The Characterization of Caterpillars with Multidimension 3

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Abstract: Let $v$ be a vertex of a connected graph $G$, and let $W = \{w_1, w_2, ..., w_k\}$ be a set of vertices of $G$. The multirepresentation of $v$ with respect to $W$ is the $k$-multiset $mr(v|W) = \{d(v,w_1), d(v,w_2), ..., d(v,w_k)\}$. A set $W$ is called a multiresolving set of $G$ if no two vertices of $G$ have the same multirepresentations with respect to $W$. The multidimension of $G$ is the minimum cardinality of a multiresolving set of $G$. In this paper, we characterize the caterpillars with multidimension 3.

Keywords: caterpillar; multirepresentation; multiresolving set; multidimension.

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1 Introduction

The distance $d(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. For an ordered set $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex $v$ of $G$, the $k$-vector

$$r(v|W) = (d(v,w_1), d(v,w_2), ..., d(v,w_k))$$

is called a representation of $v$ with respect to $W$. If every two distinct vertices of $G$ have distinct representations with respect to $W$, then the ordered set $W$ is called a resolving set of $G$. A resolving set of $G$ having a minimum cardinality is called

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A minimum resolving set or a basis of $G$ and this cardinality is the dimension of $G$, and is denoted by $\dim(G)$. To illustrate these concepts, consider a connected graph $G$ of Figure 1 with a vertex set $V(G) = \{u, v, w, x, y, z\}$.

![Image of a connected graph](image)

**Figure 1:** A connected graph $G$

We consider an ordered set $W = \{u, z\}$. There are six representations of vertices with respect to $W$:

- $r(u|W) = (0, 4)$,
- $r(v|W) = (1, 3)$,
- $r(w|W) = (3, 3)$,
- $r(x|W) = (2, 2)$,
- $r(y|W) = (3, 1)$,
- $r(z|W) = (4, 0)$.

Since the representations of two distinct vertices with respect to $W$ are distinct, it follows that $W$ is a resolving set of $G$. Since there is no 1-resolving set of $G$, it implies that $W$ is a basis of $G$, that is, $\dim(G) = 2$.

The concepts of resolving sets and minimum resolving sets have previously appeared in [1], [2] and [3]. Hulme, Shiver and Slater described in [4], [5] and [6] the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Independently, Harary and Melter [7] discovered these concepts as well. Recently, these concepts were rediscovered by Johnson [8] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. A basic problem in chemistry is to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. The structure of a chemical compound can be represented by a labeled graph whose vertex and edge labels specify the atom and bond types, respectively. Thus, a graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. More applications of these concepts to navigation of robots in networks and other areas are discussed in [9].

The foregoing discussion then gives rise to representations that is like multisets. In this case, we consider those sets $W$ of vertices of connected graphs $G$ for which any two vertices of $G$ having distinct representations with respect to $W$ in term of multisets.

Let $W = \{w_1, w_2, \ldots, w_k\}$ be a set of vertices of a connected graph $G$. For each vertex $v$ of $G$, the multirepresentation of $v$ with respect to $W$ is a $k$-multiset, which is denoted by $mr_G(v|W)$ or simply $mr(v|W)$ if the graph $G$ under consideration is clear, and defined by

$$mr(v|W) = \{d(v, w_1), d(v, w_2), \ldots, d(v, w_k)\}.$$
If \( mr(x|W) \neq mr(y|W) \) for every pair \( x, y \) of distinct vertices of \( G \), then \( W \) is called a multiresolving set of \( G \). A multiresolving set of \( G \) containing a minimum number of vertices is called a minimum multiresolving set or a multibasis of \( G \). The cardinality of multibasis is a multidimension of \( G \), which is denoted by \( \dim_M(G) \).

To illustrate these concepts, consider a connected graph \( G \) of Figure 1. As we know that the set \( W = \{ u, z \} \) is a resolving set of \( G \). However, since \( mr(v|W) = \{ 1, 3 \} = mr(y|W) \), it follows that \( W \) is not a multiresolving set of \( G \). Indeed, the set \( W' = \{ u, v, z \} \) is a multiresolving set of \( G \) with multirepresentations of the vertices of \( G \) with respect to \( W' \) as

\[
\begin{align*}
mr(u|W') &= \{ 0, 1, 4 \}, & mr(v|W') &= \{ 0, 1, 3 \}, & mr(w|W') &= \{ 2, 3, 3 \}, \\
mr(x|W') &= \{ 1, 2, 2 \}, & mr(y|W') &= \{ 1, 2, 3 \}, & mr(z|W') &= \{ 0, 3, 4 \}.
\end{align*}
\]

Since there is no multiresolving sets of cardinality 1 or 2, it follows that \( W' \) is a multibasis of \( G \), that is \( \dim_M(G) = 3 \).

Not all connected graphs have a multiresolving set and also \( \dim_M(G) \) is not defined for all connected graphs \( G \). For example, the star \( K_{1,3} \) has no multiresolving set. Therefore, \( \dim_M(K_{1,3}) \) is not defined. However, for a connected graph \( G \) of order \( n \) that \( \dim_M(G) \) is defined, every multiresolving set of \( G \) is also a resolving set of \( G \), and so

\[ 1 \leq \dim(G) \leq \dim_M(G) \leq n. \]

For every set \( W \) of vertices of a connected graph \( G \), the vertices of \( G \) whose multirepresentations with respect to \( W \) contain 0, are vertices in \( W \). On the other hand, the multirepresentations of vertices of \( G \) that do not belong to \( W \) have elements, all of which are positive. Indeed, to determine whether a set \( W \) is a multiresolving set of \( G \), the vertex set \( V(G) \) can be partitioned into \( W \) and \( V(G) \setminus W \) to examine whether the vertices in each subset have distinct multirepresentations with respect to \( W \). The multiresolving set was introduced in [10] and further studied in [11] and [12].

