On Extended Hypergeometric Function

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Abstract: The principal aim of the paper is to establish some theorems for the extended hypergeometric function due to Emine, Mehmet and Abdullah [1] in Wright function form due to E.M. Wright [2]. The result provide connection to the extended Gauss hypergeometric function due to M.A. Chaudhry et al. [3] and Gauss hypergeometric function.

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1 Introduction

The subject of hypergeometric function and its extension form has gained considerable importance and popularity in different area of the science. Extension of the some well known functions have been considerable by several authors [3 4 5 6 7]. In 2011, Özergin et al. (11, p.4606) introduced the generalized Gauss hypergeometric function $F_p^{(\alpha,\beta)}$ which is defined for $p \geq 0$ and $\Re(c) > \Re(b) > 0$ as:

$$F_p^{(\alpha,\beta)} = \sum_{n=0}^{\infty} (\alpha)_n \frac{B_p^{(\alpha,\beta)}(b + n, c - b) z^n}{B(b, c - b) n!}, \quad (1.1)$$

where

$$B_p^{(\alpha,\beta)}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} F_1 \left( \alpha; \beta; \frac{-p}{t(1-t)} \right) dt, \quad (1.2)$$
The series \( F \) satisfied the condition (1.5) then

\[
F_p^{(\alpha, \beta)}(a, b; c; z) = \frac{1}{\Gamma(b-c+1)} \int_0^1 t^{b-1}(1-t)^{c-b-1} F_1 \left( \alpha; \beta; \frac{-pt}{1-it} \right) (1-zt)^{-\alpha} dt,
\]

where

\[ \Re(p) > 0; p = 0, |\arg(1-z)| < \pi; \Re(c) > \Re(b) > 0. \]

Observe that

\[ F_p^{(\alpha, \alpha)}(a, b; c; z) = F_p(a, b; c; z) \]

and

\[ F_0^{(\alpha, \alpha)}(a, b; c; z) = 2F_1(a, b; c; z) \]

\[ F_0^{(\alpha, \beta)}(a, b; c; z) = 2F_1(a, b; c; z), \]

where \( F_p^{(\alpha, \alpha)} \) represent extended hypergeometric function due to Chaudhry et al. \cite{3} and \( 2F_1 \) represent Gauss hypergeometric function.

The generalized hypergeometric Wright function \( \psi_s(z) \) defined for \( z \in C, \alpha_i, b_j \in C \) and real \( \alpha_i, \beta_j \in \Re = (-\infty, \infty) \) \( (\alpha_i, \beta_j \neq 0; i = 1, 2, \ldots; r; j = 1, 2, \ldots, s) \) by the series

\[
\psi_s(z) \equiv \psi_s \left[ \frac{(\alpha_i, \alpha_i)_{1,r}}{(b_j, \beta_j)_{1,s}} | z \right] = \sum_{m=0}^{\infty} \prod_{i=1}^{r} \Gamma(a_i + \alpha_i, m) \prod_{j=1}^{s} \Gamma(b_j + \beta_j, m) \frac{z^m}{m!}
\]

where \( C \) is the set of the complex numbers and \( \Gamma(z) \) is the Euler gamma function \((\Re, \text{section 1.1})\) and the function (1.4) was introduced by Wright \cite{2} and is known as generalized hypergeometric Wright function \((\Re, \text{section 4.1})\). Condition for the existence of (1.4) together with its representation in terms of Mellin-Barnes integral and of the H-function were established in \cite{9}. In particular, \( \psi_s(z) \) is an entire function if there hold the condition

\[
\sum_{j=1}^{r} \beta_j - \sum_{i=1}^{r} \alpha_i > -1.
\]

2 Theorems

**Theorem 2.1.** If \( \Re(p) > 0; p = 0 \) and \( |\arg(1-z)| < \pi; \Re(c) > \Re(b) > 0 \) and satisfied the condition (1.5) then

\[
F_p^{(\alpha, \beta)}(a, b; c; 1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)} \psi_2 \left[ \begin{array}{c} (\alpha, 1), (b, -1), (c-b-a, -1) \\ (\beta, 1), (c-a, -2) \end{array} \right] - p.
\]
Proof. Using equation (1.3) is reshaped to the form
\[
F_p^{(\alpha, \beta)}(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}F_1\left(\alpha; \beta; \frac{1-p}{1-t}\right)(1-t)^{-a}dt.
\]

