A Fundamental Theorem of Co-Homomorphisms for Semirings

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Abstract: The quotient structure of a semiring with non-zero identity modulo a $Q$-strong co-ideal has been introduced and studied in [1]. In this paper, we will introduce the notions of co-homomorphisms and Maximal co-homomorphisms for semirings. Using these notions, the fundamental theorem of co-homomorphisms will be generalized to include a large class of semirings.

Keywords: semiring; co-ideal; strong co-ideal; partitioning strong co-ideal; subtractive co-ideal; co-homomorphism; maximal co-homomorphism.

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1 Introduction

P. J. Allen [2] introduced the notion of a $Q$-ideal and a construction process by which one can build the quotient structure of a semiring modulo a $Q$-ideal. Maximal homomorphisms were defined and examples of such homomorphisms were given. Using these notions, the fundamental theorem of homomorphisms for rings was generalized to include a large class of semirings. The present authors [3] have presented the notion of a $Q$-strong co-ideal $I$ in the semiring $R$ and constructed the quotient semiring $R/I$. In this paper, we extend the definition and results given by Allen to a more general $Q$-strong co-ideal case. In this paper, we introduce the notion of co-homomorphism and maximal co-homomorphism. We show if $I$ is a $Q$-strong co-ideal of semiring $R$ and $\phi : R \to R/I$ with $\phi(a) = qI$, $\phi$ is a co-homomorphism and $\phi$ is maximal if and only if $I$ is a $Q$-strong co-ideal.

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where \( q \) is the unique element of \( Q \) such that \( a \in qI \), then \( \phi \) is a maximal co-homomorphism. Also, it is shown if \( \phi \) is a co-homomorphism from the semiring \( R \) onto \( R' \) that is maximal, then \( R/\text{co-Ker}(\phi) \cong R' \).

For the sake of completeness, we state some definitions and notations used throughout. A commutative semiring \( R \) is defined as an algebraic system \((R, +, \cdot)\) such that \((R, +)\) and \((R, \cdot)\) are commutative semigroups, connected by \( ab + ac \) for all \( a, b, c \in R \), and there exists \( 0, 1 \in R \) such that \( r + 0 = r \) and \( r0 = 0r = 0 \) and \( r1 = 1r = r \) for each \( r \in R \). In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

In this paper, \( B \) denotes the boolean semiring \( \{0, 1\} \), which \( 1 + 1 = 1 \).

**Definition 1.1.** Let \( R \) be a semiring.

(1) A non-empty subset \( I \) of \( R \) is called co-ideal, denoted by \( I \trianglelefteq R \), if it is closed under multiplication and satisfies the condition \( r + a \in I \) for all \( a \in I \) and \( r \in R \) (clearly, \( 0 \in I \) if and only if \( I = R \)) ([3], [4]). A co-ideal \( I \) is called strong co-ideal if \( I \neq \emptyset \) [1].

(2) A co-ideal \( I \) of \( R \) is called subtractive if for each \( x, y \in R \) with \( x, xy \in I \), then \( y \in I \) [4].

(3) A proper co-ideal \( I \) of \( R \) is said to be maximal if \( J \) is a co-ideal of \( R \) with \( I \subseteq J \), then \( J = R \). It is known that maximal co-ideals are strong co-ideal [5].

(4) A mapping \( \varphi \) from the semiring \( R \) into the semiring \( R' \) will be called a homomorphism if \( \varphi(a + b) = \varphi(a) + \varphi(b) \) and \( \varphi(ab) = \varphi(a)\varphi(b) \) for each \( a, b \in R \). An isomorphism is a one-to-one homomorphism. The semirings \( R \) and \( R' \) will be called isomorphic (denoted by \( R \cong R' \)) if there exists an isomorphism from \( R \) onto \( R' \) [2].

**Definition 1.2.** (See [1]) A strong co-ideal \( I \) of a semiring \( R \) is called a partitioning co-ideal (= Q-strong co-ideal) if there exists a subset \( Q \) of \( R \) such that

(1) \( R = \bigcup\{qI : q \in Q\} \), where \( qI = \{qt : t \in I\} \),

(2) \( qI \cap (q2I) \neq \emptyset \) if and only if \( q1 = q2 \).

