On the Parametric Interest of the Option Price of Stock from Black-Scholes Equation

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Abstract In this paper, we studied the option price of stock from the Black-Scholes equation and discovered some parameter \( \lambda \) which is the generalization of the interest \( r \). Such \( \lambda \) is the first that named the parametric interest which is new the results. Moreover we found that such \( \lambda \) gives the conditions for the solution of the Black-Scholes equation which may be weak or strong solution.

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1. INTRODUCTION

It is well known that the Black-Scholes equation plays an important role in solving the option price of the stock which is called the Black-Scholes formula (see [1, pp. 637-659]). The Black-Scholes equation is given by

\[
\frac{\partial}{\partial t} u(s,t) + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2}{\partial s^2} u(s,t) + rs \frac{\partial}{\partial s} u(s,t) - ru(s,t) = 0 \tag{1.1}
\]

with the call payoff

\[
u(s_T, T) = (s_T - p)^+ \equiv \max(s_T - p, 0) \tag{1.2}
\]

for \( 0 \leq t \leq T \) where \( u(s,t) \) is the option price at time \( t \), \( s_T \) is the price of stock at the expiration time \( T \), \( s \) is the price of stock at time \( t \), \( r \) is the interest rate, \( \sigma \) is the volatility of stock price and \( p \) is the strike price.

They obtained the solution \( u(s,t) \) of (1.1) which is called the Black-Scholes formula and satisfies (1.2) of the form

\[
u(s,t) = sN(d_1) - pe^{-r(T-t)}N(d_2) \tag{1.3}
\]
see [2, pp. 90-91] where

\[
d_1 = \frac{\ln \left( \frac{s}{p} \right) + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}
\]

\[
d_2 = \frac{\ln \left( \frac{s}{p} \right) + (r - \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}}
\]

and \( N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \).

In this work, we studied the solution of (1.1) in the other form which is the generalization of (1.3). Starting with changing the variable \( u(s,t) = V(R,t) \) where \( R = \ln s \). Then (1.1) is transformed to equation

\[
\frac{\partial}{\partial t} V(R,t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial R^2} V(R,t) + (r - \frac{\sigma^2}{2}) \frac{\partial}{\partial R} V(R,t) - r V(R,t) = 0 \tag{1.4}
\]

with the call payoff in (1.2)

\[
V(R_T, T) = (e^{R_T} - p)^+ \tag{1.5}
\]

where \( R_T = \ln s_T \). Then we obtained the solution of (1.4) in the form

\[
u(s,t, \lambda) = (s_T - p)^+ e^{-\lambda (T-t)} \frac{X(\ln s_T)}{X(\ln s_T)} \mathcal{L}^{-1}(\xi^\alpha) \tag{1.6}
\]

where \( \mathcal{L}^{-1}(\xi^\alpha) \) is the inverse Laplace transform of \( \xi^\alpha \) and \( X(\ln s_T) \) is the function of \( \ln s_T \) and \( \alpha \) is the real number that can be obtained from the equation

\[
\sigma^2 \alpha^2 + (3 \sigma^2 - 2r) \alpha + (2 \sigma^2 - 4r + 2\lambda) = 0 \tag{1.7}
\]

with \( \lambda \) is the parametric interest.

Now consider the following cases.

(i) Suppose \( \alpha = m \) where \( m \) is nonnegative integer then we obtained the solution in (1.6) as the weak solution of the form

\[
u(s,t, \lambda) = (s_T - p)^+ e^{-\lambda (T-t)} \frac{X(\ln s_T)}{X(\ln s_T)} \delta^{(m)}(s) \tag{1.8}
\]

where \( \delta^{(m)}(s) \) is the Dirac-delta function with \( m \)-derivative with \( \delta^{(0)}(s) = \delta(s) \) and the parametric interest \( \lambda \) can be obtained from (1.7) as

\[
\lambda = \lambda(r, \sigma) = (m + 2)r - \frac{m^2 + 3m + 2}{2} \sigma^2 \tag{1.9}
\]

(ii) Suppose \( \alpha \) is a negative real number, that is \( \alpha < 0 \) then we obtained the solution is (1.6) as the strong solution or the classical solution of the form

\[
u(s,t, \lambda) = (s_T - p)^+ e^{-\lambda (T-t)} s^{-\alpha-1} \frac{X(\ln s_T)}{X(\ln s_T) \Gamma(-\alpha)} \tag{1.10}
\]
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where \( \Gamma(-\alpha) \) is the Gamma function of \(-\alpha\). In particular, if \( \alpha = -n \) for some positive integer \( n \) then (1.10) reduces to

\[
u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \left( \frac{s}{s_T} \right)^{n-1}
\]

(1.11)

with the parametric interest

\[
\lambda = (2 - n)r - \frac{(n^2 - 3n + 2)}{2} \sigma^2.
\]