Using series representation for confluent hypergeometric function \(1_F^1()\) in the inner integral and Interchanging the order of integration and summation, which is permissible under the condition, stated with the theorem due to convergence of the integrals involved in this process, we obtain
\[
= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{m=0}^{\infty} \frac{(\alpha)_m(-p)^m}{(\beta)_m m!} \int_0^1 t^{b-m-1}(1-t)^{c-a-m-1}dt. \tag{2.2}
\]

On applying the known Beta integral formula \(B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m+n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt \tag{2.3}\)
left side of equation (2.2), we have
\[
= \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)\Gamma(c-b)\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{(\alpha+m)\Gamma(b-m)\Gamma(c-b-a-m)\Gamma(\beta+m)\Gamma(c-a-2m)}{(\beta)_m m!} (-p)^m. \tag{2.4}
\]

Finally, employing the generalized hypergeometric Wright function (1.4), the desired result is obtained.

Remark 2.2. If we put \(\alpha = \beta\) in result (2.1), we obtain
\[
F_p^{(\alpha, \alpha)}(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}^2\Psi_1 \left[ \begin{array}{c} (b, -1), (c-b-a, -1) \\ (c-a, -2) \end{array} \right] | - p . \tag{2.5}
\]

Remark 2.3. If we take \(m=0\) and \(\beta = \alpha\) in equation (2.4), we arrive at
\[
F_p^{(\alpha, \alpha)}(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}. \tag{2.6}
\]

Remark 2.4. If we set \(p=0\) in our result (2.6), it reduces in it to well known result [7, p. 280]
\[
_{2}F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.7}
\]

Theorem 2.5. If \(\Re(p) > 0; p = 0 \) and \(|\text{arg}(1-z)| < \pi; \Re(c) > \Re(b) > 0\) and satisfied the condition (1.5) then
\[
F_p^{(\alpha, \beta)}(-n, a+n; c; 1) = \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(a+n)\Gamma(c-a-n)\Gamma(\alpha)} \times \Psi_2 \left[ \begin{array}{c} (\alpha, 1), (a+n, -1), (c-a, -1) \\ (\beta, 1), (c+n, -2) \end{array} \right] | - p . \tag{2.8}
\]
Proof. By using equation (1.3), we have
\[ F_p^{(\alpha, \beta)}(-n, a + n; c; 1) = \frac{\Gamma(c)}{\Gamma(a + n) \Gamma(c - a - n)} \times \int_0^1 t^{a+n-1} (1-t)^{c-a-n-1} \sum_{m=0}^{\infty} \frac{(\alpha)_m (-p)^m}{(\beta)_m m!} (1-t)^m dt. \]

Changing the order of integration and summation which is permissible under the condition stated with the theorem, we have
\[ = \frac{\Gamma(c) \Gamma(c - a - n)}{\Gamma(a + n) \Gamma(c - a - n) \Gamma(a)} \sum_{m=0}^{\infty} \frac{(\alpha)_m (-p)^m}{(\beta)_m m!} \int_0^1 \left( t^{a+n-m-1} (1-t)^{c-a-m-1} \right) dt. \]

Employing the equation (2.3) and (1.4) on the right side of the above expression, we arrive at
\[ = \frac{\Gamma(c) \Gamma(c - a - n)}{\Gamma(a + n) \Gamma(c - a - n) \Gamma(a)} \sum_{m=0}^{\infty} \frac{(\alpha)_m (1-a)_m (-p)^m}{(\beta)_m m! \Gamma(a + m) \Gamma(c + n - 2m) \Gamma(c - a - m - 2m)} \tag{2.9} \]

Which is required result. \[ \square \]

Remark 2.6. If we take \( \alpha = \beta \) in (2.8), we obtain result for extended hypergeometric function
\[ F_p^{(\alpha, \alpha)}(-n, a + n; c; 1) = \frac{\Gamma(c)}{\Gamma(a + n) \Gamma(c - a - n)} 2\Psi_1 \left[ \frac{(a+n, -1, c-a, -1)}{c+n, -2} \right]. \tag{2.10} \]