**Lemma 1.3.** (See [1]) Let \( I \) be a Q-strong co-ideal of the semiring \( R \). If \( x \in R \), then there exists a unique \( q \in Q \) such that \( xI \subseteq qI \). In particular, \( x = qa \) for some \( a \in I \).

Let \( I \) be a Q-strong co-ideal of a semiring \( R \) and let \( R/I = \{qI : q \in Q\} \). Then \( R/I \) forms a semiring under the binary operations \( \oplus \) and \( \circ \) defined as follows:

(1) \((q1I) \oplus (q2I) = q3I \), where \( q3 \) is the unique element in \( Q \) such that \((q1I + q2I) \subseteq q3I \); and

(2) \((q1I) \circ (q2I) = q3I \), where \( q3 \) is the unique element in \( Q \) such that \((q1q2)I \subseteq q3I \) (see [1]).

**Proposition 1.4.** (See [1]) Every Q-strong co-ideal \( I \) of a semiring \( R \) is subtractive.

**Lemma 1.5.** (See [5]) If \( D \) is a maximal co-ideal of a semiring \( R \), then \( R - D \) is an ideal.
2 Co-Homomorphism of semirings

We begin with the key definition of this paper.

**Definition 2.1.** Let $R$ and $R'$ be two semirings. The map $\phi : R \rightarrow R'$ is called co-homomorphism if satisfies the following conditions:

1. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$.
2. $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in R$.
3. $\phi(0) = 0$.
4. $\phi(1) = 1$.
5. If $\phi(r) = 1$ for some $r \in R$, then $\phi(a + r) = 1$ for all $a \in R$.

One can easily see that every co-homomorphism is a semiring homomorphism.

The following example shows that a homomorphism need not be a co-homomorphism.

**Example 2.2.** Let $\mathbb{Z}^+ \cup \{0\}$ be the semiring of positive integers with the usual addition and multiplication and $i$ be the identity homomorphism of semiring $\mathbb{Z}^+ \cup \{0\}$. It is clear that $i(1) = 1$ and $i(r + 1) \neq 1$ for each $r \in \mathbb{Z}^+ \cup \{0\}$. So $i$ is not a co-homomorphism.

**Proposition 2.3.** Let $D$ be a co-ideal of a semiring $R$ such that $R - D$ is an ideal of $R$. Then $D$ is a subtractive strong co-ideal of $R$.

**Proof.** Let $xy \in D$ and $x \in D$ for some $x, y \in R$. If $y \notin D$, then $y \in R - D$. By hypothesis, $R - D$ is an ideal of $R$, therefore $xy \in R - D$, a contradiction. Thus $D$ is a subtractive co-ideal of $R$. Clearly, $1 \in D$ since $D$ is a subtractive co-ideal.

The converse of Proposition 2.3 is not true, as the following example shows.

**Example 2.4.** Let $X = \{a, b, c\}$. Then $R = (P(X), \cup, \cap)$ is a semiring, where $P(X)$ is the set of all subsets of $X$. An inspection will show that $I = \{X, \{a, b\}\}$ is a $Q$-strong co-ideal of $R$, where $Q = \{\{c\}, \{a, c\}, \{b, c\}, X\}$. Thus $I$ is a subtractive co-ideal of $R$ by Proposition 1.4. It can be seen $R - I$ is not an ideal of $R$, because $\{a\}, \{b\} \in R - I$ and $\{a\} \cup \{b\} = \{a, b\} \notin R - I$.

**Proposition 2.5.** If $D$ is a maximal co-ideal of $R$, then $D$ is subtractive.

**Proof.** Apply Lemma 1.5 and Proposition 2.3.

The following example shows that the converse of Proposition 2.5 is not true.