## 2 Preliminaries

Two vertices \( u \) and \( v \) of a connected graph \( G \) are distance-similar if \( d(u, x) = d(v, x) \) for all \( x \in V(G) \setminus \{ u, v \} \). Certainly, distance similarity in \( G \) is an equivalence relation on \( V(G) \). For example, consider a complete bipartite graph \( K_{r,s} \) with partite sets \( U \) and \( V \). Every pair of vertices in the same partite set are distance-similar. Then the distance-similar equivalence classes in \( K_{r,s} \) are its partite sets \( U \) and \( V \). The following results were obtained in [10] showing the usefulness of the distance-similar equivalence class to determine the multidimensions of connected graphs.

**Theorem 2.1** ([10]). Let \( G \) be a connected graph such that \( \dim_M(G) \) is defined. If \( U \) is a distance-similar equivalence class in \( G \) with \( |U| = 2 \), then every multiresolving set of \( G \) contains exactly one vertex of \( U \).
Theorem 2.2 ([10]). If $U$ is a distance-similar equivalence class in a connected graph $G$ with $|U| \geq 3$, then $\dim_M(G)$ is not defined.

It was shown in [10] and [12] that a path is only a connected graph with multidimension 1, and there is no connected graph with multidimension 2. We state these results in the next theorems.

Theorem 2.3 ([10], [12]). Let $G$ be a connected graph. Then $\dim_M(G) = 1$ if and only if $G = P_n$, the path of order $n$.

Theorem 2.4 ([10], [12]). A connected graph has no multiresolving set of cardinality 2.

As we already mentioned, if $W$ is a multiresolving set of a connected graph $G$, then the multirepresentations of two distinct vertices of $G$ are distinct. This leads us to the fact that $W$ is also a multiresolving set of $G - v$, where $v$ is an end-vertex of $G$.

Theorem 2.5. Let $G$ be a connected graph such that $\dim_M(G)$ is defined, and let $W$ be a multiresolving set of $G$. If $v$ is an end-vertex of $G$ such that $v \notin W$, then $W$ is a multiresolving set of $G - v$.

Proof. Assume that $v$ is an end-vertex of $G$. Let $W = \{w_1, w_2, ..., w_k\}$ be a multiresolving set of $G$ that does not contain $v$. Then

$$mr_G(x|W) = \{d_G(x, w_1), d_G(x, w_2), ..., d_G(x, w_k)\}$$

and

$$mr_G(y|W) = \{d_G(y, w_1), d_G(y, w_2), ..., d_G(y, w_k)\}$$

are not the same for all vertices $x$ and $y$ of $G$. Since $v$ does not belong to $W$, it follows that

$$mr_{G-v}(x|W) = \{d_{G-v}(x, w_1), d_{G-v}(x, w_2), ..., d_{G-v}(x, w_k)\} = mr_G(x|W)$$

and

$$mr_{G-v}(y|W) = \{d_{G-v}(y, w_1), d_{G-v}(y, w_2), ..., d_{G-v}(y, w_k)\} = mr_G(y|W),$$

that is, $mr_{G-v}(x|W) \neq mr_{G-v}(y|W)$ for all vertices $x$ and $y$ of $G - v$. Hence, $W$ is a multiresolving set of $G - v$. The following is an immediate corollary of Theorem 2.5.

Corollary 2.6. Let $G$ be a connected graph such that $\dim_M(G)$ is defined, and let $W$ be a multiresolving set of $G$. If $v_1, v_2, ..., v_t \notin W$ are end-vertices of $G$, then $W$ is a multiresolving set of $G - \{v_1, v_2, ..., v_t\}$.

Next, we present a useful necessary condition for a set to be a multiresolving set.
Proposition 2.7. Let $T$ be a tree of order at least 3 containing a vertex $u$. If $W$ is a multiresolving set of $T$, then $W$ contains at least one vertex from each of $\deg_T u$ components of $T - u$, with one possible exception.

Proof. We see that $T - u$ has only one component if and only if $u$ is an end-vertex of $T$. Then we may assume, to the contrary, that there is a vertex $u$ of degree at least 2 such that $T - u$ has two components $X$ and $Y$ containing no vertex of $W$. Then there are two vertices $x$ of $X$ and $y$ of $Y$ that are adjacent to $u$ in $T$. Thus, $d(x,w) = d(u,w) + 1 = d(y,w)$ for all vertices $w$ of $W$. This implies that $mr(x|W) = mr(y|W)$, and so $W$ is not a multiresolving set of $T$. \hfill \qed

3 The Characterization of Caterpillars with Multidimension 3

A caterpillar is a tree of order at least 3, the removal of whose end-vertices produces a path called the spine of the caterpillar. A vertex of the spine of the caterpillar is called a spine-vertex. Let $T$ be a caterpillar that $\dim_M(T)$ is defined. Since any two end-vertices that are adjacent to the same spine-vertex of $T$ are distance-similar, it follows by Theorem 2.2 that there are at most two end-vertices that are adjacent to each spine-vertex of $T$. Therefore, we consider multiresolving sets of such a caterpillar. In order to do this, let us introduce some additional definitions and notation. For integers $s,k_1,k_2,...,k_s$ with $s \geq 1, 1 \leq k_1,k_s \leq 2$ and $0 \leq k_2,k_3,...,k_{s-1} \leq 2$, let $ca(k_1,k_2,...,k_s)$ be a caterpillar which is obtained from the spine $(u_1,u_2,...,u_s)$ by joining $k_i$ end-vertices to the spine-vertex $u_i$, where $1 \leq i \leq s$. Observe that, if $k_i = 0$, then there is no end-vertex joining to the spine-vertex $u_i$. Also, if $k_i = 1$, then the spine-vertex $u_i$ is adjacent to an end-vertex which is called the first end-vertex $v_i$ of $u_i$. Furthermore, if $k_i = 2$, then there are two end-vertices joining to $u_i$ that are called the first and second end-vertices of $u_i$ and denoted by $v_i$ and $w_i$, respectively. Moreover, let $\Psi$ be a set of all integers $i$ with $k_i = 2$, that is, $\Psi = \{i \in \mathbb{Z} \mid k_i = 2\}$. This is illustrated in Figure 2 for the caterpillar $ca(1,2,0,2,1,2,2)$ with $\Psi = \{2,4,6,7\}$.