(1.12)

We see that (1.8) and (1.10) are the other solutions of the Black-Scholes solution in (1.1). Particularly, for \( n = 1 \) in (1.12) we have \( \lambda = r \) that means the parametric interest \( \lambda \) is the interest rate \( r \) of the option price of stock. It follows that from (1.11)

\[
u(s, t, r) = (s_T - p)^+ e^{-r(T-t)}
\]

and at \( t = T \) we obtained \( u(s_T, T, r) = (s_T - p)^+ \) which is the call payoff in (1.2). Also for the case \( n = 2 \) in (1.12) we have \( \lambda = 0 \). It follows that from (1.11)

\[
u(s, t, 0) = (s_T - p)^+ e^{0(T-t)} \left( \frac{s}{s_T} \right) = (s_T - p)^+
\]

and at \( t = T, u(s_T, T, 0) = (s_T - p)^+ \left( \frac{s_T}{s_T} \right) = (s_T - p)^+ \) which is the call payoff in (1.2).

2. Preliminaries

The following definition and lemma are needed.

**Definition 2.1.** Given \( f \) is piecewise continuous on the interval \( 0 \leq t \leq A \) for any positive \( A \) and if there exists the real constant \( K, a \) and \( M \) such that

\[ |f(t)| \leq Ke^{at} \text{ for } t \geq M. \]

Then the Laplace transform of the \( f(t) \), denoted by \( \mathcal{L}f(t) \) is defined by

\[
\mathcal{L}f(t) = F(\xi) = \int_0^\infty e^{-\xi t} f(t) dt \tag{2.1}
\]

and the inverse Laplace transform of \( F(\xi) \) is defined by

\[
f(t) = \mathcal{L}^{-1}F(\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\xi) e^{\xi t} d\xi. \tag{2.2}
\]

**Lemma 2.2.** [3]

(i) \( \mathcal{L}\delta(t) = 1 \) where \( \delta(t) \) is the Dirac-delta function.

(ii) \( \mathcal{L}\delta^{(k)}(t) = \xi^k \) where \( \delta^{(k)}(t) \) is the Dirac-delta function with \( k \)-derivative and \( \delta^{(0)}(t) = \delta(t) \) and \( \xi > 0 \).

(iii) \( \mathcal{L}(p^t) = \frac{\Gamma(p+1)}{\xi\xi^{p+1}} \) for \( p > -1 \) and \( \xi > 0 \) where \( \Gamma(p+1) \) is the Gamma function.

If \( p \) is positive number \( n \) then \( \mathcal{L}(t^n) = \frac{n!}{\xi^{n+1}}, \xi > 0. \)

(iv) \( \mathcal{L}[t^k f(t)] = (-1)^k F^{(k)}(\xi). \)

(v) \( \mathcal{L}[f^{(k)}(t)] = \xi^k F(\xi). \)
3. Main Results

**Theorem 3.1.** Recall the Black-Scholes equation in (1.1) that
\[
\frac{\partial}{\partial t} u(s, t) + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2}{\partial s^2} u(s, t) + rs \frac{\partial}{\partial s} u(s, t) - ru(s, t) = 0 \tag{3.1}
\]
and the call payoff in (1.2) that
\[
u(s_T, T) = (s_T - p)^+ \tag{3.2}
\]
then in (1.6)
\[
u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \mathcal{L}^{-1}(\xi^\alpha) \tag{3.3}
\]
is the solution of (3.1) with the parametric interest
\[
\lambda = (\alpha + 2)r - \frac{\alpha^2 + 3\alpha + 2}{2} \sigma^2. \tag{3.4}
\]
In particular if \( \alpha = m \) where \( m \) is nonnegative integer then (3.3) becomes
\[
u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \xi^{(m)}(s) \tag{3.5}
\]
with the parametric interest
\[
\lambda = (m + 2)r - \frac{m^2 + 3m + 2}{2} \sigma^2. \tag{3.6}
\]
Also for the case \( \alpha \) is negative real number, that is \( \alpha < 0 \) then (3.3) becomes
\[
u(S, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \frac{s^{-\alpha-1}}{X(\ln s_T)} \Gamma(-\alpha). \tag{3.7}
\]
In particular, if \( \alpha \) is negative integer and suppose \( \alpha = -n \) then (3.7) reduces to
\[
u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \left( \frac{s}{s_T} \right)^{n-1} \tag{3.8}
\]
with the parametric interest
\[
\lambda = (2 - n)r - \frac{n^2 - 3n + 2}{2} \sigma^2. \tag{3.9}
\]