**Example 2.6.** Let $R$ be the set of all non-negative integers. Define $a + b = \gcd(a, b)$ and $a \times b = \operatorname{lcm}(a, b)$ (take $0 + 0 = 0$ and $0 \times 0 = 0$ ). Then $(R, +, \times)$ is easily checked to be a commutative semiring. Let $I$ be the set of all non-negative odd integers, then $I$ is a co-ideal of $R$. An inspection shows that $R - I$ is an ideal of $R$. It can be seen $I$ is not a maximal co-ideal of $R$, because $I \subsetneq R - \{0\}$ and $R - \{0\}$ is a maximal co-ideal of $R$. 
Theorem 2.7. Let $D$ be a co-ideal of $R$ such that $R - D$ is an ideal of $R$. Then there exists a co-homomorphism from $R$ onto $B$.

Proof. Let $\phi : R \to B$ with

$$
\phi(x) = \begin{cases} 
0 & \text{if } x \notin D, \\
1 & \text{if } x \in D 
\end{cases}
$$

We will show that $\phi$ is a co-homomorphism.

1. $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in R$. We consider the various possibilities for $a, b$.

   **Case 1:** $a, b \in D$. Since $D$ is a co-ideal, $a + b \in D$. So $\phi(a + b) = 1$. Also $\phi(a) + \phi(b) = 1 + 1 = 1$. Thus $\phi(a + b) = \phi(a) + \phi(b)$.

   **Case 2:** $a \notin D$ and $b \notin D$. Since $I = R - D$ is an ideal of $R$ and $a, b \in I$, $a + b \in I$ and so $a + b \notin D$. It is clear that $\phi(a + b) = \phi(a) + \phi(b) = 0$.

   **Case 3:** $(a \in D, b \notin D)$ or $(a \notin D, b \in D)$. In these two, we have $a + b \in D$. So $1 = \phi(a + b) = \phi(a) + \phi(b) = 1 + 0 = 1$.

2. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$. We consider the various possibilities for $a, b$.

   **Case 1:** $a, b \in D$. Since $D$ is a co-ideal, $ab \in D$, and so $\phi(ab) = 1$. Since $a, b \in D$, $\phi(a) = 1$ and $\phi(b) = 1$. Therefore $\phi(ab) = \phi(a)\phi(b)$.

   **Case 2:** $a \notin D$ and $b \notin D$. Since $I = R - D$ is an ideal of $R$, $ab \in I$ and so $ab \notin D$. Thus $\phi(ab) = 0$. Therefore $0 = \phi(ab) = \phi(a)\phi(b)$.

   **Case 3:** $(a \in D, b \notin D)$ or $(a \notin D, b \in D)$. Since $R - D$ is an ideal, $D$ is a subtractive co-ideal by Proposition 2.3. Therefore $ab \notin D$. So $0 = \phi(ab) = \phi(a)\phi(b)$.

3. $\phi(1) = 1$ is clear, since $1 \in D$.

4. $\phi(0) = 0$ is clear, since $0 \notin D$.

5. If $\phi(r) = 1$, then $r \in D$. Hence $a + r \in D$ for each $a \in R$. Thus $\phi(a + r) = 1$.

It is clear that $\phi$ is onto.

\[
\text{Def. 2.8. Let } R \text{ and } R' \text{ be two semirings and } \phi : R \to R' \text{ be a co-homomorphism. Set } co - Ker(\phi) = \{ r \in R : \phi(r) = 1 \}.
\]

Remark 2.9. It is clear that $co - Ker(\phi)$ is a strong co-ideal of $R$ and in Theorem 2.7, $co - Ker(\phi) = \{ x \in R : \phi(x) = 1 \} = D$.

\[
\text{Def. 2.10. A co-homomorphism } \phi \text{ with } co - Ker(\phi) = K \text{ from a semiring } R \text{ onto the semiring } R' \text{ is said to be maximal if for each } a \in R' \text{ there exists } g_a \in \phi^{-1}(\{a\}) \text{ such that } xK \subseteq g_aK \text{ for each } x \in \phi^{-1}(\{a\}).
\]
Example 2.11. Let \( R = \mathbb{Z}^+ \cup \{0\} \) be the semiring of positive integers and \( \phi : R \to B \) with
\[
\phi(x) = \begin{cases} 
0 & \text{if } x = 0, \\
1 & \text{if } x \in R - \{0\}.
\end{cases}
\]