![Figure 2: The caterpillar ca(1,2,0,2,1,2,2)](image)

For integer $s$ with $1 \leq s \leq 2$, the caterpillars $ca(k_1)$ and $ca(k_1,k_2)$ are shown in Figure 3 where the vertices of multibasis of these caterpillars are indicated by
solid vertices. Notice that \( \text{ca}(2) \cong P_3, \text{ca}(1,1) \cong P_4 \) and \( \text{ca}(1,2) \cong \text{ca}(2,1) \). This implies that there is no caterpillar having multidimension 3, where \( s = 1 \), and there are two distinct caterpillars having multidimension 3, where \( s = 2 \). For \( s = 3 \), it is routine to verify that \( \text{ca}(1,0,2) \cong \text{ca}(2,0,1), \text{ca}(1,1,1), \text{ca}(1,1,2) \cong \text{ca}(2,1,1), \text{ca}(2,0,2) \) and \( \text{ca}(2,1,2) \) are caterpillars having multidimension 3. For \( s \geq 4 \), we are prepared to establish a characterization of a caterpillar \( \text{ca}(k_1,k_2,\ldots,k_s) \) with multidimension 3. In order to do this, we first present several preliminary results.

**Proposition 3.1.** Let \( s, \alpha, \beta \) be integers with \( s \geq 4 \) and \( 1 \leq \alpha < \beta \leq s \), and let \( W \) be a set of vertices of a caterpillar \( \text{ca}(k_1,k_2,\ldots,k_s) \) containing one of \( \{v_1, w_1\} \) and one of \( \{v_s, w_s\} \). If \( \text{mr}(u_\alpha|W) = \text{mr}(u_\beta|W) \) or \( \text{mr}(v_\alpha|W) = \text{mr}(v_\beta|W) \), then \( 1 \leq \alpha \leq \lfloor \frac{s}{2} \rfloor \) and \( \beta = s - \alpha + 1 \).

**Proof.** (i) Suppose that \( \text{mr}(u_\alpha|W) = \text{mr}(u_\beta|W) \). Without loss of generality, assume that \( W \) contains \( v_1 \) and \( v_s \). For \( 1 \leq \alpha < \beta \leq \lfloor \frac{s}{2} \rfloor \), since \( d(u_\alpha, v_s) = s - \alpha + 1 \) and \( d(u_\beta, v_s) = s - \beta + 1 \) are the maximum elements of \( \text{mr}(u_\alpha|W) \) and \( \text{mr}(u_\beta|W) \), respectively, it follows that \( \alpha = \beta \), which is a contradiction. For \( \lfloor \frac{s}{2} \rfloor + 1 \leq \alpha < \beta \leq s \), since \( d(u_\alpha, v_1) = \alpha \) and \( d(u_\beta, v_1) = \beta \) are the maximum elements of \( \text{mr}(u_\alpha|W) \) and \( \text{mr}(u_\beta|W) \), respectively, it follows that \( \alpha = \beta \), a contradiction is produced. Thus, \( 1 \leq \alpha \leq \lfloor \frac{s}{2} \rfloor \) and \( \lfloor \frac{s}{2} \rfloor + 1 \leq \beta \leq s \). Moreover, since \( d(u_\alpha, v_s) = s - \alpha + 1 \) and \( d(u_\beta, v_1) = \beta \) are the maximum elements of \( \text{mr}(u_\alpha|W) \) and \( \text{mr}(u_\beta|W) \), respectively, it follows that \( \beta = s - \alpha + 1 \), as we claimed. (ii) can be obtained in a manner similar to that used in the proof of (i). \( \square \)

**Proposition 3.2.** Let \( s, \gamma, \delta \) be integers with \( s \geq 4 \) and \( 1 \leq \gamma, \delta \leq s \), and let \( W \) be a set of vertices of a caterpillar \( \text{ca}(k_1,k_2,\ldots,k_s) \) containing one of \( \{v_1, w_1\} \) and one of \( \{v_s, w_s\} \). Then

(i) if \( 1 \leq \gamma < \delta \leq s \) and \( \text{mr}(v_\gamma|W) = \text{mr}(u_\delta|W) \), then \( 1 \leq \gamma \leq \lfloor \frac{s}{2} \rfloor \) and \( \delta = s - \gamma + 2 \), and

(ii) if \( 1 \leq \delta \leq \gamma \leq s \) and \( \text{mr}(v_\gamma|W) = \text{mr}(u_\delta|W) \), then \( \lceil \frac{s}{2} \rceil + 1 \leq \gamma \leq s \) and \( \delta = s - \gamma \).
Proof. (i) Suppose that \(1 \leq \gamma < \delta \leq s\) and \(mr(v_\gamma\mid W) = mr(u_\delta\mid W)\). Without loss of generality, let us assume that \(W\) contains \(v_1\) and \(v_s\). If \(1 \leq \gamma < \delta \leq \lceil \frac{s}{2} \rceil\), then \(d(v_\gamma, v_s) = s - \gamma + 2\) and \(d(u_\delta, v_s) = s - \delta + 1\) are the maximum elements of \(mr(v_\gamma\mid W)\) and \(mr(u_\delta\mid W)\), respectively. Therefore, \(\delta = \gamma - 1\), that is, \(\gamma > \delta\), which gives a contradiction. If \(\lceil \frac{s}{2} \rceil + 1 \leq \gamma < \delta \leq s\), then \(d(v_\gamma, v_1) = \gamma + 1\) and \(d(u_\delta, v_1) = \delta\) are the maximum elements of \(mr(v_\gamma\mid W)\) and \(mr(u_\delta\mid W)\), respectively. Thus, \(\delta = \gamma + 1\). Since \(d(v_\gamma, v_s) = s - \gamma + 2\) belongs to \(mr(v_\gamma\mid W)\), there is a vertex \(w\) for which \(w = u_{2\delta-s-3}\) or \(v_{2\delta-s-2}\) or \(w_{2\delta-s-2}\) such that \(d(u_\delta, w) = s - \gamma + 2\). Moreover, since \(d(v_\gamma, w) = d(u_\delta, w) = s - \gamma + 2\), it follows that \(mr(v_\gamma\mid W)\) contains \(s - \gamma + 2\)'s more than \(mr(u_\delta\mid W)\) does, which is impossible. Therefore, \(1 \leq \gamma \leq \lceil \frac{s}{2} \rceil\) and \(\lceil \frac{s}{2} \rceil + 1 \leq \delta \leq s\). Moreover, since \(d(v_\gamma, v_s) = s - \gamma + 2\) and \(d(u_\delta, v_1) = \delta\) are the maximum elements of \(mr(v_\gamma\mid W)\) and \(mr(u_\delta\mid W)\), respectively, it follows that \(\delta = s - \gamma + 2\), as we claimed. For (ii), the statement may be proven in the same way as (i), and therefore such proof is omitted.

