Left and Right Magnifying Elements in Certain Linear Transformation Semigroups

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Abstract: An element $a$ in a semigroup $S$ is called a left (right) magnifying element in $S$ if $aM = S$ ($Ma = S$) for some proper subset $M$ of $S$. In this paper, we determine whether or not the linear transformation semigroups with infinite nullity and co-rank have left and right magnifying elements, and provide a characterization if such elements exist.

Keywords: left (right) magnifying element; linear transformation semigroups; nullity; co-rank.

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1 Introduction

In 1964, Ljapin [1] introduced the notion of left and right magnifying elements in a semigroup, that is, an element $a$ in a semigroup $S$ is said to be a left (right) magnifying element if there is a proper subset $M$ of $S$ such that $aM = S$ ($Ma = S$). Furthermore, a left (right) magnifying element $a$ in $S$ is said to be strong if there is a proper subsemigroup $M$ such that $aM = S$ ($Ma = S$). These were introduced by Tolo [2] in 1969. It is obvious that strongly left (right) magnifying elements in $S$ are left (right) magnifying elements.

Magnifying elements in semigroups have long been studied and many properties have also been investigated. In 1992, Catino and Migliorini [3] characterized when a bisimple semigroup contains left magnifying elements. Gutan [4] showed that semigroups which contain magnifying elements are factorizable. In 1994, Mag-
ill [5] provided necessary and sufficient conditions for elements in linear transformation semigroups to be left or right magnifying elements. Chinram and Buapradist [6] extended, in 2018, the results of Magill by considering linear transformations with invariant subspaces and gave characterization on these semigroups.

Let \( V \) be a vector space and \( L(V) \) the semigroup, under composition, of all linear transformations on \( V \). For any \( \alpha \in L(V) \), let \( \ker \alpha \) and \( \text{im} \alpha \) denote the kernel of \( \alpha \) and the image of \( \alpha \), respectively. The dimension of a subspace \( U \) of \( V \) is denoted by \( \text{dim} U \). The quotient of \( V \) modulo by a subspace \( U \) is written as \( V/U \).

We are interested in studying left or right magnifying elements in some subsemigroups of \( L(V) \). For an infinite dimensional vector space \( V \), let

\[
OM(V) = \{ \alpha \in L(V) \mid \text{dim}(\ker \alpha) \text{ is infinite} \},
\]

\[
OE(V) = \{ \alpha \in L(V) \mid \text{dim}(V/\text{im} \alpha) \text{ is infinite} \}.
\]

These are subsemigroups of \( L(V) \), see [7] for more details. Note that \( \text{dim}(\ker \alpha) \) and \( \text{dim}(V/\text{im} \alpha) \) are called the nullity of \( \alpha \) and the co-rank of \( \alpha \), respectively. Observe that the identity map is not an element in both \( OM(V) \) and \( OE(V) \). Moreover, \( OM(V) \) does not contain any injective linear transformations on \( V \), and similarly, surjective linear transformations on \( V \) are not contained in \( OE(V) \).

In [5], the author characterized that \( L(V) \) contains left (right) magnifying elements if and only if \( \text{dim} V \) is infinite. Moreover, in case \( \text{dim} V \) is infinite, he provided necessary and sufficient conditions when elements in \( L(V) \) are left or right magnifying in \( L(V) \), see below.

**Theorem 1.1** ([5]). Let \( \alpha \in L(V) \) where \( \text{dim} V \) is infinite. The following statements hold.

1. \( \alpha \) is a left magnifying element if and only if \( \alpha \) is surjective but not injective.
2. \( \alpha \) is a right magnifying element if and only if \( \alpha \) is injective but not surjective.

Below is a useful property that will be used in our results.

**Proposition 1.2** ([8]). Let \( \alpha \in L(V) \) and let \( B_1 \) be a basis of \( \ker \alpha \), \( B \) a basis of \( V \) containing \( B_1 \). Then

(i) for each \( v_1, v_2 \in B \setminus B_1 \), \( v_1 = v_2 \) if and only if \( \alpha(v_1) = \alpha(v_2) \);

(ii) \( \alpha(B \setminus B_1) \) is a basis of \( \text{im} \alpha \).