It can be checked that \( \phi \) is a co-homomorphism. Put \( q_1 = 1 \) and \( q_0 = 0 \). Then for each \( x \in R - \{0\} \), we have \( x(co - Ker(\phi)) \subseteq q_1(co - Ker(\phi)) \) and for \( x = 0, x(co - Ker(\phi)) \subseteq q_0(co - Ker(\phi)) \). Therefore \( \phi \) is a maximal co-homomorphism.

Proposition 2.12. Let \( R \) be a semiring and \( I \) be a \( Q \)-strong co-ideal of \( R \). If \( \phi : R \to R/I \) with \( \phi(a) = qI \), where \( q \) is the unique element of \( Q \) such that \( a \in qI \), then \( \phi \) is a maximal co-homomorphism.

Proof. We prove the proposition in six steps.

(1) \( \phi(ab) = \phi(a) \circ \phi(b) \) for all \( a, b \in R \). Let \( q_1, q_2, q \) be elements of \( Q \) such that \( ab \in qI, a \in q_1I \) and \( b \in q_2I \). Hence \( \phi(a) = q_1I, \phi(b) = q_2I \) and \( \phi(ab) = qI \). Let \( q' \in Q \) such that \( q_1q_2I \subseteq q'I \) then \( \phi(a) \circ \phi(b) = q_1I \circ q_2I = q'I \). We will show that \( q = q' \). Since \( ab = q_1q_2I \subseteq q_1q_2I \subseteq q'I \), \( ab \in (q'I) \cap (qI) \) and so \( q = q' \). Therefore \( \phi(ab) = \phi(a) \circ \phi(b) \).

(2) \( \phi(a + b) = \phi(a) \oplus \phi(b) \), for all \( a, b \in R \). Let \( q \in I \) such that \( a + b \in qI \), then \( \phi(a + b) = qI \). Let \( q_1 \in Q \) and \( q_2 \in Q \) such that \( a \in q_1I \) and \( b \in q_2I \), then \( \phi(a) = q_1I \) and \( \phi(b) = q_2I \). Let \( q' \in Q \) such that \( q_1I + q_2I \subseteq q'I \), then \( \phi(a) \oplus \phi(b) = q_1I \oplus q_2I = q'I \). Since \( a + b \in q_1I + q_2I \), \( a + b \in q'I \) and hence \( a + b \in (q'I) \cap (qI) \). Therefore \( q = q' \) and so \( \phi(a + b) = \phi(a) \oplus \phi(b) \).

(3) \( \phi(0) = 0 \). Let \( q_0 \in Q \) be unique element such that \( 0 \in q_0I \). Therefore \( \phi(0) = q_0I \) where \( q_0I \) is zero element of \( R/I \).

(4) \( \phi(1) = 1 \) is clear.

(5) Let \( \phi(r) = q_rI = I \) where \( q_rI \) is the identity element of \( R/I \), then by definition of \( \phi \), \( r \in I \). Thus for each \( a \in R \), \( a + r \in I \) (since \( I \) is a co-ideal), and hence \( \phi(a + r) = I \) as desired.

(6) It is clear that \( co - Ker(\phi) = I \). Since \( I \) is a \( Q \)-strong co-ideal, for each \( qI \in R/I \) and \( x \in \phi^{-1}(qI), xI \subseteq qI \). Thus \( \phi \) is a maximal co-homomorphism.

Lemma 2.13. Let \( \phi \) be a co-homomorphism from the semiring \( R \) onto semiring \( R' \). If \( \phi \) is maximal, then \( co - Ker(\phi) = K \) is a \( Q \)-strong co-ideal of \( R \).