An argument similar to the one used in the proof of Propositions 3.1 and 3.2 establishes the following results.

**Proposition 3.3.** Let \(s, \alpha, \beta\) be integers with \(s \geq 4\) and \(1 \leq \alpha < \beta \leq s\), and let \(W\) be a set of vertices of a caterpillar \(ca(k_1, k_2, \ldots, k_s)\) containing \(u_1\) and one of \(\{v_s, w_s\}\) except \(v_1\) and \(w_1\). If \(mr(u_\alpha\mid W) = mr(u_\beta\mid W)\) or \(mr(v_\alpha\mid W) = mr(v_\beta\mid W)\), then \(1 \leq \alpha \leq \lceil \frac{s}{2} \rceil\) and \(\beta = s - \alpha + 2\).

**Proposition 3.4.** Let \(s, \gamma, \delta\) be integers with \(s \geq 4\) and \(1 \leq \gamma, \delta \leq s\), and let \(W\) be a set of vertices of a caterpillar \(ca(k_1, k_2, \ldots, k_s)\) containing \(u_1\) and one of \(\{v_s, w_s\}\) except \(v_1\) and \(w_1\). Then

(i) if \(1 \leq \gamma < \delta \leq s\) and \(mr(v_\gamma\mid W) = mr(u_\delta\mid W)\), then \(1 \leq \gamma \leq \lceil \frac{s}{2} \rceil\) and \(\delta = s - \gamma + 3\), and

(ii) if \(1 \leq \delta \leq \gamma \leq s\) and \(mr(v_\gamma\mid W) = mr(u_\delta\mid W)\), then \(\lceil \frac{s}{2} \rceil + 1 \leq \gamma \leq s\) and \(\delta = s - \gamma + 1\).

For an even integer \(s \geq 4\), let \(T_1\) be a caterpillar \(ca(k_1, k_2, \ldots, k_s)\) such that \(\Psi = \{1, r, s\}\), where \(r \in \{2, 3, \ldots, s - 1\}\). In particular, the caterpillar \(T_1 = ca(2, 0, 2, 1, 0, 1, 0, 2)\) is shown in Figure 4.

Figure 4: The caterpillar \(T_1 = ca(2, 0, 2, 1, 0, 1, 0, 2)\) with \(\Psi = \{1, 3, 8\}\)
For an odd integer \( s \geq 5 \), let \( T_2 \) be a caterpillar \( \text{ca}(k_1, k_2, ..., k_s) \) such that \( \Psi = \{1, r, s\} \), where

\[
\begin{align*}
    r & \in \begin{cases} 
        \{2, 3, ..., s - 1\} - \{3, \frac{s+1}{2}, s - 2\} & \text{if } s \equiv 1 \pmod{4}, \\
        \{2, 3, ..., s - 1\} - \{3, \frac{s+1}{2}, \frac{s+3}{2}, s - 2\} & \text{if } s \equiv 3 \pmod{4}.
    \end{cases}
\end{align*}
\]

For example, the caterpillar \( T_2 = \text{ca}(2, 0, 1, 2, 0, 1, 1, 0, 2) \) is illustrated in Figure 5.

![Figure 5: The caterpillar \( T_2 = \text{ca}(2, 0, 1, 2, 0, 1, 1, 0, 2) \) with \( \Psi = \{1, 4, 9\} \)](image)

For an odd integer \( s \geq 9 \), let \( T_3 \) be a caterpillar \( \text{ca}(k_1, k_2, ..., k_s) \) such that \( \Psi = \{1, 3, s\} \) and \( k_{2s-1} = 0 \), or \( \Psi = \{1, s - 2, s\} \) and \( k_{2s-3} = 0 \). For an odd integer \( s \geq 11 \) and \( s \equiv 3 \pmod{4} \), let \( T_4 \) be a caterpillar \( \text{ca}(k_1, k_2, ..., k_s) \) such that \( \Psi = \{1, \frac{s-1}{2}, s\} \) and \( k_{s+3} = 0 \), or \( \Psi = \{1, \frac{s-1}{2}, s\} \) and \( k_{s+3} = 0 \).

**Proposition 3.5.** A caterpillar \( T_i \), where \( 1 \leq i \leq 4 \) has multidimension 3.

**Proof.** For each integer \( i \) with \( 1 \leq i \leq 4 \), we show that every caterpillar \( T_i \) has multidimension 3. We verify this for \( T_2 \) only since the proof for \( T_1, T_3 \) and \( T_4 \) uses an argument similar to the one for \( T_2 \). First, we verify that \( W = \{w_1, w_r, w_s\} \) is a multiresolving set of \( T_2 \), where \( r \) satisfies the condition (3.1). Without loss of generality, we may assume that \( 2 \leq r \leq \lceil \frac{s}{2} \rceil \). The multirepresentations of vertices of \( W \) with respect to \( W \) are \( \text{mr}(w_1|W) = \{0, r + 1, s + 1\} \), \( \text{mr}(w_r|W) = \{0, r + 1, s - r + 2\} \) and \( \text{mr}(w_s|W) = \{0, s - r + 2, s + 1\} \). Since \( r \notin \{1, \frac{s+1}{2}\} \), it follows that these 3-multisets are distinct. Next, we claim that \( \text{mr}(x|W) \neq \text{mr}(y|W) \) for all vertices \( x, y \in V(T_2) - W \). Suppose, contrary to our claim, that \( \text{mr}(x|W) = \text{mr}(y|W) \) for some vertices \( x, y \in V(T_2) - W \). We consider three cases.