Remark 2.7. If we set \( \alpha = \beta, m=0 \) and using the formula \( (a)_k = \frac{(-1)^k}{(1-a)_k}, k = 1, 2, ... \) in equation (2.9). Then we get
\[ F_p^{(\alpha, \alpha)}(-n, a + n; c; 1) = (-1)^n \frac{(1-c+a)_n}{(c)_n}. \tag{2.11} \]

Remark 2.8. If we set \( p=\alpha \) in equation (2.11), we get the following result \( [10], p. 283] \)
\[ 2F_1(-n, a + n; c; 1) = (-1)^n \frac{(1-c+a)_n}{(c)_n}. \tag{2.12} \]

Theorem 2.9. If \( \Re(p) > 0; p = 0 \) and \( \arg(1 - z) \leq \pi; \Re(c) > \Re(b) > 0 \) and satisfied the condition (1.5) then
\[ F_p^{(\alpha, \beta)} \left[ \frac{1}{2} n, -\frac{1}{2} n + \frac{1}{2} b + \frac{1}{2} 1 \right] \]
\[ = \frac{2^n \Gamma(2b) \Gamma(b + \frac{1}{2} n + \frac{1}{2}) \Gamma(\beta)}{\Gamma(b) \Gamma(2b + n) \Gamma(\frac{1}{2} - \frac{1}{2} n) \Gamma(\alpha)} 2\Psi_2 \left[ \frac{\left( \frac{1}{2} - \frac{1}{2} n, -1 \right), (b + n - 1), (\alpha, 1)}{\left( \beta, 1, \frac{1}{2} + \frac{1}{2} n + b, -2 \right)} \right]. \tag{2.13} \]
Proof. With the equation (1.3), we get

\[ F_p^{(\alpha,\beta)} \left[ -\frac{1}{2} n, -\frac{1}{2} n + \frac{1}{2} b + \frac{1}{2}; 1 \right] \]

\[ = \frac{\Gamma(b + \frac{1}{2})}{\Gamma(-\frac{1}{2} n + \frac{1}{2}) \Gamma(b + \frac{1}{2} + \frac{1}{2} n - \frac{1}{2})} \]

\[ \times \int_0^1 t^{-\frac{1}{2} n + \frac{1}{2} - 1} (1 - t)^{b + \frac{1}{2} + \frac{1}{2} n - \frac{1}{2} - 1} {}_1 F_1 \left( \alpha; \beta; \frac{-p}{t(1-t)} \right) (1 - t)^{\frac{1}{2} n} dt \]

\[ = \frac{\Gamma(b + \frac{1}{2})}{\Gamma(-\frac{1}{2} n + \frac{1}{2}) \Gamma(b + \frac{1}{2} + \frac{1}{2} n - \frac{1}{2})} \]

\[ \times \int_0^1 t^{-\frac{1}{2} n + \frac{1}{2} - 1} (1 - t)^{b + \frac{1}{2} + \frac{1}{2} n - \frac{1}{2} - 1} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{(-p)^k}{(t(1-t))^{k+1}} (1 - t)^{\frac{1}{2} n} dt \]

\[ = \frac{\Gamma(b + \frac{1}{2})}{\Gamma(-\frac{1}{2} n + \frac{1}{2}) \Gamma(b + \frac{1}{2} + \frac{1}{2} n - \frac{1}{2})} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{(-p)^k}{k!} \]

\[ \times \int_0^1 t^{-\frac{1}{2} n - k - 1} (1 - t)^{b + \frac{1}{2} n - k + \frac{1}{2} n - 1} dt. \]

By using (2.3), above equation becomes

\[ = \frac{\Gamma(b + \frac{1}{2})}{\Gamma(-\frac{1}{2} n + \frac{1}{2}) \Gamma(b + \frac{1}{2} + \frac{1}{2} n - \frac{1}{2})} \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{(-p)^k}{k!} \frac{(1 - t)^{\frac{1}{2} n}}{(b + n - k)} \frac{(1 - t)^{\frac{1}{2} n}}{(b + n - k)} \frac{(-p)^k}{k!}. \] (2.14)