Proof. As \( \phi \) is a maximal co-homomorphism, for each \( a \in R' \) there exists \( q_a \in \phi^{-1}(\{a\}) \) such that \( xK \subseteq q_aK \) for each \( x \in \phi^{-1}(\{a\}) \). First, we show that \( R = \cup \{q_aK : a \in R'\} \). Let \( r \in R \), then \( \phi(r) \in R' \). Let \( \phi(r) = b \). Then \( r \in \phi^{-1}(...}
Lemma 2.14. Let $R$, $R'$, $\phi$ and $Q$ be as stated in Lemma 2.13 and let $q_a$, $q_b$ and $q_c$ be elements in $Q$ and $K = co - Ker(\phi)$.

(1) If $(q_aK + q_bK) \subseteq q_cK$, then $a + b = c$.

(2) If $q_aq_bK \subseteq q_cK$, then $ab = c$.

Proof. (1) Since $q_a + q_b \in (q_aK + q_bK) \subseteq q_cK$, there exists $k \in K$ such that $q_a + q_b = q_c$. Thus $a + b = \phi(q_a) + \phi(q_b) = \phi(q_a + q_b) = \phi(q_c) = \phi(q_c)\phi(k) = c$.

(2) It can be proved by a similar way as in (1).

Theorem 2.15. If $\phi$ is a co-homomorphism from the semiring $R$ onto $R'$ that is maximal, then $R/\text{co} - \text{Ker}(\phi) \cong R'$.

Proof. Let $co - \text{Ker}(\phi) = K$. By Lemma 2.13, $K$ is a $Q$-strong co-ideal and $R = \cup\{q_aK : a \in R'\}$. Let $\hat{\phi} : R/K \rightarrow R'$ with $\hat{\phi}(q_aK) = a$ (for each $x \in \phi^{-1}\{a\}, xK \subseteq q_aK$). Let $q_aK = q_bK$. Since $K$ is a $Q$-strong co-ideal, $q_a = q_b$. So $a = \phi(q_a) = \phi(q_b) = b$. Thus $\hat{\phi}$ is well-defined. Now we show $\hat{\phi}$ is a isomorphism.

(1) $\tilde{\phi}(q_aK \cap q_bK) = \tilde{\phi}(q_aK)\tilde{\phi}(q_bK)$. Let $q_c \in Q$ such that $q_aq_bK \subseteq q_cK$. Then $q_aK \subseteq q_cK$. By Lemma 2.14, $ab = c$ and so $\tilde{\phi}(q_aK \cap q_bK) = \tilde{\phi}(q_cK)\tilde{\phi}((q_aK))$.

(2) $\tilde{\phi}(q_aK \oplus q_bK) = \tilde{\phi}(q_aK)\tilde{\phi}(q_bK)$. Let $q_c \in Q$ such that $q_aK \oplus q_bK \subseteq q_cK$, then $q_aK \oplus q_bK \subseteq q_cK$. By Lemma 2.14, $a + b = c$. Thus $\tilde{\phi}(q_aK \oplus q_bK) = \tilde{\phi}(q_cK)\tilde{\phi}(q_aK)\tilde{\phi}(q_bK)$.

(3) $\tilde{\phi}$ is monomorphism. Let $\tilde{\phi}(q_aK) = \tilde{\phi}(q_cK)$. Hence $a = b$. Since for each $x \in \phi^{-1}\{a\}, xK \subseteq q_bK$, we have $q_aK \subseteq q_bK$. Similarly $q_bK \subseteq q_aK$. Hence $q_aK = q_bK$.

(4) $\tilde{\phi}$ is epimorphism. Let $a \in R'$. Since $\phi$ is epic, $\phi^{-1}\{a\} \neq \emptyset$. Since $\phi$ is maximal, there exists $q_a \in Q$ such that $q_a \in \phi^{-1}\{a\}$ and for each $x \in \phi^{-1}\{a\}$, $xK \subseteq q_aK$. Thus $\tilde{\phi}(q_aK) = a$. Thus $\tilde{\phi}$ is epic.

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References


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