**Case 1.** \( x \) and \( y \) are spine-vertices.

Let \( x = u_\alpha \) and \( y = u_\beta \), where \( 1 \leq \alpha < \beta \leq s \). Then by Proposition 3.1

\[
1 \leq \alpha \leq \lceil \frac{s}{2} \rceil \text{ and } \beta = s - \alpha + 1.
\]

Thus, \( \text{mr}(u_\beta|W) = \{s - \beta + 1, \beta - r + 1, \beta\} = \{\alpha, s - \alpha - r + 2, s - \alpha + 1\} \). Since \( \text{mr}(u_\alpha|W) = \{\alpha, |\alpha - r| + 1, s - \alpha + 1\} \), it follows that \( |\alpha - r| + 1 = s - \alpha - r + 2 \). If \( \alpha \geq r \), then \( 2\alpha = s + 1 \), and so \( \alpha = \beta \) which is impossible. If \( \alpha < r \), then \( r = \frac{s+1}{2} \), a contradiction.

**Case 2.** \( x \) and \( y \) are first end-vertices.

Let \( x = v_\alpha \) and \( y = v_\beta \), where \( 1 \leq \alpha < \beta \leq s \). Then by Proposition 3.1

\[
1 \leq \alpha \leq \lceil \frac{s}{2} \rceil \text{ and } \beta = s - \alpha + 1.
\]

Thus, \( \text{mr}(v_\beta|W) = \{s - \beta + 2, \beta - r + 2, \beta + 1\} = \{s - \alpha + 1, s - \alpha - 1, s + 1\} \). Since \( \text{mr}(v_\alpha|W) = \{s - \alpha, |\alpha - r| + 1, s - \alpha + 1\} \), it follows that \( |\alpha - r| + 1 = s - \alpha - r + 2 \). If \( \alpha \geq r \), then \( 2\alpha = s + 1 \), and so \( \alpha = \beta \) which is impossible. If \( \alpha < r \), then \( r = \frac{s+1}{2} \), a contradiction.
Proposition 3.7. A caterpillar \(\{\alpha + 1, s - \alpha - r + 3, s - \alpha + 2\}\). Since \(\text{mr}(v_\alpha|W) = \{\alpha + 1, |\alpha - r| + 2, s - \alpha + 2\}\), it follows that \(|\alpha - r| + 2 = s - \alpha - r + 3\). If \(\alpha \geq r\), then \(2\alpha = s + 1\), and so \(\alpha = \beta\), which cannot occur. If \(\alpha < r\), then \(r = \frac{s - 1}{2}\), that is also a contradiction.

Case 3. \(x\) is a first end-vertex and \(y\) is a spine-vertex.

Let \(x = v_\gamma\) and \(y = u_\delta\), where \(1 \leq \gamma, \delta \leq s\). We consider two subcases.

Subcase 3.1. \(1 \leq \gamma < \delta \leq s\).

Then by Proposition 3.2 (i), \(1 \leq \gamma \leq \left\lceil \frac{s}{2} \right\rceil\) and \(\delta = s - \gamma + 2\). Thus, \(\text{mr}(u_\delta|W) = \{s - \gamma + 1, s - \gamma - r + 3, s - \gamma + 2\}\). Since \(\text{mr}(v_\gamma|W) = \{\gamma + 1, |\gamma - r| + 2, s - \gamma + 2\}\), it follows that \(|\gamma - r| + 2 = \gamma - 1\) and \(\gamma + 1 = s - \gamma - r + 3\).

If \(\gamma \geq r\), then \(r = 3\), which is impossible. If \(\gamma < r\), then \(s = 4(\gamma - 2) + 3\), that is, \(s \equiv 3 \mod 4\). Also, we obtain that \(2r = s - 1\), and then \(r = \frac{s - 1}{2}\), which is a contradiction.

Subcase 3.2. \(1 \leq \delta \leq \gamma \leq s\).

Then by Proposition 3.2 (ii), \(\left\lceil \frac{s}{2} \right\rceil + 1 \leq \gamma \leq s\) and \(\delta = s - \gamma\). Thus, \(\text{mr}(v_\gamma|W) = \{s - \gamma + 2, s - \gamma - r + 2, s - \gamma + 1\}\). Since \(\text{mr}(u_\delta|W) = \{\delta, |\delta - r| + 1, s - \delta + 1\}\), it follows that \(|\delta - r| + 1 = \delta + 2\) and \(\delta = s - \delta - r + 2\).

Consequently, \(|\delta - r| = s - \delta - r + 3\). If \(\delta \geq r\), then \(2\delta = s + 3\), which cannot occur. If \(\delta < r\), then \(2r = s + 3\), a contradiction.

Therefore, \(\text{mr}(x|W) \neq \text{mr}(y|W)\) for all \(x, y \in V(T_2) - W\), that is, \(W\) is a multiresolving set of \(T_2\) and so \(\dim_M(T_2) \leq 3\). Since \(T_2\) is not a path, it follows by Theorems 2.3 and 2.4 that \(\dim_M(T_2) \geq 3\). Hence, \(\dim_M(T_2) = 3\).

The following corollary is an immediate consequence of Proposition 3.5.

