Some Structural Properties of Vector Valued Orlicz Sequence Space

M. K. Özdemir and İ. Solak

Abstract : In this work, we introduce the vector valued sequence space $F(X_k, M, p, s)$ and study the closed subspace of it. We examine various algebraic and topological properties of this space and also investigate some inclusion relations on it.

Keywords : Orlicz function, Orlicz sequence space, vector valued sequence space, paranormed space.

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1 Introduction

Orlicz sequence spaces are one of the most natural generalizations of classical spaces $\ell_p$, $p \geq 1$. They were first considered by W. Orlicz in 1936. Afterwards, J. Lindenstrauss and L. Tzafriri [4] used the idea of Orlicz function $M$ to construct the sequence space $\ell_M$ of all sequences of scalars $(x_n)$ such that

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0.$$ 

The space $\ell_M$ becomes a Banach space which is called an Orlicz sequence space. The space $\ell_M$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $M(x) = x^p$, $(1 \leq p < \infty)$. In the present note, we introduce and examine some properties of a sequence space defined by using Orlicz function $M$, which generalizes the well known Orlicz sequence space $\ell_M$. Before introducing this sequence space, let us give some basic concepts:

An algebra $X$ is a linear space together with an internal operation of multiplication of elements of $X$, such that $xy \in X$, $x(yz) = (xy)z$, $x(y+z) = xy + xz$, $(x+y)z = xz + yz$ and $\lambda(xy) = (\lambda x)y = x(\lambda y)$, for any scalar $\lambda$, and a normed algebra is an algebra which is normed, as a linear space, and in which $\|xy\| \leq \|x\|\|y\|$ for all $x, y$; [6].

Let $F$ be a sequence space and $x, y$ be the arbitrary elements of $F$. Then $F$ is called a sequence algebra if it is closed under the multiplication defined by $xy = (x_k y_k)$. The space $F$ is called normal or solid if $y = (y_k) \in F$ whenever
\[ |y_k| \leq |x_k|, \quad k \in \mathbb{N}, \text{ for some } x = (x_k) \in F. \] If \( F \) is both normal and sequence algebra then it is called a normal sequence algebra. For example, \( w, \ell_\infty, c_0 \) and \( \ell_p \) \((0 < p < \infty)\) are normal sequence algebras. \( c \) is a sequence algebra but not normal.

A norm \( \|\cdot\| \) on a normal sequence space \( F \) is said to be absolutely monotone if \( x = (x_k), y = (y_k) \in F \) and \( |x_k| \leq |y_k| \) for all \( k \in \mathbb{N} \) implies \( \|x\| \leq \|y\| \), \( [5] \). The norm
\[
\|x\|_\infty = \sup |x_k|
\]
over \( \ell_\infty \), \( c \), \( c_0 \) and the norm
\[
\|x\| = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}
\]
over \( \ell_p \) for \( p \geq 1 \) are absolutely monotone.

We recall [3, 4] that an Orlicz function is a function \( M : [0, \infty) \rightarrow [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0 \), \( M(x) > 0 \) for all \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \). An Orlicz function \( M \) can always be represented in the following integral form:
\[
M(x) = \int_{0}^{x} p(t) \, dt,
\]
where \( p \), known as the kernel of \( M \).

We remark that \( M_1 + M_2 \) and \( M_1 \circ M_2 \) are Orlicz functions when \( M_1 \) and \( M_2 \) are Orlicz functions.

An Orlicz function \( M \) is said to satisfy the \( \Delta_2 \)-condition for all values of \( u \) if there exists a constant \( K > 0 \) such that \( M(2u) \leq KM(u) \), \( u \geq 0 \). It is easy to see always that \( K \geq 2 \). The \( \Delta_2 \)-condition is equivalent to the inequality \( M(\ell u) \leq K(\ell)M(u) \) which holds for all values of \( u \) and \( \ell > 1 \); \([3]\).

We now introduce the vector valued sequence space \( F(X_k, M, p, s) \) using Orlicz function \( M \).

