Option Pricing under a Mean Reverting Process with Jump-Diffusion and Jump Stochastic Volatility

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Abstract: An alternative option pricing model is proposed, in which the asset prices follow the jump-diffusion and exhibits mean reversion. The stochastic volatility follows the jump-diffusion with mean reversion. We find a formulation for the European-style option in terms of characteristic functions.

Keywords: jump-diffusion model; stochastic volatility; characteristic function; option pricing; mean reverting.

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1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with filtration \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\). All processes that we shall consider in this section will be defined in this space. An asset price model with stochastic volatility has been defined by Heston [1] which has the following dynamics:

\[
\begin{align*}
    dS_t &= S_t(\mu dt + \sqrt{v_t}dW^S_t), \\
    dv_t &= \kappa(\theta - v_t)dt + \sigma \sqrt{v_t}dW^v_t,
\end{align*}
\]

where \(S_t\) is the asset price, \(\mu \in \mathbb{R}\) is the rate of return of the asset price, \(v_t\) is the volatility of asset returns, \(\kappa > 0\) is the rate at which the volatility reverts...
toward its long-term mean, \( \theta \in \mathbb{R} \) is the mean long-term volatility, \( \sigma > 0 \) is the volatility of the volatility process, \( W_t^S \) and \( W_t^v \) are standard Brownian motions corresponding to the processes \( S_t \) and \( v_t \), respectively, with constant correlation \( \rho \). Bate [2] introduced the jump-diffusion stochastic volatility model by adding log normal jump \( Y_t \) to the Heston stochastic volatility model. In the original formulation of Bate, the model has the following form:

\[
dS_t = S_t(\mu dt + \sqrt{v_t}dW_t^S) + S_t - Y_t dN_t^S, \tag{1.3}
\]

where \( N_t^S \) is the Poisson process which corresponds to the underlying asset \( S_t \), \( Y_t \) is a proportion of jump size of the asset price \( (1.1) \) with log normal distribution and \( S_t - \) means that there is a jump in the value of the process before the jump is used on the left-hand side of the formula. Eraker et al. [3] extended Bate’s work by incorporating jumps into the volatility model, i.e.

\[
dv_t = \kappa(\theta - v_t)dt + \sigma \sqrt{v_t}dW_t^v + Z_t dN_t^v \tag{1.4}
\]

Eraker et al. [3] developed a likelihood-based estimation strategy and provided estimates of parameters, spot volatility, jump times, and jump sizes using S&P 500 and Nasdaq 100 index returns. Moreover, they examined the volatility structure of the S&P and Nasdaq indices and indicated that models with jumps in volatility are preferred over those without jumps in volatility. But they did not provide a closed-form formula for the price of a European call option.

Empirical evidence on mean reversion in financial assets has been produced by Cecchetti et al. [4] and Bessembinder et al. [5], respectively. It has been documented that currency exchange rates also exhibit mean reversion. Jorion and Sweeney [6] show how the real exchange rates revert to their mean levels and Sweeney [7] provides empirical evidence of mean reversion in G-10 nominal exchange rates. Mean reversion also appears in some stock prices as evidenced by Poterba and Summers [8].

In this paper, we consider the problem of finding a closed-form formula for a European call option where the asset price follows mean reverting jump-diffusion and the stochastic volatility with jump.

The rest of this paper is organized as follows. In Section 2, we briefly discuss model descriptions for option pricing. Deriving a formula for a characteristic function is presented in Section 3. Finally, a closed-form formula for a European call option in terms of characteristic functions is presented.

2 Model Descriptions

It is assumed that a risk-neutral probability measure \( \mathcal{M} \) exists. The asset price \( S_t \) under this measure follows a mean reverting jump-diffusion process, and the volatility \( v_t \) follows mean reverting with jump, i.e. our models are governed
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by the following dynamics:

\[
\begin{align*}
    dS_t &= b \left( a - \ln S_t - \frac{\lambda_S m}{b} \right) dt + \sqrt{v_t} S_t dW_t^S + S_t dN_t^S \\
    dv_t &= \kappa(\theta - v_t) dt + \sigma \sqrt{v_t} dW_t^v + Z_t dN_t^v
\end{align*}
\]  

(2.1)  

(2.2)

where \( S_t, v_t, \kappa, \theta, \sigma, W_t^S \) and \( W_t^v \) are defined as above, \( a \in \mathbb{R} \) is the mean of long-term asset price return, \( b > 0 \) is the rate at which the asset price return reverts toward its long-term mean, \( N_t^S \) and \( N_t^v \) are independent Poisson processes with constant intensities \( \lambda^S \) and \( \lambda^v \) respectively. \( Y_t \) and \( Z_t \) are proportional jump sizes of the asset price (2.1) and the jump size of the volatility process (2.2) respectively.

