On Common Fixed Point of Nonself Nonexpansive Mappings for Multistep Iteration in Banach Spaces

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Abstract: Suppose that $E$ be a uniformly convex Banach space, let $K$ be a nonempty closed convex subset of $E$ with $P$ as a nonexpansive retraction. Let $T: K \rightarrow E$ be a given nonself mapping. The modified multistep iterative scheme $\{x_n\}$ is defined by (1.10). We establish the weak convergence of a sequence of a modified multistep iteration of an nonself $I$-nonexpansive map in a Banach space which satisfies Opial’s condition.

Keywords: Mann, Ishikawa, Noor and Multistep iterations; nonself nonexpansive maps

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1. Introduction and preliminaries

Let $E$ be a normed linear space, $K$ a nonempty, convex subset of $E$, and $T$ a self map of $K$. Three most popular iteration procedures for obtaining fixed points of $T$, if they exist, are Mann iteration [12], defined by

$$u_1 \in K, \quad u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n, \quad n \geq 1 \tag{1.1}$$

Ishikawa iteration [13], defined by

$$z_1 \in K, \quad z_{n+1} = (1 - \alpha_n)z_n + \alpha_n Ty_n, \quad y_n = (1 - \beta_n)z_n + \beta_n Tz_n, \quad n \geq 1, \tag{1.2}$$

Noor iteration [14], defined by

$$v_1 \in K, \quad v_{n+1} = (1 - \alpha_n)v_n + \alpha_n Tw_n, \quad w_n = (1 - \beta_n)v_n + \beta_n Tw_n, \quad t_n = (1 - \gamma_n)v_n + \gamma_n Tv_n, \quad n \geq 1, \tag{1.3}$$

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for certain choices of \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \subset [0,1] \).

The multistep iteration \([15]\), arbitrary fixed order \( p \geq 2 \), defined by

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy^1_n, \\
y^i_n = (1 - \beta^i_n)x_n + \beta^i_nTy^{i+1}_n, \quad i = 1, 2, ..., p - 2 \\
y^{p-1}_n = (1 - \beta^{p-1}_n)x_n + \beta^{p-1}_nTx_n.
\]

The sequence \( \{\alpha_n\} \) is such that for all \( n \in \mathbb{N} \)

\[
\{\alpha_n\} \subset (0, 1), \quad \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty
\]

and for all \( n \in \mathbb{N} \)

\[
\{\beta^{i}_n\} \subset [0, 1), \quad 1 \leq i \leq p - 1, \quad \lim_{n \to \infty} \beta^{i}_n = 0.
\]

In the above taking \( p = 3 \) in (1.4) we obtain iteration (1.3). Taking \( p = 2 \) in

(1.4) we obtain iteration (1.2).

The following definitions and statements will be needed for the proof of our
theorem.

Let \( K \) be a subset of normed linear space \( E = (E, \|\cdot\|) \) and \( T \) self-mappings of

\( K \). Then \( T \) is called nonexpansive on \( K \) if

\[
\|Tx - Ty\| \leq \|x - y\| 
\]
for all \( x, y \in K \). Let \( F(T) := \{x \in K : Tx = x\} \) be denoted as the set of fixed
points of a mapping \( T \).

Let \( K \) be a subset of a normed linear space \( E = (E, \|\cdot\|) \) and \( T \) and \( I \) self-
mappings of \( K \). Then \( T \) is called \( I \)-nonexpansive on \( K \) if

\[
\|Tx - Ty\| \leq \|Ix - Iy\| 
\]
for all \( x, y \in K \) \([9]\).

\( T \) is called \( I \)-quasi-nonexpansive on \( K \) if

\[
\|Tx - f\| \leq \|Ix - f\| 
\]
for all \( x, y \in K \) and \( f \in F(T) \cap F(I) \).