Let \( X_k \) be seminormed space over the complex field \( \mathbb{C} \) with seminorm \( q_k \) for each \( k \in \mathbb{N} \), and \( F \) be a normal sequence algebra with absolutely monotone norm \( \|\cdot\|_F \) and having a Schauder basis \( (e_k) \), where \( e_k = (0, \ldots, 0, 1, 0, \ldots) \), with 1 in \( k \)-th place. Let \( p = (p_k) \) be any sequence of strictly positive real numbers and \( s \) be any non-negative real number. By \( s(X_k) \), we denote the linear space of all sequences \( x = (x_k) \) with \( x_k \in X_k \) for each \( k \in \mathbb{N} \) under the usual coordinatewise operations:
\[
\alpha x = (\alpha x_k) \quad \text{and} \quad x + y = (x_k + y_k)
\]
for each \( \alpha \in \mathbb{C} \). Let \( x \in s(X_k) \) and \( \lambda = (\lambda_k) \) is a scalar sequence such that
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\( \lambda x = (\lambda_k x_k) \). We define for an Orlicz function \( M \),

\[
F(X_k, M, p, s) = \{ x = (x_k) \in s(X_k) : x_k \in X_k \text{ for each } k \text{ and } \\
\left(k^{-s} \left[ M \left( \frac{q_k(x_k)}{\rho} \right) \right]^{p_k}\right) \in F \text{ for some } \rho > 0 \}.
\]

Y. Yılmaz, M. K. Özdemir and İ. Solak [8] introduced a generalization of Minkowski Inequality to normal sequence algebras with absolutely monotone seminorm. We will use Lemma 1 which states this extension to put forward a topology of the space \( F(X_k, M, p, s) \). For \( x = (x_k) \in F(X_k, M, p, s) \), we define

\[
g(x) = \inf \left\{ \rho^{\rho_n/H} > 0 : \left\| k^{-s} \left[ M \left( \frac{q_k(x_k)}{\rho} \right) \right]^{p_k} \right\|_F^{1/H} \leq 1, n \in \mathbb{N} \right\}, \tag{1.1}
\]

where \( H = \max(1, \sup p_k) \). It is shown that \( F(X_k, M, p, s) \) turns out to be a complete paranormed space with the paranorm defined by (1.1) whenever the seminormed space \( X_k \) is complete under the seminorm \( q_k \) for each \( k \in \mathbb{N} \).

It can be seen that for suitable choice of the sequence space \( F \), the seminormed space \( X_k \), the sequence of strictly positive real numbers \( (p_k) \), \( s \geq 0 \) and Orlicz function \( M \), the space \( F(X_k, M, p, s) \) reduces to the many number of known ordinary sequence spaces and as well as vector valued sequence spaces, as a particular case. For example, choosing \( F \) to be \( \ell_1 \), \( X_k = X \) (a vector space over \( \mathbb{C} \)) and \( q_k = q \) to be a seminorm on \( X \) in \( F(X_k, M, p, s) \) one gets the scalar valued sequence space \( \ell_M(p, q, s) \) defined by Ç. A. Bektaş & Y. Altın [1].

If \( X_k \) is normed space, \( p_k = 1 \) for each \( k \in \mathbb{N} \) and \( s = 0 \), then the class \( F(X_k, M, p, s) \) gives the class \( F(X_k, M) \) defined by D. Ghosh & P. D. Srivastava [2]. Furthermore, if \( F = \ell_1 \), \( X_k = \mathbb{C} \) and \( s = 0 \) in \( F(X_k, M, p, s) \), then one obtains the space \( \ell_M(p) \) defined by S. D. Parashar & B. Choudhary [7]. Thus, the generalized sequence space \( F(X_k, M, p, s) \) yields several spaces studied by several authors.

2 Linear Topological Structure of \( F(X_k, M, p, s) \)

Now, we examine some algebraic and topological properties of \( F(X_k, M, p, s) \) and investigate some inclusion relations on it. In order to discuss the properties of \( F(X_k, M, p, s) \), we assume that \( (p_k) \) is bounded. We will henceforth denote by \( h \) and \( C \), the real numbers \( \sup p_k \) and \( \max (1, 2^{h-1}) \), respectively.

Theorem 2.1 \( F(X_k, M, p, s) \) is a linear space over the complex field \( \mathbb{C} \).