Suppose that \( Y_t \) and \( Z_t \) are independent and identically distributed sequences with densities \( \phi_{Y_t}(y) := \phi_Y(y) \), \( \phi_{Z_t}(z) := \phi_Z(z) \) and \( EY_t := m \). Moreover, we assume that the jump processes \( N_t^S \) and \( N_t^v \) are independent of standard Brownian motions \( W_t^S \) and \( W_t^v \).

Assume that the asset price \( S_t \) and the volatility \( v_t \) satisfy equations (2.1) and (2.2) respectively. Let \( L_t = \ln S_t \), by the jump-diffusion chain rule, \( \ln S_t \) satisfies the SDE

\[
    dL_t = b \left( a - L_t - \frac{\lambda_S m}{b} - \frac{v_t}{2b} \right) dt + \sqrt{v_t} dW_t^S + \ln(1 + Y_t) dN_t^S. \tag{2.3}
\]

3 Characteristic Functions

We denote the characteristic function for \( L_T = \ln S_T \) as

\[
    f(x : t, l, v) = E_M[e^{ixL_T}|L_t = l, v_t = v] \tag{3.1}
\]

where \( 0 \leq t \leq T \) and \( i = \sqrt{-1} \). Here \( L_t \) is the mean reverting asset price process with jumps specified by (2.3) and \( v_t \) is the volatility process specified by (2.2). The generalized Feynman-Kac theorem [9] implies that \( f(x : t, l, v) \) solves the following partial integro-differential equation (PIDE):

\[
    0 = \frac{\partial f}{\partial t} + b \left( a - l - \frac{\lambda^S m}{b} - \frac{v}{2b} \right) \frac{\partial f}{\partial l} \\
    + \kappa(\theta - v) \frac{\partial f}{\partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 f}{\partial v^2} + \rho \sigma v \frac{\partial^2 f}{\partial l \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 f}{\partial v^2} \\
    + \lambda^S \int_{\mathbb{R}} [f(x : t, l + y, v) - f(x : t, l, v)] \phi_Y(y) dy \\
    + \lambda^v \int_{\mathbb{R}} [f(x : t, l, v + z) - f(x : t, l, v)] \phi_Z(z) dz. \tag{3.2}
\]
Lemma 3.1. Suppose that \( L_t \) follows the dynamics in (2.3). Then the characteristic function for \( L_T \) can be written in the form
\[
f(x : t, l, v) = \exp[B(t, T) + C(t, T)l + D(t, T)v + ixl],
\]
(3.3)

where
\[
B(t, T) = \left( \frac{\lambda S_m}{b} - a \right) ix(e^{-b(T-t)} - 1) - \theta \kappa \int_t^T D(s, T)ds \\
+ (T - t) \lambda S \int_\mathbb{R} [e^{ixy} - 1] \phi_Y(y)dy \\
+ (T - t) \lambda \int_\mathbb{R} [e^{xD(t, T)} - 1] \phi_Z(z)dz,
\]
\[
C(t, T) = ix(e^{-b(T-t)} - 1),
\]
\[
D(t, T) = U(e^{-b(T-t)}) + \frac{e^{-\kappa(T-t)}V(e^{-b(T-t)})}{- \frac{1}{U(1)} + \frac{\sigma^2}{2b} \int_1^{e^{-b(T-t)}} h \hat{\varphi}^{-1}V(h)dh},
\]
\[
U(h) = \frac{2bh \left( \sqrt{1 - \rho^2} - \rho i \right) \frac{\sigma^2}{2b} \Phi(a^*, b^*, \frac{h}{\zeta}) + \frac{\sigma^2}{b^*} \Phi(a^* + 1, b^* + 1, \frac{h}{\zeta})}{\Phi(a^*, b^*, \frac{1}{\zeta})},
\]
\[
V(h) = \frac{\Phi^2(a^*, b^*, \frac{1}{\zeta}) e^{(\sqrt{1 - \rho^2} + \rho i)(1-h)}}{\Phi^2(a^*, b^*, \frac{1}{\zeta})},
\]
\[
h = e^{-b(T-t)},
\]
\[
a^* = \frac{b^*}{T} \left( \sqrt{\rho^2 - 1} + \rho \right) + \frac{\sigma}{\sqrt{\rho^2 - 1}},
\]
\[
b^* = 1 - \frac{\kappa}{b^*},
\]
\[
\zeta = \frac{-b}{ax \sqrt{1 - \rho^2}}
\]
and \( \Phi(\cdot, \cdot, \cdot) \) is the degenerated hypergeometric function.