Corollary 3.6. Let \(T\) be a caterpillar \(\text{ca}(k_1, k_2, ..., k_s)\) such that \(T \cong T_1\), where \(1 \leq i \leq 4\) with \(\Psi = \{1, r, s\}\). Then \(W\) is a multibasis of \(T\) if and only if \(W = \{x_1, x_r, x_s\}\), where \(x_i \in \{v_1, w_1\}\) for \(i = 1, r, s\).

For an integer \(s \geq 4\), let \(T_5\) be a caterpillar \(\text{ca}(k_1, k_2, ..., k_s)\) such that \(\Psi = \{p, s\}\) or \(\Psi = \{1, q\}\), where \(1 \leq p < q \leq s\).

Proposition 3.7. A caterpillar \(T_5\) has multidimension 3.

Proof. First, suppose that \(\Psi = \{p, s\}\), where \(1 \leq p \leq s - 1\). Since \(T_5\) is not a path, it follows by Theorems 2.3 and 2.4 that \(\dim_M(T_5) \geq 3\). We consider two cases.

Case 1. \(p = 1\).

We show that \(W = \{u_1, w_1, w_s\}\) is a multiresolving set of \(T_5\). The multirepresentations of vertices of \(W\) with respect to \(W\) are \(\text{mr}(u_1|W) = \{0, 1, s\}\), \(\text{mr}(w_1|W) = \{0, 1, s + 1\}\) and \(\text{mr}(w_s|W) = \{0, s, s + 1\}\). Thus, these 3-multisets are distinct.

Next, we claim that \(\text{mr}(x|W) \neq \text{mr}(y|W)\) for all \(x, y \in V(T_5) - W\).

Assume, contrary to our claim, that \(\text{mr}(x|W) = \text{mr}(y|W)\) for some \(x, y \in V(T_5) - W\). We consider three subcases.

Subcase 1.1. \(x\) and \(y\) are spine-vertices.

Let \(x = u_\alpha\) and \(y = u_\beta\), where \(1 \leq \alpha < \beta \leq s\). Then by Proposition 3.1 \(1 \leq \alpha \leq \left\lceil \frac{s}{2} \right\rceil\) and \(\beta = s - \alpha + 1\). Thus, \(\text{mr}(u_\beta|W) = \{s - \beta + 1, \beta - 1, \beta\} = \{\alpha, s - \alpha, s - \alpha + 1\}\). Since \(\text{mr}(u_\alpha|W) = \{\alpha, \alpha - 1, s - \alpha + 1\}\), it follows that \(\alpha - 1 = s - \alpha\) and so \(\alpha = \beta\), which cannot occur. Therefore, \(\text{mr}(x|W) \neq \text{mr}(y|W)\) for all \(x, y \in V(T_5) - W\).
which is impossible.

**Subcase 1.2.** $x$ and $y$ are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.1, $1 \leq \alpha \leq \lfloor \frac{s}{2} \rfloor$ and $\beta = s - \alpha + 1$. Thus, $mr(v_\beta)W = \{s - \beta + 2, \beta + 1\} = \{\alpha + 1, s - \alpha + 1, s - \alpha + 2\}$. Since $mr(v_\alpha)W = \{\alpha + 1, \alpha, s - \alpha + 2\}$, it follows that $\alpha = s - \alpha + 1$ and so $\alpha = \beta$, this is also a contradiction.

**Subcase 1.3.** $x$ is a first end-vertex and $y$ is a spine-vertex.

Let $x = v_\gamma$ and $y = u_\delta$, where $1 \leq \gamma, \delta \leq s$. We consider two subcases.

**Subcase 1.3.1.** $1 \leq \gamma < \delta \leq s$.

Then by Proposition 3.2 (i), $1 \leq \gamma \leq \lfloor \frac{s}{2} \rfloor$ and $\delta = s - \gamma + 2$. Since $mr(u_\delta)W = \{s - \delta + 1, \delta - 1, \delta\} = \{\gamma - 1, s - \gamma + 1, s - \gamma + 2\}$ and $mr(v_\gamma)W = \{\gamma + 1, \gamma, s - \gamma + 2\}$, it follows that $mr(v_\gamma)W \neq mr(u_\delta)W$, which is impossible.

**Subcase 1.3.2.** $1 \leq \delta \leq \gamma \leq s$.

Then by Proposition 3.2 (ii), $\lfloor \frac{s}{2} \rfloor + 1 \leq \gamma \leq s$ and $\delta = s - \gamma$. Since $mr(v_\gamma)W = \{s - \gamma + 2, \gamma, \gamma + 1\} = \{\beta + 2, s - \delta, s - \delta + 1\}$ and $mr(u_\delta)W = \{\delta, \delta - 1, s - \delta + 1\}$, it follows that $mr(v_\gamma)W \neq mr(u_\delta)W$, this is also a contradiction.

Therefore, $mr(x)W \neq mr(y)W$ for all vertices $x, y \in V(T_5) - W$, that is, $W$ is a multiresolving set of $T_5$. Hence, $\dim_M(T_5) \leq 3$, and so $\dim_M(T_5) = 3$, where $p = 1$.

**Case 2.** $p \geq 2$.

We consider two subcases.

**Subcase 2.1.** $s$ is even.

With the aid of Theorem 2.5 and Corollary 3.6 since $T_5 \cong T_1 - w_1$ and $W = \{v_1, w_p, w_s\}$ is a multiresolving set of $T_1$, it follows that $W$ is a multiresolving set of $T_5$. Therefore, $\dim_M(T_5) \leq 3$, and so $\dim_M(T_5) = 3$, where $p \geq 2$ and $s$ is even.

**Subcase 2.2.** $s$ is odd.

We consider two subcases.

**Subcase 2.2.1.** $p = 2$.

By Theorem 2.5 and Corollary 3.6 since $T_5 \cong T_2 - w_1$ and $W = \{v_1, w_p, w_s\}$ is a multiresolving set of $T_2$, it follows by Theorem 2.5 that $W$ is a multiresolving set of $T_5$. Therefore, $\dim_M(T_5) \leq 3$, and so $\dim_M(T_5) = 3$, where $p = 2$ and $s$ is odd.