Now using Legendre's duplication formula

\[ \Gamma(b) \Gamma(b + \frac{1}{2}) = 2^{1-2b} \sqrt{\pi} \Gamma(2b) \]

in (2.14), we arrive at

\[ = \frac{2^{1-2b} \sqrt{\pi} \Gamma(2b)}{\Gamma(b) \Gamma(b + \frac{1}{2})} \frac{\Gamma(b + \frac{1}{2} n + \frac{1}{2})}{2^{1-2b-n} \sqrt{\pi} \Gamma(b + 2n)} \]

\[ \times \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{(-p)^k}{(b + n - k)} \frac{(1 - t)^{\frac{1}{2} n}}{(b + n - k)} \frac{(-p)^k}{k!} \frac{(1 - t)^{\frac{1}{2} n}}{(b + n - k)} \frac{(-p)^k}{k!}. \]

\[ \times \frac{\Gamma(\alpha + k)}{\Gamma(\beta + k)} \frac{\Gamma(b + n - k)}{\Gamma(b + n - k)} \frac{(-p)^k}{k!}. \] (2.15)

Finally, using (1.4) we arrive at required result.
Remark 2.10. If we take $\alpha = \beta$, in (2.13), these results reduce to

$$F^{(\alpha, \alpha)}_p \left[ -\frac{1}{2} n, -\frac{1}{2} n + \frac{1}{2}; b + \frac{1}{2}; 1 \right] = \frac{2^n \Gamma(2b) \Gamma(b + \frac{1}{2} n + \frac{1}{2})}{\Gamma(b) \Gamma(2b + n) \Gamma(\frac{1}{2} - \frac{1}{2} n)} 2^\Psi_1 \left[ \left( \frac{1}{2} - \frac{1}{2} n, -1 \right), \left( \frac{1}{2} + \frac{1}{2} n + b, -2 \right) \right] | - p \right].$$  (2.16)

Remark 2.11. On the other hand, when $\alpha = \beta$ and $k=0$ in (2.15), we get the result

$$F^{(\alpha, \alpha)}_p \left[ -\frac{1}{2} n, -\frac{1}{2} n + \frac{1}{2}; b + \frac{1}{2}; 1 \right] = \frac{2^n (b)_n}{(2b)_n}.  (2.17)$$

Remark 2.12. If we set $p=0$ in result (2.17) reduces immediately to Gauss hypergeometric function ([10], p. 283)

$$2F_1\left[ -\frac{1}{2} n, -\frac{1}{2} n + \frac{1}{2}; b + \frac{1}{2}; 1 \right] = \frac{2^n (b)_n}{(2b)_n}. (2.18)$$

Theorem 2.13. If $\Re(p) > 0; p = 0$ and $\arg(1 - z) < \pi$; $\Re(c) > \Re(b) > 0$ and satisfied the condition (1.5) then

$$F^{(\alpha, \beta)}(\alpha, \beta; -n, 1 - b - n; c; 1) = \frac{\Gamma(c) \Gamma(\beta)}{\Gamma(1 - b - n) \Gamma(c - 1 + b + n) \Gamma(\alpha) \Gamma(c - 1 + b + n + 1)} \times 2F_1 \left[ \left( \alpha, 1 \right), \left( 1 - b - n, -1 \right), \left( c - 1 + b + 2n, -1 \right) \right] \left( \beta, 1, (c + n, -2) \right) | - p \right].$$  (2.19)

Proof. From (1.3), we get

$$F^{(\alpha, \beta)}(-n, 1 - b - n; c; 1) = \frac{\Gamma(c) \Gamma(\beta)}{\Gamma(1 - b - n) \Gamma(c - 1 + b + n) \Gamma(\alpha) \Gamma(c - 1 + b + n + 1)} \times 2F_1 \left[ \left( \alpha, \beta; -\frac{p}{t(1 - t)} \right) (1 - t)^n dt. \right.$$

$$= \frac{\Gamma(c) \Gamma(\beta)}{\Gamma(1 - b - n) \Gamma(c - 1 + b + n) \Gamma(\alpha) \Gamma(c - 1 + b + n + 1)} \times \sum_{m=0}^{\infty} \frac{(\alpha)_m (-p)_m}{(\beta)_m m!} \times \int_0^1 t^{1 - b - n - m - 1} (1 - t)^{c - 1 + b + 2n - m - 1} dt.$$

The simplification of above equation gives

$$= \frac{\Gamma(c) \Gamma(\beta)}{\Gamma(1 - b - n) \Gamma(c - 1 + b + n) \Gamma(\alpha)} \times \sum_{m=0}^{\infty} \frac{(\alpha) \Gamma(1 - b - n - m) \Gamma(c - 1 + b + 2n - m) (-p)_m}{\Gamma(\beta) \Gamma(c + n - 2m)} \frac{m!}. \quad (2.20)$$