Proof. Let \( x = (x_k), y = (y_k) \in F(X_k, M, p, s) \) and \( \alpha, \beta \in \mathbb{C} \). So, there exist \( \rho_1, \rho_2 > 0 \) such that

\[
\left( k^{-s} \left[ M \left( \frac{q_k(x_k)}{\rho_1} \right) \right]^{p_k} \right), \left( k^{-s} \left[ M \left( \frac{q_k(y_k)}{\rho_2} \right) \right]^{p_k} \right) \in F.
\]
Let $\rho_3 = \max (2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M$ is non-decreasing and convex,

$$k^{-s} \left[ M \left( \frac{q_k (\alpha x_k + \beta y_k)}{\rho_3} \right) \right]^{p_k} \leq k^{-s} \left[ M \left( \frac{|\alpha| q_k (x_k)}{\rho_3} + |\beta| q_k (y_k) \right) \right]^{p_k} \leq k^{-s} \left[ M \left( \frac{q_k (x_k)}{\rho_1} \right) + M \left( \frac{q_k (y_k)}{\rho_2} \right) \right]^{p_k} \leq C \left\{ k^{-s} \left[ M \left( \frac{q_k (x_k)}{\rho_1} \right) \right]^{p_k} + k^{-s} \left[ M \left( \frac{q_k (y_k)}{\rho_2} \right) \right]^{p_k} \right\}.$$ 

Since $F$ is a normal space, we have

$$\left( k^{-s} \left[ M \left( \frac{q_k (\alpha x_k + \beta y_k)}{\rho_3} \right) \right]^{p_k} \right) \in F$$

which shows that $\alpha x + \beta y \in F(X_k, M, p, s)$. □

**Theorem 2.2** $F(X_k, M, p, s)$ is a topological linear space, paranormed by

$$g(x) = \inf \left\{ \rho^{p_n/H} > 0 : \left\| \left( k^{-s} \left[ M \left( \frac{q_k (x_k)}{\rho} \right) \right]^{p_k} \right) \right\|_F^{1/H} \leq 1, n \in \mathbb{N} \right\},$$

where $H = \max (1, h)$.

To prove this theorem we need the following lemma.

**Lemma 1** Let $F$ be a normal sequence algebra, $\| \cdot \|_F$ be an absolutely monotone seminorm on $F$ and let $p > 1$. Then

$$\left\| (u + v)^p \right\|_F^{1/p} \leq \|u^p\|_F^{1/p} + \|v^p\|_F^{1/p},$$

for every $u = (u_n), v = (v_n) \in F$; [8].

**Proof.** [Proof of Theorem 2.2] Let $x = (x_k), y = (y_k) \in F(X_k, M, p, s)$. It is easy to see that $g(x) = g(-x)$ and $g(\theta) = 0$ for $\theta = (\theta_1, \theta_2, \ldots)$ the null element of $F(X_k, M, p, s)$ (where $\theta_i$ is the zero element of $X_i$ for each $i$).

We shall now show the subadditivity of $g$. By taking $\alpha = \beta = 1$ in Theorem 2.1, we have
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\[ k^{-s} \left[ M \left( \frac{q_k(x_k + y_k)}{\rho_3} \right) \right]^{p_k} \leq k^{-s} \left[ M \left( \frac{q_k(x_k)}{\rho_1} \right) + M \left( \frac{q_k(y_k)}{\rho_2} \right) \right]^{p_k} \]

\[ = \left( k^{-s/3} \left[ M \left( \frac{q_k(x_k)}{\rho_1} \right) + M \left( \frac{q_k(y_k)}{\rho_2} \right) \right]^{p_k/3} \right)^H \]

\[ \leq \left( k^{-s/3} \left[ M \left( \frac{q_k(x_k)}{\rho_1} \right) \right]^{p_k/3} \right)^H + k^{-s/3} \left[ M \left( \frac{q_k(y_k)}{\rho_2} \right) \right]^{p_k/3} \right)^H \]

Considering Lemma 1, we get

\[ \left\| \left( k^{-s} \left[ M \left( \frac{q_k(x_k + y_k)}{\rho_3} \right) \right]^{p_k} \right)^{1/3} \right\|_F \leq \left\| \left( k^{-s} \left[ M \left( \frac{q_k(x_k)}{\rho_1} \right) \right]^{p_k/3} \right)^{1/3} \right\|_F + \left\| \left( k^{-s} \left[ M \left( \frac{q_k(y_k)}{\rho_2} \right) \right]^{p_k/3} \right)^{1/3} \right\|_F \]

which means that \( g(x + y) \leq g(x) + g(y) \).