Let \( B \) be a basis of \( V \) and \( u \in V \). A linear transformation on \( V \) can be defined on \( B \). Now let \( \{B_1, B_2\} \) be a partition of \( B \). For \( \alpha \in L(V) \) defined by \( \nu \alpha = u \) and \( \nu \alpha = v_w \) for all \( v \in B_1 \) and \( w \in B_2 \), we write

\[
\alpha = \begin{pmatrix} B_1 & w \\ u & v_w \end{pmatrix}_{w \in B_2}.
\]

We use this notation for an abbreviation of describing many linear transformations all along in this paper.
2 Left and right magnifiers in $OM(V)$ and $OE(V)$

Throughout this section, let $V$ be an infinite dimensional vector space over a field. Our purpose is to provide a necessary and sufficient conditions for an element in $OM(V)$ and $OE(V)$ to be left or right magnifying elements. It has seen from Theorem 1.1 that a linear map in $L(V)$ that is surjective but not injective is a left magnifying element in $L(V)$. In $OM(V)$, every element is not injective. We first show a necessary and sufficient condition for element in $OM(V)$ to be a left magnifying element in $OM(V)$.

**Theorem 2.1.** Let $\alpha \in OM(V)$. Then $\alpha$ is a left magnifying element in $OM(V)$ if and only if $\alpha$ is surjective.

**Proof.** Assume that $\alpha$ is a left magnifying element in $OM(V)$. Then $\alpha M = OM(V)$ for some proper subset $M$ of $OM(V)$. Let $B$ be a basis of $V$ and let $\{B_1, B_2\}$ be a partition of $B$ such that $|B| = |B_1| = |B_2|$. Thus there is a bijection $\phi : B_2 \to B$. Define a linear transformation $\beta$ in $L(V)$ by

$$\beta = \begin{pmatrix} B_1 & v \\ 0 & \phi(v) \end{pmatrix}_{v \in B_2}.$$

It can be seen that $\beta \in OM(V)$. Hence there exists $\gamma \in M$ such that $\alpha \gamma = \beta$. To show $\alpha$ is surjective, let $v \in B$. Hence $v = \phi(u_v) = \beta(u_v) = \alpha \gamma(u_v) = \alpha(\gamma(u_v))$ for some $u_v \in B_2$, so $\alpha$ is surjective.

Now suppose that $\alpha$ is surjective. Let

$$M = \{ \gamma \in OM(V) \mid \gamma \text{ is not surjective} \}.$$

Next, let $\beta \in OM(V)$ and $B_1$ a basis of $\ker \beta$. Extend it to a basis $B$ of $V$. Note that for each $v \in B \setminus B_1$, there is $u_v \in V$ such that $\alpha(u_v) = \beta(v)$ since $\alpha$ is surjective. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} B_1 & v \\ 0 & u_v \end{pmatrix}_{v \in B \setminus B_1}.$$

Thus $\gamma \in OM(V)$ since $\dim(\ker \gamma) = |B_1|$ is infinite. As $\alpha \in OM(V)$, we get $\gamma$ is not surjective and hence $\gamma \in M$. Observe that for any $v \in B_1$, $\alpha \gamma(v) = 0 = \beta(v)$. Moreover, for any $v \in B \setminus B_1$, we have $\alpha \gamma(v) = \alpha(u_v) = \beta(v)$. Therefore, $\alpha$ is a left magnifying element in $OM(V)$.

**Remark 2.1.** The set $M$ defined in the proof of the sufficiency of Theorem 2.1 is a subsemigroup of $OM(V)$. To show this, let $\gamma_1, \gamma_2 \in M$. Then they are not surjective. It follows that $\gamma_1 \gamma_2$ is also not surjective.
Hence we conclude a characterization for elements in $OM(V)$ to be strongly left magnifiers.