Which is required result.  \[ \square \]
Remark 2.14. By putting $\alpha = \beta$ in (2.19) we shall have the result

$$F_p^{(0,\alpha)}(-n,1-b-n;c;1) = \frac{\Gamma(c)}{\Gamma(1-b-n)\Gamma(c-1+b+n)} \times 2\Psi_1 \left[ \left( 1-b-n,-1, \right) \left( c-1+b+2n,-1 \right) \mid -p \right].$$  \hspace{1cm} (2.21)

Remark 2.15. On setting $\beta = \alpha$, $m=0$ and using the calculation

$$\frac{\Gamma(c-1+b+2n)}{\Gamma(c-1+b+n)} = \frac{(c-1+b)_{2n}\Gamma(c-1+b)}{(c-1+b)_n\Gamma(c-1+b)} = \frac{(c+b-1)_{2n}}{(c+b-1)_n}$$

in result (2.20) we can produce the result

$$F_p^{(0,\alpha)}(-n,1-b-n;c;1) = \frac{(c-1+b)_{2n}}{(c)_n(c-1+b)_n}. \hspace{1cm} (2.22)$$

Remark 2.16. On taking $p=0$ in result (2.22), we obtain the known result [7, p. 283]

$$2F_1(-n,1-b-n;c;1) = \frac{(c-1+b)_{2n}}{(c)_n(c-1+b)_n}. \hspace{1cm} (2.23)$$

Theorem 2.17. If $\Re(p) > 0; p = 0$ and $|\text{arg}(1-z)| < \pi$; $\Re(c) > \Re(b) > 0$ and satisfied the condition (1.5) then

$$F_p^{(a,\beta)}[-n,b;c;1] = \frac{\Gamma(c)\Gamma(b)}{\Gamma(b)\Gamma(c-b)\Gamma(a)} {2\Psi_2} \left[ \left( \alpha,1 \right), \left( b,-1 \right), \left( c-b+n,-1 \right) \mid -p \right].$$  \hspace{1cm} (2.24)

Proof. By using the equation (1.3), equation (1.4) and proceeding similarly to the proof of above theorems, we obtain

$$F_p^{(a,\beta)}[-n,b;c;1] = \frac{1}{B(b,c-b)} \int_0^1 (1-t)^{c-b+n-1} F_1 \left( \alpha;\beta;\frac{-p}{t(1-t)} \right) dt$$

$$= \frac{1}{B(b,c-b)} \int_0^1 (1-t)^{c-b+n-1} \sum_{m=0}^{\infty} \frac{(a)_m}{(\beta)_m} \frac{(-p)^m}{m!} \frac{1}{B(b,c-b)} \int_0^1 (1-t)^{c-b+n-m-1} dt$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m}{(\beta)_m} \frac{(-p)^m}{m!} \frac{1}{B(b,c-b)} \int_0^1 (1-t)^{c-b+n-m-1} dt$$

$$= \frac{\Gamma(c)\Gamma(b)}{\Gamma(b)\Gamma(c-b)\Gamma(a)} \sum_{m=0}^{\infty} \frac{\Gamma(b-m)\Gamma(c-b+n-m)\Gamma(a+m)}{\Gamma(c+n-2m)\Gamma(\beta+m)}. \hspace{1cm} (2.25)$$

This complete the proof. \hfill \Box

Remark 2.18. When $\alpha = \beta$ in (2.24) reduces to an elegant result

$$F_p^{(a,a)}[-n,b;c;1] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} 2\Psi_1 \left[ \left( \frac{b-1}{c+n-1}, \frac{c-b+n,-1}{c+n,-2} \right) \mid -p \right].$$  \hspace{1cm} (2.25)
Remark 2.19. If we take \( m=0 \) and \( \alpha = \beta \) in result (2.25), we obtain
\[
F_p^{(\alpha,\alpha)}[-n,b;c;1] = \frac{(c-b)_n}{(c)_n}.
\] (2.26)

Remark 2.20. On setting \( p=0 \), in result (2.27) we arrive at given result (\[10\], p.283)
\[
\, _2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n}.
\] (2.27)

References


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