**Subcase 2.2.2.** $p \geq 3$.

Let $W = \{u_1, w_p, w_s\}$. The multirepresentations of vertices of $W$ with respect to $W$ are $mr(u_1)W = \{0, p, s\}$, $mr(w_p)W = \{0, p, s - p + 2\}$ and $mr(w_s)W = \{0, s - p + 2, s\}$, Thus, these 3-multisets are distinct. Next, we claim that $mr(x)W \neq mr(y)W$ for all vertices $x, y \in V(T_5) - W$. Suppose, contrary to our claim, that $mr(x)W = mr(y)W$ for some vertices $x, y \in V(T_5) - W$. We consider three subcases.

**Subcase 2.2.2.1.** $x$ and $y$ are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.3, $1 \leq \alpha \leq \lfloor \frac{s}{2} \rfloor$ and $\beta = s - \alpha + 2$. Thus, $mr(u_\beta)W = \{s - \beta + 1, |\beta - p| + 1, \beta - 1\} = \{|\alpha - 1|, |\beta - p| + 1, s - \alpha + 1\}$. Since $mr(u_\alpha)W = \{|\alpha - 1|, |\alpha - p| + 1, s - \alpha + 1\}$, it follows that $|\alpha - p| + 1 = |\beta - p| + 1$. If $p \leq \alpha$ or $\beta \leq p$, then $\alpha = \beta$, which is impossible. If $\alpha < p < \beta$, then $s = 2p - 2$, contradicting the fact that $s$ is odd.

**Subcase 2.2.2.2.** $x$ and $y$ are first end-vertices.
Let \( x = v_\alpha \) and \( y = v_\beta \), where \( 1 \leq \alpha < \beta \leq s \). Then by Proposition 3.3, \( 1 \leq \alpha \leq \left\lceil \frac{s}{2} \right\rceil \) and \( \beta = s - \alpha + 2 \). Thus, \( mr(v_\beta|W) = \{s - \beta + 2, |\beta - p| + 2, \beta\} = \{\alpha, |\beta - p| + 2, s - \alpha + 2\} \). Since \( mr(v_\alpha|W) = \{\alpha, |\alpha - p| + 2, s - \alpha + 2\} \), it follows that \( |\alpha - p| + 2 = |\beta - p| + 2 \). By the same argument as the proof in Subcase 2.2.2.1., we obtain a contradiction.

**Subcase 2.2.2.3.** \( x \) is a first end-vertex and \( y \) is a spine-vertex. Let \( x = v_\gamma \) and \( y = u_\delta \), where \( 1 \leq \gamma, \delta \leq s \). There are two possibilities:

1) \( 1 \leq \gamma < \delta \leq s \).

Then by Proposition 3.4 (i), \( \left\lceil \frac{s}{2} \right\rceil+1 \leq \gamma \leq s \) and \( \delta = s - \gamma + 3 \). Thus, \( mr(u_\delta|W) = \{s - \delta + 1, |\delta - p| + 1, s - \delta + 1\} = \{\gamma - 2, |\delta - p| + 1, s - \gamma + 2\} \). Since \( mr(v_\gamma|W) = \{\gamma, |\gamma - p| + 2, s - \gamma + 2\} \), it follows that \( |\gamma - p| + 2 = \gamma - 2 \) and \( \gamma = |\delta - p| + 1 \). Consequently, \( |\gamma - p| + 3 = |\delta - p| \). If \( p \leq \gamma \), then \( 2\gamma = s \), contradicting the fact that \( s \) is odd. If \( \gamma < p < \delta \), then \( 2p = s \), a contradiction. If \( \delta \leq p \), then \( 2\gamma - 6 = s \), this is also a contradiction.

2) \( 1 \leq \delta \leq \gamma \leq s \).

Then by Proposition 3.4 (ii), \( \left\lceil \frac{s}{2} \right\rceil+1 \leq \gamma \leq s \) and \( \delta = s - \gamma + 1 \). Thus, \( mr(v_\gamma|W) = \{s - \gamma + 2, |\gamma - p| + 2, \gamma\} = \{\delta + 1, |\gamma - p| + 2, s - \delta + 1\} \). Since \( mr(u_\delta|W) = \{\delta - 1, |\delta - p| + 1, s - \delta + 1\} \), it follows that \( |\delta - p| + 1 = \delta + 1 \) and \( \delta - 1 = |\gamma - p| + 2 \). Consequently, \( |\delta - p| = |\gamma - p| + 3 \). If \( p < \delta \), then \( s = 2\gamma + 2 \), contradicting the fact that \( s \) is odd. If \( \delta \leq p \leq \gamma \), then \( s = 2p - 4 \), a contradiction. If \( \gamma < p \), then \( s = 2\delta + 2 \), this is also a contradiction.

Therefore, \( \dim_M(T_5) \leq 3 \), and so \( \dim_M(T_5) = 3 \), where \( p \geq 3 \) and \( s \) is odd. Similarly, for \( \Psi = \{1, q\} \), where \( 2 \leq q \leq s \), \( \dim_M(T_5) = 3 \) can be proven in the same manner as well.

For an integer \( s \geq 4 \), let \( T_5 \) be a caterpillar \( ca(k_1, k_2, \ldots, k_s) \) such that \( \Psi = \{r\} \), where \( r \in \{1, 2, \ldots, s\} \). For an integer \( s \geq 4 \), let \( T_7 \) be a caterpillar \( ca(k_1, k_2, \ldots, k_s) \) such that \( \Psi = \emptyset \) and \( k_r = 1 \), where \( r \in \{2, 3, \ldots, s - 1\} \). Combining Theorem 2.5 and Proposition 3.7, we arrive yet another result.

**Proposition 3.8.** A caterpillar \( T_i \), where \( 6 \leq i \leq 7 \) has multidimension 3.

Caterpillars with multidimension 3 are completely characterized, as we present next.