**Proof.** By changing the variable \( u(s, t) = V(R, T) \) where \( R = \ln s \) then (3.1) is transformed to the equation
\[
\frac{\partial}{\partial t} V(R, t) + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial R^2} V(R, t) + \left(r - \frac{\sigma^2}{2}\right) \frac{\partial}{\partial R} V(R, t) - rV(R, t) = 0. \tag{3.10}
\]
By the method of separation of variable, let \( V(R, t) = X(R)U(t) \), then \( \frac{\partial}{\partial t} V(R, t) = X(R)U'(t), \frac{\partial}{\partial R} V(R, t) = X'(R)U(t) \) and \( \frac{\partial^2}{\partial R^2} V(R, t) = X''(R)U(t) \) then substitute into (3.10). Then we obtained
\[
X(R)U'(t) + \frac{1}{2} \sigma^2 X''(R)U(t) + \left(r - \frac{\sigma^2}{2}\right) X'(R)U(t) - rX(R)U(t) = 0
\]
\[
\frac{U'(t)}{U(t)} + \frac{1}{2} \sigma^2 \frac{X''(R)}{X(R)} + \left( r - \frac{\sigma^2}{2} \right) \frac{X'(R)}{X(R)} - r = 0.
\]

Let
\[
\frac{U'(t)}{U(t)} + \frac{1}{2} \sigma^2 \frac{X''(R)}{X(R)} + \left( r - \frac{\sigma^2}{2} \right) \frac{X'(R)}{X(R)} - r = \lambda
\]
where \( \lambda \) is a parameter.

Now consider \( \frac{U'(t)}{U(t)} = \lambda \) then \( U(t) = C e^{\lambda t} \). Now we compute the constant \( C \).

Since \( u(s, t) = V(R, t) = X(R)U(t) \) and the call payoff \( u(s_T, T) = (s_T - p)^+ \), hence \( X(\ln s_T)U(T) = (s_T - p)^+ \) and \( U(T) = C e^{\lambda T} \). It follows that \( C = \frac{U(T)}{e^{\lambda T}} = \frac{(s_T - p)^+}{X(\ln s_T)e^{\lambda T}} \).

Thus we have
\[
U(t) = \frac{(s_T - p)^+ e^{\lambda t}}{X(\ln s_T)} e^{\lambda T} = \frac{(s_T - p)^+}{X(\ln s_T)} e^{-\lambda(T-t)}. \tag{3.11}
\]

Next consider \( \frac{1}{2} \sigma^2 \frac{X''(R)}{X(R)} + \left( r - \frac{\sigma^2}{2} \right) \frac{X'(R)}{X(R)} - r + \lambda = 0 \) and let \( X(R) = X(\ln s) = y(s) \).

Then \( X'(R) = \frac{dy(s)}{dR} = \frac{dy(s)}{ds} \frac{ds}{dR} = sy'(s) \) and \( X''(R) = s^2y''(s) + sy'(s) \) thus
\[
\sigma^2[s^2y''(s) + sy'(s)] + (2r - \sigma^2) sy'(s) - (2r - 2\lambda)y(s) = 0
\]
or
\[
\sigma^2 s^2y''(s) + 2rsy'(s) - (2r - 2\lambda)y(s) = 0. \tag{3.12}
\]

The equation (3.12) is the Euler’s equation of order 2. Take the Laplace transform of (2.1) to (3.12) and use (iv) and (v) of Lemma 2.2. Where \( \mathcal{L}y(s) = Y(\xi) \) then
\[
\sigma^2 \frac{d^2}{d\xi^2} [\xi^2 Y(\xi)] + (-1)2r \frac{d}{d\xi} [\xi Y(\xi)] - (2r - 2\lambda)Y(\xi) = 0
\]
or
\[
\sigma^2 \xi^2 Y''(\xi) + (4\sigma^2 - 2r)\xi Y'(\xi) + (2\sigma^2 - 4r + 2\lambda)Y(\xi) = 0 \tag{3.13}
\]
which is also the Euler’s equation of order 2. Let \( y(\xi) = \xi^\alpha \) and substitute into (3.13) then
\[
[\sigma^2 \alpha (\alpha - 1) + (4\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda)]\xi^\alpha = 0.
\]

Thus we have
\[
\sigma^2 \alpha^2 + (3\sigma^2 - 2r)\alpha + (2\sigma^2 - 4r + 2\lambda) = 0. \tag{3.14}
\]

The real number \( \alpha \) and the parametric interest \( \lambda \) can be obtained from (3.14). Since \( Y(\xi) = \xi^\alpha \), hence \( y(s) = \mathcal{L}^{-1}Y(\xi) = \mathcal{L}^{-1}(\xi^\alpha) \) where \( \mathcal{L}^{-1} \) is the inverse Laplace transform.

Since \( u(s, t) = V(R, t) = X(\ln s)U(t) = y(s)U(t) \), thus from (3.11)
\[
u(s, t, \lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \mathcal{L}^{-1}(\xi^\alpha) \tag{3.15}
\]
where \( u(s, t, \lambda) \) is the function of \( s, t \) and \( \lambda \).
Thus we obtained (3.3) as required.