Finally, we show that the scalar multiplication is continuous. Let \( \lambda \) be any complex number. By (1.1), we have

\[ g(\lambda x) = \inf \left\{ \rho^{p_n/3} > 0 : \left\| \left( k^{-s} \left[ M \left( \frac{q_k(\lambda x_k)}{\rho} \right) \right]^{p_k} \right)^{1/3} \right\|_F \leq 1, \ n \in \mathbb{N} \} \]

Then

\[ g(\lambda x) = \inf \left\{ (|\lambda| r)^{p_n/3} > 0 : \left\| \left( k^{-s} \left[ M \left( \frac{q_k(x_k)}{r} \right) \right]^{p_k} \right)^{1/3} \right\|_F \leq 1, \ n \in \mathbb{N} \} \]

where \( r = \rho / |\lambda| \). Since \( |\lambda|^{p_n} \leq \max (1, |\lambda|^{\sup p_n}) \), we have

\[ g(\lambda x) = \max (1, |\lambda|^{\sup p_n})^{1/3} \]

\[ \cdot \inf \left\{ r^{p_n/3} > 0 : \left\| \left( k^{-s} \left[ M \left( \frac{q_k(x_k)}{r} \right) \right]^{p_k} \right)^{1/3} \right\|_F \leq 1, \ n \in \mathbb{N} \} \]

which converges to zero whenever \( x \) converges to zero in \( F(X_k, M, p, s) \).

Suppose that \( \lambda_n \to 0 \) and \( x \) is fixed in \( F(X_k, M, p, s) \). Then,

\[ t = (t_k) = \left( k^{-s} \left[ M \left( \frac{q_k(x_k)}{\rho} \right) \right]^{p_k} \right) \in F \]
for some $\rho > 0$. For arbitrary $\varepsilon > 0$, let $N$ be a positive integer such that

$$
\left\| t - \sum_{k=1}^{N} t_k e_k \right\|_F = \left\| \sum_{k=N+1}^{\infty} t_k e_k \right\|_F < \left( \frac{\varepsilon}{2} \right)^H,
$$

since $(e_k)$ is a Schauder basis for $F$. Let $0 < \|\lambda\| < 1$, using convexity of $M$ and absolutely monotonicity of $\|\cdot\|_F$ we get

$$
\left\| \sum_{k=N+1}^{\infty} k^{-s} \left[ M \left( \frac{q_k (\lambda x_k)}{\rho} \right) \right] e_k \right\|_F \leq \left\| \sum_{k=N+1}^{\infty} k^{-s} \left[ |\lambda| M \left( \frac{q_k (x_k)}{\rho} \right) \right] e_k \right\|_F < \left( \frac{\varepsilon}{2} \right)^H.
$$

Since $M$ is continuous everywhere in $[0, \infty)$, then

$$
f(u) =: \sum_{k=1}^{N} k^{-s} \left[ M \left( \frac{q_k (ux_k)}{\rho} \right) \right] e_k \right\|_F
$$

is continuous at 0. So there is $0 < \delta < 1$ such that $f(u) < (\varepsilon/2)^H$ for $0 < u < \delta$. Let $K$ be a positive integer such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$

$$
\left\| \sum_{k=1}^{N} k^{-s} \left[ M \left( \frac{q_k (\lambda_n x_k)}{\rho} \right) \right] e_k \right\|_F^{1/H} < \frac{\varepsilon}{2}.
$$

Thus

$$
\left\| \left( k^{-s} \left[ M \left( \frac{q_k (\lambda_n x_k)}{\rho} \right) \right] e_k \right) \right\|_F^{1/H} < \frac{\varepsilon}{2}
$$

for $n > K$, so that $g(\lambda x) \to 0$ as $\lambda \to 0$.

This completes the proof of Theorem 2.2. \(\square\)

**Remark 1** It can be easily verified that when $F = \ell_1$, $(X_k, q_k) = (C, |\cdot|)$, $p_k = 1$ for each $k \in \mathbb{N}$ and $s = 0$ the paranorms defined on $F(X_k, M, p, s)$ and $\ell_M(p)$ are the same, and also taking $q_k = \|\cdot\|_{X_k}$, $p_k = 1$ for each $k \in \mathbb{N}$ and $s = 0$ in (1.1), one obtains the norm of $F(X_k, M)$.

**Theorem 2.3** $F(X_k, M, p, s)$ is complete with the paranorm (1.1) if $X_k$ is complete under the seminorm $q_k$ for each $k \in \mathbb{N}$.

