2.2 to obtain to following corollary.

Corollary 2.3. Let $S$ be an AG*-groupoid such that $S^2 = S$. If $a$ is a cancellative element of $S$, then $a$ is a product of two cancellative elements of $S$.

Proposition 2.4. Let $S$ be an AG*-groupoid and $a, b \in S$. If $a$ is a left cancellative element of $S$ and $a = ab$, then $ab = ba$ and $b^2 = b$.

Proof. Suppose that $a$ is left cancellative and $a = ab$. Then $ab = (ab)b = b(ab) = ba$. Since

$$ab^2 = a(bb) = (ab)(bb) = (ba)(bb) = (bb)(ab) = (bb)a = (ab)b = ab,$$

we have $b^2 = b$ by assumption.

Proposition 2.5. Let $S$ be an AG*-groupoid. If $a$ is a left cancellative element of $S$ and $a^2 = a$, then $a$ is a left identity of $S$.

Proof. Let $b \in S$. Since $ab = (aa)b = a(ab)$ and $a$ is left cancellative, we deduce that $b = ab$. This shows that $a$ is a left identity of $S$.

In fact, if $e$ is a right identity of an AG-groupoid $S$, then $ab = (ae)b = (be)a = ba$ for all $a, b \in S$. Hence $S$ is commutative. If $e$ is a left identity of $S$, then $e$ is right cancellative. To prove this, let $a, b \in S$ be such that $ae = be$. Then $a = ea = (ee)a = (ae)e = (be)e = (ee)b = eb = b$.

Lemma 2.6. Let $e$ be an element of an AG*-groupoid $S$. Then $e$ is a right identity of $S$ if and only if $e$ is a left identity of $S$. In this case, $S$ is commutative.

Proof. As mentioned, if $e$ is a right identity of $S$, then $S$ is commutative. Hence $e$ is a left identity of $S$.

For the converse, assume that $e$ is a left identity of $S$. From the above observation, $e$ is right cancellative. Let $a \in S$. Since $(ae)e = e(ae) = ae$, we deduce that $ae = a$. This shows that $e$ is a right identity of $S$.

Note that Lemma 2.6 does not hold for AG-groupoid. The counterexample is given as follows.
Example 2.7. Let \( S = \{a, b, c, d, e\} \) with the following Cayley table as shown in Table 1.

<table>
<thead>
<tr>
<th>·</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>e</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>d</td>
<td>e</td>
<td>b</td>
<td>d</td>
</tr>
<tr>
<td>e</td>
<td>a</td>
<td>d</td>
<td>e</td>
<td>b</td>
<td>e</td>
</tr>
</tbody>
</table>

Table 1.

By routine calculation to prove that \( S \) is an AG-groupoid but not an AG*-groupoid since \((cd)e \neq d(ce)\). We see that \( b \) is a left identity element in \( S \) but not right identity.

The next corollary follows directly from Lemma 2.6 and Theorem 1.3

**Corollary 2.8.** If \( S \) is an AG*-groupoid with left identity, then \( S \) is a commutative semigroup.

The quoted results will be indispensable for our proof.

**Theorem 2.9.** [4] Let \( S \) be an AG*-groupoid. Then the following statements hold:

(i) \( a^2(bc) = (a^2)b \) \( \) for all \( a, b, c \in S \).
(ii) \( (ab)c^2 = a(bc^2) \) for all \( a, b, c \in S \).
(iii) \( (ab^2)c = a(b^2c) \) for all \( a, b, c \in S \).

**Theorem 2.10.** Let \( a \) be an element of an AG*-groupoid \( S \) with \( a^2 = a \) and let

\[ Q_a = \{ x \in S \mid ax = x \} \]

Then

(i) \( Q_a \) is a commutative monoid.
(ii) \( Q_a \) has the identity element.
(iii) \( Q_a \) is an ideal of \( S \).

**Proof.** Since \( a \in Q_a \), it is a nonempty subset of \( S \). Let \( x, y \in Q_a \). Then \( ax = x \) and \( ay = y \), and hence \( a(xy) = (aa)(xy) = (ax)(ay) = xy \). So \( xy \in Q_a \). By virtue of Theorem 2.9 (ii), we have

\[ xy = (ax)(ay) = (yx)(aa) = x(y(aa)) = (xy)(aa) = (ay)(ax) = yx. \]

These prove that \( Q_a \) is a commutative semigroup having \( a \) as its identity. Hence (i) and (ii) hold. Let \( s \in S \) and \( x \in Q_a \). Then

\[ a(sx) = (aa)(sx) = (as)(ax) = (as)x = (x)s = a(sx) = sx \]

which implies that \( sx \in Q_a \). Also, we have \( a(xs) = (xa)s = xs \). Hence (iii) holds.

\[ \square \]
An element $a$ of an AG-groupoid $S$ is called:

- a left regular element of $S$ if $a = xa^2$ for some $x \in S$.
- a right regular element of $S$ if $a = a^2x$ for some $x \in S$.
- a completely regular element of $S$ if $a$ is regular, left regular and right regular.
- a $(2,2)$-regular element of $S$ if $a = (a^2x)a^2$ for some $x \in S$.
- a weakly regular element of $S$ if $a = (ax)(ay)$ for some $x, y \in S$.
- a left quasi regular element of $S$ if $a = (xa)(ya)$ for some $x, y \in S$.
- an intra-regular element of $S$ if $a = (xa)2y$ for some $x, y \in S$.
- a strongly regular element of $S$ if $a = (ax)a$ and $ax = xa$ for some $x \in S$.