Proof. From (3.1), it is clear that
\[
f(x : T, l, v) = e^{ixl}
\]
which is the boundary condition of PIDE (3.2). This implies that
\[
B(T, T) = C(T, T) = D(T, T) = 0.
\]
(3.5)
Substituting (3.3) in (3.2) and using the fact that the function \( f \) is never zero, we obtain
\[ 0 = \left[ B_t + (ba - \lambda S m)(C + ix) + \kappa \theta D \right. \]
\[ + \lambda^S \int_{\mathbb{R}} [e^{ixy} - 1] \phi_Y(y) dy + \lambda^v \int_{\mathbb{R}} [e^{iz} - 1] \phi_Z(z) dz \]
\[ + [C_t - b(C + ix)]l \]
\[ + [D_t + \frac{1}{2}(C + ix) + \frac{1}{2}(C + ix)^2 - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho \sigma (C + ix)D]v \]
\[ + [F_t - b(C + ix)] \]
\[ \left. + \frac{1}{2}(C + ix) \right] \int_{\mathbb{R}} [\phi_Y(y) dy] \]
\[ + [G_t + \frac{1}{2}(C + ix) + \frac{1}{2}(C + ix)^2 - \kappa G + \frac{1}{2}\sigma^2 G^2 + \rho \sigma (C + ix)G]v \]
\[ \int_{\mathbb{R}} [\phi_Z(z) dz] \]
\[ = 0 \]  
(3.6)

where \( B_t, C_t \) and \( D_t \) are the partial derivatives with respect to \( t \) of functions \( B, C \) and \( D \) respectively.

This reduces the problem to one of solving three, much simpler, ordinary differential equations:

\[ B_t + (ba - \lambda S m)(C + ix) + \kappa \theta D + \lambda^S \int_{\mathbb{R}} [e^{ixy} - 1] \phi_Y(y) dy \]
\[ + \lambda^v \int_{\mathbb{R}} [e^{iz} - 1] \phi_Z(z) dz = 0 \]  
(3.7)

\[ C_t - b(C + ix) = 0 \]  
(3.8)

\[ D_t + \frac{1}{2}(C + ix)(C + ix - 1) - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho \sigma (C + ix)D = 0 \]  
(3.9)

subject to boundary conditions (3.5).

The solution to equation (3.8) with the boundary condition \( C(T, T) = 0 \) is given by

\[ C(t, T) = ix(e^{-b(T-t)} - 1). \]  
(3.10)

We now consider equation (3.9). Substituting (3.10) in (3.9), one gets

\[ D_t + \frac{1}{2} \left[ ix e^{-b(T-t)} \right] \left[ ix e^{-b(T-t)} - 1 \right] - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho \sigma ix e^{-b(T-t)} = 0. \]

Hence,

\[ D_t = -\frac{1}{2}\sigma^2 D^2 + \left[ \kappa - \rho \sigma ix e^{-b(T-t)} \right] D + \frac{1}{2} \left[ \sigma^2 e^{-2b(T-t)} + ix e^{-b(T-t)} \right]. \]  
(3.11)

Let \( h = e^{-b(T-t)} \) and we define a new function \( \hat{D}(h(t), T) := D(t, T) \). Then

\[ \frac{\partial D(t, T)}{\partial t} = \frac{\partial \hat{D}(h, T)}{\partial h} \frac{\partial h}{\partial t} = b e^{-b(T-t)} \frac{\partial \hat{D}(h, T)}{\partial h}. \]  
(3.12)
Substituting (3.12) into (3.11), we obtain the following Riccati equation

\[
\frac{\partial \hat{D}}{\partial h} = -\frac{1}{2bh}\sigma^2 \hat{D}^2 + \left(\frac{\kappa}{bh} - \frac{\rho \sigma i}{b}\right) \hat{D} + \frac{1}{2b} \left(x^2 h + ix\right).
\]  