**Corollary 2.2.** Let $\alpha \in OM(V)$. Then $\alpha$ is a strongly left magnifying elements if and only if $\alpha$ is surjective.

Therefore the following fact is true.

**Corollary 2.3.** Any left magnifying elements in $OM(V)$ are strong.

We provide an example of a left magnifying element in $OM(V)$ as follows.

**Example 2.4.** Let $B$ be a basis of $V$ and let $\{B_1, B_2\}$ be a partition of $B$ such that $|B| = |B_1| = |B_2|$ and $B_0$ a finite subset of $B_1$. Then there exists a bijection $\phi$ from $B_2$ to $B \setminus B_0$. Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 \setminus B_0 & v & w \\ 0 & v & \phi(w) \end{pmatrix}_{v \in B_0, w \in B_2}.$$

This map is clearly in $OM(V)$ and surjective. By Theorem 2.1, $\alpha$ is a left magnifying element in $OM(V)$.

Note that any elements in $OM(V)$ are not injective. Next, we find that $OM(V)$ has no right magnifying elements in $OM(V)$.

**Theorem 2.5.** $OM(V)$ has no right magnifying elements.

*Proof.* Let $\alpha \in OM(V)$ and $B_1$ be a basis of $\ker \alpha$. Extend $B_1$ to a basis $\mathcal{B}$ of $V$. Since $B_1$ is infinite, there is a partition $\{B'_1, B''_1\}$ of $B_1$ such that $|B_1| = |B'_1| = |B''_1|$. Define $\beta \in OM(V)$ by

$$\beta = \begin{pmatrix} B'_1 & v \\ 0 & v \end{pmatrix}_{v \in B \setminus B'_1}.$$

Then, for any $\emptyset \neq M \subseteq OM(V)$ and $\gamma \in M$, $\gamma \alpha(v) = 0$ but $\beta(v) = v \neq 0$ for all $v \in B'_1$. Hence $\alpha$ is not a right magnifying element. \hfill $\square$

**Corollary 2.6.** $OM(V)$ has no strongly right magnifying elements.

We note that $OE(V)$ has no surjective elements. The following result is obtained.
Theorem 2.7. \(OE(V)\) has no left magnifying elements.

Proof. Let \(\alpha \in OE(V)\) and \(C_1\) a basis of \(im\ \alpha\). Extend \(C_1\) to a basis \(C\) of \(V\). Since \(\alpha \in OE(V)\), we have \(C \setminus C_1\) is infinite. Let \(u \in C \setminus C_1\). Define \(\beta \in L(V)\) by

\[
\beta = \begin{pmatrix} C \setminus \{u\} & u \\ 0 & u \end{pmatrix}.
\]

As \(\dim(V/im\ \beta) = |C \setminus \{u\}| = |C|\) is infinite, we obtain \(\beta \in OE(V)\). It is easy to see that, for any \(\emptyset \neq M \subseteq OE(V)\), \(\alpha \gamma \neq \beta\) for all \(\gamma \in M\) since \(u \notin C_1\) and \(C_1\) is a basis of \(im\ \alpha\). Hence \(\alpha\) is not a left magnifying element in \(OE(V)\).

Corollary 2.8. \(OE(V)\) has no strongly left magnifying elements.

We next show that an injective linear transformation in \(OE(V)\) is a right magnifying element and vice versa.

Theorem 2.9. Let \(\alpha \in OE(V)\). Then \(\alpha\) is a right magnifying element if and only if \(\alpha\) is injective.

Proof. To show the necessity, suppose that \(\alpha\) is a right magnifying element in \(OE(V)\). Then there exists \(M \subseteq OE(V)\) such that \(M\alpha = OE(V)\). Claim that \(\alpha\) is injective. Let \(u \in \ker\ \alpha\). Define \(\beta \in L(V)\) by

\[
\beta = \begin{pmatrix} w & u \\ 0 & u \end{pmatrix}_{w \notin \ker\ \alpha}.
\]

Thus \(\beta \in OE(V)\). It follows that there is \(\gamma \in M\) such that \(\gamma\alpha = \beta\). Hence \(u = \beta(u) = \gamma\alpha(u) = 0\), so \(\alpha\) is injective.