Next, to show that regular, left regular, right regular, completely regular, $(2,2)$-regular, weakly regular, left quasi regular, intra-regular and strongly regular coincide in any AG*-groupoids. The following lemmas are needed.

**Lemma 2.11.** Let $S$ be an AG*-groupoid. Then $(ab)(ba) = (ba)(ab)$ for all $a, b \in S$.

*Proof.* Let $a, b \in S$. Then $(ab)(ba) = ((ba)b)a = b((ba)a) = b((aa)b) = b(aa)b = ((ab)a)b = (ba)(ab)$. \hfill \Box

**Lemma 2.12.** Let $S$ be an AG*-groupoid and $a \in S$. If $a = (ax)a$ for some $x \in S$, then $ax$ is an idempotent of $S$.

*Proof.* Suppose that $a = (ax)a$ for some $x \in S$. Then

$$ax = ((ax)a)x = (x(aa))x = (aa)(xx) = (ax)(ax).$$

This shows that $ax$ is idempotent. \hfill \Box

**Theorem 2.13.** Let $a$ be an element of an AG*-groupoid $S$. Then the following statements are equivalent.

1. $a$ is regular of $S$.
2. $a$ is left regular of $S$.
3. $a$ is right regular of $S$.
4. $a$ is completely regular of $S$.
5. $a$ is $(2,2)$-regular of $S$.
6. $a$ is weakly regular of $S$.
7. $a$ is left quasi regular of $S$.
8. $a$ is intra-regular of $S$.
(ix) \( a \) is strongly regular of \( S \).

Proof. Only sample proofs are necessary.

(ii) \( \Rightarrow \) (iii) Suppose that \( a \) is left regular of \( S \). Then \( a = xa^2 \) for some \( x \in S \).

From Lemma 2.11 we have
\[
a = x(aa) = (ax)a = ((xa)(ax))a = ((ax)(xa))a = x((aa)a)x = a^2x.
\]

Hence \( a \) is right regular.

(iii) \( \Rightarrow \) (iv) Suppose that \( a \) is right regular of \( S \). Then there exists \( x \in S \) such that \( a = a^2x \). From Theorem 2.9 we have
\[
a = (aa)x = (((aa)x)a)x = ((aa)(aa)x) = ((aa)(aa)x) = (ax)(aa)x = (aa)^2 = xa^2.
\]

This shows that \( a \) is regular and left regular. Thus \( a \) is completely regular.

(iv) \( \Rightarrow \) (v) Suppose that \( a \) is completely regular of \( S \). Then \( a \) is right regular.

Hence \( a = a^2x \) for some \( x \in S \). From Theorem 2.9 we have
\[
a = (aa)x = (xa)a = (a^2x)a = (a^2)(a^2)x = (a^2)(a^2)x = (a^2)(a^2)x = a^2(xa^2)
\]
\[
= (aa)x^2 = (a^2)(a^2)x^2 = (a^2)(a^2)x^2 = a^2(xa^2)
\]

This shows that \( a \) is \((2,2)\)-regular.

(v) \( \Rightarrow \) (vi) Suppose that \( a \) is \((2,2)\)-regular of \( S \). Then \( a = (a^2x)a^2 \) for some \( x \in S \). Since \( a = (a^2x)a^2 = ((aa)x)(aa) = (a(ax))(aa) \), we have that \( a \) is weakly regular of \( S \).

(vi) \( \Rightarrow \) (vii) Suppose that \( a \) is left quasi regular of \( S \). Then \( a = (xa)(ya) \) for some \( x, y \in S \). Since \( a = (xa)(ya) = (aa)(yx) = (yx)(aa) = (ya^2)x \), we have that \( a \) is intra-regular.

(vii) \( \Rightarrow \) (viii) Suppose that \( a \) is intra-regular. Then \( a = (xa^2)y \) for some \( x, y \in S \). Since \( a = (xa^2)y = x(a^2y) = (a^2)(a^2)y = (yx)(aa) = (a(yx))a \), we have that \( a \) is regular.

(i) \( \Rightarrow \) (ix) Suppose that \( a \) is regular of \( S \). Then \( a = (ax)a \) for some \( x \in S \). It suffices to show that \( ax = xa \). By Lemma 2.12 we have that \( ax \) is idempotent of \( S \). Then \( ax = (ax)(ax) = (xx)(aa) = x'(axa) = x((ax)a) = xa \).

Hence the theorem is completely proved. \( \qed \)

Lemma 2.12 and Theorem 2.13 are not true in an AG-groupoid shown in Table 2.