Let \( E \) be a real Banach space. A subset \( K \) of \( E \) is said to be a retract of \( E \) if there exists a continuous map \( P : E \to K \) such that \( Px = x \) for all \( x \in K \). A map \( P : E \to E \) is said to be a retraction if \( P^2 = P \). It follows that if a map \( P \) is
a retraction, then \( Py = y \) for all \( y \) in the range of \( P \). A set \( K \) is optimal if each
point outside \( K \) can be moved to be closer to all points of \( K \). Note that every
nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets
are closed and convex. However, every closed convex subset of a Hilbert space is
optimal and also a nonexpansive retract.
Recall that a Banach space $E$ is said to satisfy Opial’s condition [6] if, for each sequence $\{x_n\}$ in $E$, the condition $x_n \to x$ implies that
\[ \lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\| \] (1.8)
for all $y \in E$ with $y \neq x$.

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space $H$: if $K$ is a closed and convex subset of $H$ and $T$ has a fixed point, then for every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of $T$. It was also shown by Pazy [7] that if $H$ is a real Hilbert space and $\left(\frac{1}{n}\right) \sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [2] and Dotson [3] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [5] in metric spaces which we adapt to a normed space as follows: $T$ is called a quasi-nonexpansive mapping provided
\[ \|Tx - f\| \leq \|x - f\| \] (1.9)
for all $x \in K$ and $f \in F(T)$.

**Remark 1.1.** From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive. There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn and Williamson [8]. Their analysis was related to the convergence of Mann iterates studied by Dotson [3]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [4]. In [10], the weakly convergent theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [11], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In [16], Rhoades and Temir considered $T$ and $I$ self-mappings of $K$, where $T$ is an $I$-nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of $T$ and $I$. More precisely, they proved the following theorem.

**Theorem 1.2. (Rhoades and Temir [16]):** Let $K$ be a closed convex bounded subset of uniformly convex Banach space $E$, which satisfies Opial’s condition, and let $T, I$ self-mappings of $K$ with $T$ be an $I$-nonexpansive mapping, $I$ a nonexpansive on $K$. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Mann iterates converges weakly to common fixed point of $F(T) \cap F(I)$.

In the above theorem, $T$ remains self-mapping of a nonempty closed convex subset
K of a uniformly convex Banach space. If, however, the domain \( K \) of \( T \) is a proper subset of \( E \) and \( T \) maps \( K \) into \( E \) then, the iteration formula (1.1) may fail to be well defined. One method that has been used to overcome this in the case of single operator \( T \) is to introduce a retraction \( P : E \to K \) in the recursion formula (1.1) as follows: \( u_1 \in K, \)

\[
u_{n+1} = (1 - \alpha_n)u_n + \alpha_n PTu_n, \quad n \geq 1,
\]

In [17], Kızıltunc and Ozdemir considered \( T \) and \( I \) nonself-mappings of \( K \), where \( T \) is an \( I \)-nonexpansive mapping. They established the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of \( T \) and \( I \). More precisely, they proved the following theorem.

**Theorem 1.3. (Kızıltunc and Ozdemir [17]):** Let \( K \) be a closed convex bounded subset of uniformly convex Banach space \( E \), which satisfies Opial’s condition, and let \( T \), \( I \) nonself mappings of \( K \) with \( T \) be an \( I \)-nonexpansive mapping, \( I \) a nonexpansive on \( K \). Then, for \( x_0 \in K \), the sequence \( \{x_n\} \) of modified Ishikawa iterates converges weakly to a common fixed point of \( F(T) \cap F(I) \).

In this study, we consider \( T \) and \( I \) nonself mappings of \( K \), where \( T \) is an \( I \)-nonexpansive mappings. We establish the weak convergence of the sequence of modified multistep iterates to a common fixed point of \( T \) and \( I \).