Now for the case \( \alpha = m \) where \( m \) is nonnegative integer and from (ii) of Lemma 2.2
\( \mathcal{L} \delta^{(m)}(s) = \xi^m \). Thus \( \delta^{(m)}(s) = \mathcal{L}^{-1}(\xi^m) \). It follows that
\[
u(s, t, \lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)}}{X(\ln s_T)} \delta^{(m)}(s).
\]
Thus we obtained (3.5) as required.

Next for the case \( \alpha < 0 \) and from (iii) of Lemma 2.2
\[
\mathcal{L} s^p = \frac{\Gamma(p + 1)}{\xi^{p+1}} = \Gamma(p + 1)\xi^{-p-1}, \text{ for } p > -1.
\]
Let \( \alpha = -p - 1 \) then \( p = -\alpha - 1 \). Thus \( \mathcal{L}(s^{-\alpha-1}) = \Gamma(-\alpha)\xi^{\alpha} \). It follows that \( \mathcal{L}^{-1}(\xi^{\alpha}) = \frac{s^{-\alpha-1}}{\Gamma(-\alpha)} \).
Thus from (3.5),
\[
u(s, t, \lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)} s^{-\alpha-1}}{X(\ln s_T) \Gamma(-\alpha)}.
\]
Thus we obtained (3.7) as required. In particular, if \( \alpha \) is negative integer and suppose \( \alpha = -n \) then \( u(s, t, \lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)} s^{-n-1}}{X(\ln s_T) \Gamma(n)} \). Since the call payoff \( u(s_T, T, \lambda) = (s_T - p)^+ \) at \( t = T \) hence \( (s_T - p)^+ \frac{e^{-\lambda(T-t)} s^{-n-1}}{X(\ln s_T) \Gamma(n)} = (s_T - p)^+ \). It follows that \( X(\ln s_T) = \frac{s_T^{n-1}}{\Gamma(n)} \). Thus we have \( u(s, t, \lambda) = (s_T - p)^+ \frac{e^{-\lambda(T-t)} \Gamma(n)}{\Gamma(n)} \frac{s_T^{n-1}}{s_T^{n-1}} s^{-n-1} \) or \( u(s, t, \lambda) = (s_T - p)^+ e^{-\lambda(T-t)} \left( \frac{s}{s_T} \right)^{n-1} \) with the parametric interest from (3.14) with \( \alpha = -n \)
\[
\lambda = (2 - n)r - \frac{(n^2 - 3n + 2)}{2} \sigma^2.
\]
Thus we obtained (3.8) as required. Now consider the call payoff \( u(s_T, T, \lambda) \) at \( t = T \) then from (3.8),
\[
u(s_T, T, \lambda) = (s_T - p)^+ e^{-\lambda(T-T)} \left( \frac{s_T}{s} \right) = (s_T - p)^+
\]
with the same as the call payoff in (1.2)
\[
u(s_T, T) = (s_T - p)^+.
\]
We see that even \( u(s, t) \) in (1.3) is different from \( u(s, t, \lambda) \) in (1.11) but they have the same call payoff.

Now consider (3.8) with \( n = 1 \) and we have \( \lambda = r \) in (3.9) with \( n = 1 \). Then (3.8) reduces to
\[
u(s, t, r) = (s_T - p)^+ e^{-r(T-t)}
\]
with call payoff \( u(s_T, T, r) = (s_T - p)^+ \) at \( t = T \).

Moreover for \( n = 2 \) in (3.9) \( \lambda = 0 \) and again from (3.8)

\[
u(s, t, 0) = (s_T - p) \left( \frac{s}{s_T} \right)
\]

with the call payoff

\[
u(s_T, T, 0) = (s_T - p)^+ \left( \frac{s_T}{s_T} \right) = (s_T - p)^+.
\]

We see that the parametric interest \( \lambda \) is the generalization of the interest rate \( r \) that appears in the option price \( u(S, t) \).

4. Conclusion

The main point of this research is focusing on the parametric interest \( \lambda \) which gives the conditions of the solutions of the Black-Scholes equation to be weak or strong solution that appear in (1.8), (1.10) and (1.11) of the Introduction part.

Moreover \( \lambda \) is the generalization of the interest rate \( r \). Thus from (3.9) for \( n = 1 \) we have \( \lambda = r \). Thus we obtained from (3.8)

\[
u(s, t, r) = (s_T - p)^+ e^{-r(T-t)}
\]

or \((s_T - p)^+ = u(s, t, r)e^{r(T-t)}\). That means the call payoff \((s_T - p)^+\) is equal to the the option price \( u(s, t, r) \) put in the bank with the interest rate \( r \) at the time \( T - t \).

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References