(3.13)

We shall solve the second order ODE (3.13) together with the initial condition \(\hat{D}(1, T) = 0\). Let

\[
\hat{D}(h, T) = \frac{2bh w'(h)}{\sigma^2 w(h)}
\]

(3.14)

and taking the derivative of (3.14) with respect to \(h\), one gets

\[
\frac{\partial \hat{D}}{\partial h} = \left[\sigma^2 w(h) \frac{\partial}{\partial h} (2bh w'(h)) - 2bh w'(h) \frac{\partial}{\partial h} (\sigma^2 w(h))\right] \frac{1}{\sigma^4 w^2(h)}
\]  

(3.15)

Substituting (3.14) and (3.15) into (3.13), we have

\[
h w''(h) - \left[\frac{\kappa}{b} - 1 - h(\frac{\rho \sigma i}{b})\right] w'(h) - \left[\frac{x^2 \sigma^2 h}{4b^2} + \frac{ix \sigma^2}{4b^2}\right] w(h) = 0.
\]  

(3.16)

The ODE (3.16) has a general solution of the form [10],

\[
w(h) = e^{(\sqrt{\rho^2 - \rho^2 - \rho^2}) \frac{\hat{D}}{\sigma h}} \left[C_1 \Phi(a^*, b^*, \frac{h}{\zeta}) + C_2 h^{1-b^*} \Phi(a^* - b^* + 1, 2 - b^*, \frac{h}{\zeta})\right],
\]  

(3.17)

where

\[
a^* = \frac{(\sqrt{\rho^2 - 1} + \rho) \frac{\hat{D}}{\sigma h} + \sigma}{\sqrt{\rho^2 - 1}}
\]

\[
b^* = 1 - \frac{\kappa}{b},
\]

and

\[
\zeta = \frac{-b}{\sigma x \sqrt{1 - \rho^2}}.
\]

Here \(C_1\) and \(C_2\) are constants to be determined from the boundary conditions. \(\Phi(a, b, z)\) is the degenerated hypergeometric function which has the following Kummer’s series expansion

\[
\Phi(a, b, z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k z^k}{(b)_k k!},
\]

where

\[
(a)_k = a(a + 1) \cdots (a + k - 1).
\]
If we let $C_1 = 1$ and $C_2 = 0$ in (3.17) then a particular solution for (3.16) is

$$w(h) = e^{(\sqrt{1-\rho^2-\rho i})\frac{b}{\zeta} h} \Phi(a^*, b^*, \frac{h}{\zeta}).$$

Using the transformation (3.14), Wong and Lo [11] show that a particular solution for (3.13) is

$$U(h) = \frac{2bh}{\sigma^2} \left( \sqrt{1-\rho^2-\rho i} \frac{\Phi(a^*, b^*, \frac{h}{\zeta})}{\Phi(a^*, b^*, \frac{1}{\zeta})} + \frac{e^{-\rho/\zeta}}{\Phi(a^*, b^*, \frac{1}{\zeta})} \Phi(a^* + 1, b^* + 1, \frac{h}{\zeta}) \right),$$

which can be used to obtain the general solution for (3.13) as follows

$$\hat{D}(h) = U(h) + \frac{e^{b^* h} \Phi(a^*, b^*, \frac{1}{\zeta})}{e^{b^* - 1} \Phi(a^*, b^*, \frac{1}{\zeta})} \int_1^T \frac{e^{-\rho/\zeta}}{\Phi(a^*, b^*, \frac{1}{\zeta})} \Phi(a^* + 1, b^* + 1, \frac{h}{\zeta}) e^{-2(\sqrt{1-\rho^2-\rho i} \frac{b}{\zeta}(h-1))} dh.$$

We now consider the final ordinary differential equation (3.7). Substituting (3.18) and (3.10) in (3.7), we have

$$B_t(t, T) = (\lambda^S m - ba)ixe^{-(T-t)} - \kappa \theta D(t, T) - \lambda^S \int_\mathbb{R} [e^{ixy} - 1] \phi_Y(y) dy - \lambda^v \int_\mathbb{R} [e^{izD} - 1] \phi_Z(z) dz.$$ 