**Theorem 3.9.** For an integer \( s \geq 4 \), let \( T \) be a caterpillar \( ca(k_1, k_2, \ldots, k_s) \). Then \( T \) has multidimension 3 if and only if \( T \cong T_i \), where \( i \in \{1, 2, \ldots, 7\} \).

**Proof.** The preceding results provide the sufficient condition for a caterpillar \( T \) having multidimension 3. To show the necessary condition, suppose that \( T \) has a multidimension 3. By Theorem 2.4, it implies that \( |\Psi| \leq 3 \). For \( |\Psi| = 0 \), there is an integer \( r \) with \( 2 \leq r \leq s - 1 \) such that \( k_r = 1 \), for otherwise \( T \) is a path, contradicting the fact that \( \dim_M(T) = 3 \). Hence, \( T \cong T_7 \). For \( |\Psi| = 1 \), obviously, \( T \cong T_6 \). It remains therefore only to consider \( |\Psi| = 2 \) and \( |\Psi| = 3 \).

For \( |\Psi| = 2 \), we claim that \( \Psi \) contains at least one of \( \{1, s\} \). Suppose, contrary to our claim, that \( \Psi \) contains neither 1 nor \( s \). Let \( \Psi = \{r_1, r_2\} \), where \( 2 \leq r_1 <
four vertices, this is a contradiction. Thus, $\Psi$ contains at least one of $T$ and so contradicting the fact that $mr$ multibasis of $T - u_{r_1}$, it follows by Proposition 2.7 that there is a vertex of a multibasis $W$ belonging to the component containing the spine-vertex $u_{r_1-1}$. Similarly, since there are $\deg_T u_{r_2} = 4$ distinct components of $T - u_{r_2}$, there is a vertex of $W$ belonging to the component containing the spine-vertex $u_{r_2+1}$. Therefore, $W$ contains at least four vertices, this is a contradiction. Thus, $\Psi$ contains at least one of $\{1, s\}$, that is, $T \cong T_5$.

For $|\Psi| = 3$, we show that $\Psi$ contains both 1 and $s$. Assume, to the contrary, that $\Psi$ does not contain 1 or $s$, say 1. Let $\Psi = \{r_1, r_2, r_3\}$, where $2 \leq r_1 < r_2 < r_3 \leq s$. Then $W = \{w_{r_1}, w_{r_2}, w_{r_3}\}$ is a multibasis of $T$. Notice that $\deg_T u_{r_1} = 4$, that is, there are four distinct components of $T - u_{r_1}$. However, both $w_{r_2}$ and $w_{r_3}$ must belong to the same component containing the spine-vertex $u_{r_1+1}$, contradicting Proposition 2.7 that $w_{r_1}$, $w_{r_2}$ and $w_{r_3}$ cannot belong to the same component of $T - u_{r_1}$. Thus, $\Psi$ contains 1 and $s$. We may assume without loss of generality that $\Psi = \{1, r, s\}$ with $2 \leq r \leq \lfloor \frac{s}{2} \rfloor$. Then $W = \{w_1, w_r, w_s\}$ is a multibasis of $T$. If $s$ is even, then $T \cong T_1$. We may assume that $s$ is odd. If $r = \lfloor \frac{s}{2} \rfloor$, then $mr(w_1|W) = mr(w_s|W)$, which is impossible. Thus $2 \leq r \leq \frac{s-1}{2}$.

Next, we consider two cases according to whether $s$ is congruent to 1 or 3 modulo 4.

Case 1. $s \equiv 1 \pmod{4}$.

If $r \neq 3$, then $T \cong T_2$. For $r = 3$, since $r \leq \frac{s-1}{2}$, it follows that $s \geq 9$. Next, we claim that $k_{\frac{s}{2}-1} = 0$. Suppose, contrary to our claim, that $k_{\frac{s}{2}-1} \geq 1$. Then $mr(v_{\frac{s}{2}-1}|W) = \{\frac{s+5}{2}, \frac{s+1}{2}, \frac{s+3}{2}\} = mr(u_{\frac{s}{2}-1}|W)$, contradicting the fact that $W$ is a multibasis of $T$. Hence, $k_{\frac{s}{2}-1} = 0$, and so $T \cong T_3$.

Case 2. $s \equiv 3 \pmod{4}$.

If $r \neq 3$, then $T \cong T_2$. For $r = 3$, we claim that $k_{\frac{s}{2}+1} = 0$. Suppose, contrary to our claim, that $k_{\frac{s}{2}+1} \geq 1$. Then $mr(v_{\frac{s}{2}+1}|W) = \{\frac{s+3}{2}, \frac{s+1}{2}, \frac{s+5}{2}\} = mr(u_{\frac{s}{2}+1}|W)$, contradicting the fact that $W$ is a multibasis of $T$, as we claimed. Hence, $s \geq 11$, and so $T \cong T_3$. For $r = \frac{s}{2}-1 \geq 4$, since $r \leq \frac{s-1}{2}$, it follows that $s \geq 11$. Next, we claim that $k_{\frac{s}{2}+3} = 0$. Suppose, contrary to our claim that $k_{\frac{s}{2}+3} \geq 1$. Then $mr(v_{\frac{s}{2}+3}|W) = \{\frac{s+5}{2}, \frac{s+1}{2}, \frac{s+3}{2}\} = mr(u_{\frac{s}{2}+3}|W)$, contradicting the fact that $W$ is a multibasis of $T$. Hence, $k_{\frac{s}{2}+3} = 0$, and so $T \cong T_3$. \hfill \Box

4 Final Remarks

For an integer $s \geq 2$, let $T$ be a caterpillar $ca(k_1, k_2, ..., k_r)$ of order $n$ such that $\Psi \neq \emptyset$ and $\dim_M(T)$ is defined. It then follows by Theorem 2.1 that

$$|\Psi| \leq \dim_M(T) \leq n - |\Psi|.$$ 

Moreover, by Corollary 3.6, caterpillars $T_1, T_2, T_3$ and $T_4$ also illustrate the sharpness of this lower bound. It would be interesting to determine whether this upper bound is sharp or not.
References


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