For the sufficiency, suppose that \(\alpha\) is injective. Let \(B\) be a basis of \(V\). Then \(C_1 := \alpha(B)\) is a basis of \(im\ \alpha\) and let \(C\) be a basis of \(V\) containing \(C_1\). Now let

\[
M = \{\gamma \in OE(V) \mid v \in \ker\ \gamma\ \text{for all} \ v \in C \setminus C_1\}.
\]

Then \(M \subseteq OE(V)\). Next, let \(\beta \in OE(V)\). Note that if \(v \in C_1\), there is \(u_v \in B\) such that \(\alpha(u_v) = v\). Define \(\gamma \in L(V)\) by

\[
\gamma = \begin{pmatrix} C \setminus C_1 & v \\ 0 & \beta(u_v) \end{pmatrix}_{v \in C_1}.
\]

Then \(\gamma \in M\) and \(\dim(im\ \gamma) \leq \dim(im\ \beta)\). This implies that \(\dim(V/im\ \gamma) \geq \dim(V/im\ \beta)\) and so \(\gamma \in OE(V)\) since \(\dim(V/im\ \beta)\) is infinite. Hence, for each \(v \in B\), \(\gamma\alpha(v) = \gamma(\alpha(v)) = \beta(v)\). Therefore, \(\alpha\) is a right magnifying element in \(OE(V)\).
We give an example of \( \gamma \in OE(V) \) that is not an element in the set \( M \) in the proof of the above theorem.

**Example 2.10.** We still use notations in the proof of the sufficiency of Theorem 2.9. Since \( \alpha \in OE(V) \), we have \( C \setminus C_1 \) is infinite. Let \( u \in C \setminus C_1 \). Define \( \gamma \in L(V) \) by

\[
\gamma = \begin{pmatrix}
C \setminus \{u\} & u \\
0 & u
\end{pmatrix}.
\]

Then \( \gamma \in OE(V) \) and \( u \in C \setminus C_1 \) but \( u \notin \ker \gamma \). Hence \( \gamma \notin M \). This guarantees that \( M \) is a proper subset of \( OE(V) \).

**Remark 2.2.** In the proof of Theorem 2.9, the set \( M \) is a subsemigroup of \( OE(V) \). To show this, let \( \gamma_1, \gamma_2 \in M \) and \( v \in C \setminus C_1 \). Then \( v \in \ker \gamma_2 \). It follows that \( \gamma_1 \gamma_2(v) = 0 \) and thus \( v \in \ker(\gamma_1 \gamma_2) \). Hence \( \gamma_1 \gamma_2 \in M \).

Therefore, a characterization for strongly magnifying elements in \( OE(V) \) can be described by Theorem 2.9 and Remark 2.2.

**Corollary 2.11.** Let \( \alpha \in OE(V) \). Then \( \alpha \) is a strongly right magnifying element if and only if \( \alpha \) is injective.

**Corollary 2.12.** Any right magnifying elements in \( OE(V) \) are strong.

For the sake of completeness, we provide an example of \( \alpha \in OE(V) \) which is injective.

**Example 2.13.** Let \( B \) be a basis of \( V \). There is a partition \( \{B_1, B_2\} \) of \( B \) such that \( |B| = |B_1| = |B_2| \). Let \( \phi : B \rightarrow B_1 \) be a bijection. Then define \( \alpha \in L(V) \) by

\[
\alpha = \begin{pmatrix}
v \\
\phi(v)
\end{pmatrix}_{v \in B}.
\]

Then \( \dim(V/\text{im } \alpha) = |B \setminus B_1| = |B_2| \) is infinite. Hence \( \alpha \in OE(V) \) and injective. Therefore, by Theorem 2.9, \( \alpha \) is a right magnifying element in \( OE(V) \).

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References


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