**Example 2.14.** Let \( S = \{a, b, c, d\} \) and the binary operation \( \cdot \) defined on \( S \) as follows:

\[
\begin{array}{cccc}
  \cdot & a & b & c & d \\
  a & b & b & d & d \\
  b & b & b & b & b \\
  c & a & b & c & d \\
  d & a & b & a & b \\
\end{array}
\]

Table 2.
It is a routine matter to verify that $S$ is an AG-groupoid not an AG*-groupoid since $(ac)d \neq c(ad)$ and we see that $a$ is regular since $a = (ad)a$. But $a$ is not right regular of $S$ and $ad$ is not idempotent of $S$.

To characterize the regular elements of an AG*-groupoid, the following theorems are shown.

**Theorem 2.15.** Let $S$ be a cancellative AG*-groupoid and $a, b \in S$. If $ab$ is regular, then $a$ and $b$ are regular of $S$.

**Proof.** Let $ab$ be regular of $S$. By Theorem 2.13, $ab$ is right regular of $S$. Then $ab = ((ab)(ab))x$ for some $x \in S$. Since

$$ab = ((ab)(ab))x = (b(a(ab)))x = (x(a(ab)))b$$

and by cancellativity of $b$, we have $a = x(a(ab)) = (ax)(ab)$. This implies that $a$ is weakly regular of $S$. Hence $a$ is regular. Since

$$ab = ((ax)(ab))b = ((aa)(xb))b = (aa)((xb)b) = a((xb)b)$$

and by cancellativity of $a$, we have that $b = a((xb)b) = a(bb)x = (a(bb))x$. This shows that $b$ is intra-regular of $S$. By Theorem 2.13, $b$ is regular.

**Corollary 2.16.** Let $S$ be a cancellative AG*-groupoid and $a \in S$. If $a = (ax)a$ for some $x \in S$, then $x$ is regular of $S$.

**Proof.** Suppose that $a = (ax)a$ for some $x \in S$. By Lemma 2.12, $ax$ is idempotent of $S$ which implies that $ax$ is regular. It follows directly from Theorem 2.15 that $x$ is regular.

As consequence of Proposition 1.7 and Theorem 2.15, the following result follows immediately.

**Corollary 2.17.** Let $S$ be a cancellative AG*-groupoid and $a, b \in S$. Then $a$ and $b$ are regular if and only if $ab$ is regular.

Finally, we define a relation $\rho$ on a cancellative AG*-groupoid $S$ by

$$a\rho b \quad \text{if and only if} \quad ab = ba$$

for all $a, b \in S$. Then we have

**Theorem 2.18.** Let $S$ be a cancellative AG*-groupoid. Then the following statements hold:

(i) $\rho$ is an equivalence relation.

(ii) $\rho$ is a congruence.

(iii) $S/\rho$ is a cancellative AG*-groupoid.
(iv) $S/\rho$ is a commutative AG*-groupoid.

Proof. (i) Clearly, $\rho$ is reflexive and symmetry. Let $a,b,c \in S$ be such that $(a,b),(b,c) \in \rho$. Then $ab = ba$ and $bc = cb$. Since
\[(ac)b = c(ab) = c(ba) = (bc)a = (cb)a = (ab)c = (ba)c = (ca)b\]
and by cancellativity of $b$, we have $ac = ca$. This shows that $\rho$ is transitive. Hence $\rho$ is an equivalence relation on $S$.

(ii) Let $a,b \in S$ be such that $(a,b) \in \rho$. To show that $(ac,bc) \in \rho$ for all $c \in S$. Let $c \in S$. Then $(ac)(bc) = (ab)(cc) = (ba)(cc) = (bc)(ac)$ and $(ca)(cb) = (cc)(ab) = (cc)(ba) = (cb)(ca)$. These show that $(ac,bc),(ca,cb) \in \rho$. Hence $\rho$ is a congruence.

(iii) Let $a,b,c \in S$ be such that $(a\rho,b\rho) = (a\rho)(c\rho)$. Then $(ab,ac) \in \rho$. To show that $b\rho = c\rho$, let $x \in b\rho$. Then $xb = bx$ and hence $(xb)a = (bx)a$. Since
\[(ab)x = (xb)a = (bx)a = (ax)b = x(ab),\]
we deduce that $x \in ab\rho = ac\rho$. Then $x(ac) = (ac)x$. Since
\[(cx)a = (ax)c = x(ac) = (ac)x = (xc)a.\]
and by cancellativity of $a$, we have $cx = xc$. It shows that $x \in c\rho$. That is, $b\rho \subseteq c\rho$. Similary, we also have $c\rho \subseteq b\rho$. Hence $b\rho = c\rho$. This proves that $a\rho$ is left cancellative of $S/\rho$. By Theorem 2.11, $a\rho$ is cancellative of $S/\rho$.

(iv) Let $a\rho,b\rho \in S/\rho$. By Lemma 2.11, we have that $(ab)(ba) = (ba)(ab)$. It follows that $(ab)a\rho(ba)$. Therefore $(a\rho)(b\rho) = (b\rho)(a\rho)$.

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References


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