Let \( E \) be a normed linear space, \( K \) be a nonempty convex subset of \( E \) with \( P \) as a nonexpansive retraction. Let \( T : K \to E \) be a given nonself mapping. The modified multistep iterative scheme \( \{x_n\} \) is defined by, arbitrary fixed order \( p \geq 2 \)

\[
x_{n+1} = P \left( (1 - \alpha_n)x_n + \alpha_n Ty_n \right),
\]

\[
y_i^n = P \left( (1 - \beta_i^n)x_n + \beta_i^n Ty_{i+1} \right), \quad i = 1, 2, ..., p-2
\]

\[
y_{p-1}^{p-1} = P \left( (1 - \beta_{p-1}^n)x_n + \beta_{p-1}^n Tx_n \right),
\]

where the sequence \( \{\alpha_n\} \) is such that for all \( n \in \mathbb{N} \),

\[
\{\alpha_n\} \subset (0, 1), \quad \lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty
\]

and for all \( n \in \mathbb{N} \),

\[
\{\beta_i^n\} \subset [0, 1), \quad 1 \leq i \leq p-1, \quad \lim_{n \to \infty} \beta_i^n = 0.
\]

Clearly, if \( T \) is self maps, then (1.10) reduces to an iteration scheme (1.4).

**2. The main result**

**Theorem 2.1.** Let \( K \) be a closed convex bounded subset of uniformly convex Banach space \( E \), which satisfies Opial’s condition, and let \( T \), \( I \) nonself mappings of \( K \) with \( T \) be an \( I \)-nonexpansive mapping, \( I \) a nonexpansive on \( K \). Then, for \( x_0 \in K \), the sequence \( \{x_n\} \) of modified multistep iterates converges weakly to common fixed point of \( F(T) \cap F(I) \).
**Proof.** If \( F(T) \cap F(I) \) is nonempty and a singleton, then the proof is complete. We will assume that \( F(T) \cap F(I) \) is not a singleton.

\[
\|x_{n+1} - f\| = \|P ((1 - \alpha_n)x_n + \alpha_n T y_n^1) - f\|
\]

\[
\leq \|(1 - \alpha_n)x_n + \alpha_n T y_n^1 - f\|
\]

\[
\leq \|(1 - \alpha_n)(x_n - f) + \alpha_n \left[ P ((1 - \beta_n^1)x_n + \beta_n^1 T y_n^2) - f\right]\|
\]

\[
\leq \|(1 - \alpha_n)(x_n - f) + \alpha_n \left[ (1 - \beta_n^1)x_n + \beta_n^1 T y_n^2 - f\right]\|
\]

\[
\leq (1 - \alpha_n) \|x_n - f\| + \alpha_n \beta_n^1 \| (1 - \beta_n^2) x_n - f\| - \alpha_n T x_n - f\|
\]

\[
\leq (1 - \alpha_n) \|x_n - f\| + \alpha_n \beta_n^1 \| (1 - \beta_n^2) x_n - f\| - \alpha_n T x_n - f\|
\]

Thus, for \( \alpha_n \neq 0 \) and \( \beta_n^i \neq 0 \), \( \{\|x_n - f\|\} \) is a nonincreasing sequence. Then, \( \lim_{n \to \infty} \|x_n - f\| \) exists.

Now we show that \( \{x_n\} \) converges weakly to a common fixed point of \( T \) and \( I \). The sequence \( \{x_n\} \) contains a subsequence which converges weakly to a point in \( K \). Let \( \{x_{n_k}\} \) and \( \{x_{m_k}\} \) be two subsequences of \( \{x_n\} \) which converge weakly to \( f \) and \( q \), respectively. We will show that \( f = q \). Suppose that \( E \) satisfies
Opial’s condition and that \( f \neq q \) is in weak limit set of the sequence \( \{x_n\} \). Then \( \{x_{m_k}\} \to f \) and \( \{x_{m_k}\} \to q \), respectively. Since \( \lim_{n \to \infty} \|x_n - f\| \) exists for any \( f \in F(T) \cap F(I) \) by Opial’s condition, we conclude that

\[
\lim_{n \to \infty} \|x_n - f\| = \lim_{k \to \infty} \|x_{n_k} - f\| \quad (1)
\]

\[
< \lim_{k \to \infty} \|x_{n_k} - q\| = \lim_{j \to \infty} \|x_{m_j} - q\|
\]

\[
< \lim_{j \to \infty} \|x_{m_j} - f\|
\]

\[
= \lim_{n \to \infty} \|x_n - f\|.
\]

This is a contradiction. Thus \( \{x_n\} \) converges weakly to an element of \( F(T) \cap F(I) \).

References


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