Integrating both sides of the above equation and invoking the condition $B(T, T) = 0$, we obtain

$$B(t, T) = \left( \frac{\lambda^S m}{b} - a \right) ix(e^{-b(T-t)} - 1) - \kappa \theta \int_t^T D(s, T) ds$$

$$+ (T-t)\lambda^S \int_\mathbb{R} [e^{ixy} - 1] \phi_Y(y) dy$$

$$+ (T-t)\lambda^v \int_\mathbb{R} [e^{izD} - 1] \phi_Z(z) dz.$$ 

We can conclude that the characteristic function of the mean reverting process (2.3) with stochastic volatility (2.2) is

$$f(x : t, l, v) = e^{B(t, T) + C(t, T)x + D(t, T)v + ixl},$$

where $B(t, T), C(t, T)$ and $D(t, T)$ are as given in the Lemma.
4 A formula for European Option Pricing

Let $C$ denote the price at time $t$ of a European style call option on the current price of the underlying asset $S_t$ with strike price $K$ and expiration time $T$.

The terminal payoff of a European call option on the underlying stock price $S_T$ with strike price $K$ is
\[ \max(S_T - K, 0). \]
This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \leq K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate $r$ is constant over the lifetime of the option, the price of the European call at time $t$ is equal to the discounted conditional expected payoff
\[ C(t, S_T) = e^{-r(T-t)} E_M[\max(S_T - K, 0)|F_t]. \]
Assume that $t = 0$ and we define $L_T = \ln S_T$ and $k = \ln K$. Moreover, we express the call price option $C(0, S_T)$ as a function of the log of the strike price $K$ rather than the terminal log asset price $S_T$. The initial call value $C_T(k)$ is related to the risk-neutral density $q_T(l)$ by
\[ C_T(k) = e^{-rT} \int_{k}^{\infty} (e^l - e^k)q_T(l)dl, \tag{4.1} \]
where $q_T(l)$ is the density function of the random variable $L_T$. It was mentioned by Carr and Madan [12] that $C_T(k)$ is not square integrable. To obtain a square integrable function, they introduced the modified call price function $c_T(k)$ defined by
\[ c_T(k) = e^{\alpha k}C_T(k) \tag{4.2} \]
for some constant $\alpha > 0$ that makes $c_T(k)$ is square integrable in $k$ over the entire real line and a good choice of $\alpha$ is that one fourth of the upper bound $E[S_T^{\alpha+1}] < \infty$. Consider the Fourier transform of $c_T(k)$
\[ \psi_T(u) = \int_{-\infty}^{\infty} e^{iku}c_T(k)dk \]
\[ = \int_{-\infty}^{\infty} e^{iku} \int_{k}^{\infty} e^{\alpha k}e^{-rT}(e^l - e^k)q_T(l)dl dk \]
\[ = \int_{-\infty}^{\infty} e^{-rT}q_T(l) \int_{-\infty}^{l} (e^{l+\alpha k} - e^{(1+\alpha)k})e^{iku}dk dl \]
\[ = \int_{-\infty}^{\infty} e^{-rT}q_T(l) \left[ \frac{e^{(\alpha+1+iu)l}}{\alpha + iu} - \frac{e^{(\alpha+1+iu)l}}{\alpha + iu + 1} \right] dl \]
\[ = e^{-rT} \int_{-\infty}^{\infty} \left[ \frac{(\alpha + iu)e^{(\alpha+1+iu)l} + e^{(\alpha+1+iu)l} - (\alpha + iu)e^{(\alpha+1+iu)l}}{(\alpha + iu)(\alpha + iu + 1)} \right] q_T(l)dl \]
\[ = e^{-rT} \int_{-\infty}^{\infty} \left[ \frac{e^{(\alpha+1+iu)l}}{\alpha^2 + 2\alpha iu - u^2 + \alpha + iu} \right] q_T(l)dl \]
where \( f \) is the characteristic function defined in Lemma 3.1.

Hence, the European call prices at time \( t = 0 \) with strike price \( k = \ln K \) can then be numerically obtained by using the inverse transform:

\[
C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iu(k-\alpha l)} \psi_T(u) du = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-iu(k-\alpha l)} \frac{f(x=u-(\alpha+1)i:t,l,v)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u} du. \tag{4.3}
\]

Integration (4.3) is a direct Fourier transform and lends itself to an application of the Fast Fourier Transform (FFT).

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References


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