**Proof.** Let $(x^i)$ be any Cauchy sequence in $F(X_k, M, p, s)$. We get by (1.1) that

$$
\left\| \left( k^{-s} \left[ M \left( \frac{q_k (x^i_k - x^j_k)}{g(x^i - x^j)} \right) \right] e_k \right) \right\|_F^{1/H} \leq 1.
$$
Since $F$ is a normal space and $(e_k)$ is a Schauder basis of $F$, it follows that

$$
k^{-s} \left[ M \left( \frac{q_k(x_k^i - x_k^j)}{g(x^i - x^j)} \right) \right]^{p_k} \|e_k\|_F \leq \left\| k^{-s} \left[ M \left( \frac{q_k(x_k^i - x_k^j)}{g(x^i - x^j)} \right) \right]^{p_k} \right\|_F \leq 1.
$$

We choose $\gamma$ with $\gamma^H \|e_k\|_F > 1$ and $x_0 > 0$, such that $\gamma^H \|e_k\|_F \frac{x_0^H}{2} \left[ p \left( \frac{x_0}{2} \right) \right]^{p_k} \geq 1$, where $p$ is the kernel associated with $M$. Hence,

$$
k^{-s} \left[ M \left( \frac{q_k(x_k^i - x_k^j)}{g(x^i - x^j)} \right) \right]^{p_k} \|e_k\|_F \leq \gamma^H \|e_k\|_F \frac{x_0^H}{2} \left[ p \left( \frac{x_0}{2} \right) \right]^{p_k}
$$

for each $k \in \mathbb{N}$. Using the integral representation of Orlicz function $M$, we get

$$
k^{-s} \left[ q_k(x_k^i - x_k^j) \right]^{p_k} \leq \gamma^H x_0^H \left[ g(x^i - x^j) \right]^H.
$$

(2.1)

For given $\varepsilon > 0$, we choose an integer $i_0$ such that $g(x^i - x^j) < \frac{\varepsilon^{1/H}}{\gamma x_0}$ for all $i, j > i_0$.

(2.2)

From (2.1) and (2.2) we get

$$
k^{-s} \left[ q_k(x_k^i - x_k^j) \right]^{p_k} < \varepsilon \text{ for all } i, j > i_0.
$$

and so,

$$q_k(x_k^i - x_k^j) < \varepsilon \text{ for all } i, j > i_0.
$$

Hence, there exists a sequence $x = (x_k)$ such that $x_k \in X_k$ for each $k \in \mathbb{N}$ and $q_k(x_k^i - x_k) < \varepsilon$ as $i \to \infty$,

for each fixed $k \in \mathbb{N}$. For given $\varepsilon > 0$, choose an integer $n > 1$ such that $g(x^i - x^j) < \varepsilon/2$, for all $i, j > n$ and a $\rho > 0$, such that $g(x^i - x^j) < \rho < \varepsilon/2$. Since $F$ is a normal space and $(e_k)$ is a Schauder basis of $F$,

$$
\left\| \sum_{k=1}^{n} k^{-s} \left[ M \left( \frac{q_k(x_k^i - x_k^j)}{\rho} \right) \right]^{p_k} e_k \right\|_F \leq \left\| k^{-s} \left[ M \left( \frac{q_k(x_k^i - x_k^j)}{\rho} \right) \right]^{p_k} \right\|_F \leq 1.
$$

Since $M$ is continuous, so by taking $j \to \infty$ and $i, j > n$ in the above inequality we get

$$
\left\| \sum_{k=1}^{n} k^{-s} \left[ M \left( \frac{q_k(x_k^i - x_k)}{2\rho} \right) \right]^{p_k} e_k \right\|_F < 1.
$$
Letting $n \to \infty$, we get $g(x^i - x) < 2\rho < \varepsilon$ for all $i > n$. That is to say that $(x^i)$ converges to $x$ in the paranorm of $F(X_k, M, p, s)$. Now, we should show that $x \in F(X_k, M, p, s)$. Since $x^i = (x^i_k) \in F(X_k, M, p, s)$, there exists a $\rho > 0$ such that

$$\left(k^{-s} \left[M \left(\frac{q_k(x^i_k)}{\rho}\right)\right]^{p_k}\right) \in F.$$ 

Since $q_k(x^i_k - x_k) \to 0$ as $i \to \infty$, for each fixed $k$ we can choose a positive number $\delta_k^i$ satisfying $0 < \delta_k^i < 1$ such that

$$k^{-s} \left[M \left(\frac{q_k(x^i_k - x_k)}{\rho}\right)\right]^{p_k} < \delta_k^i k^{-s} \left[M \left(\frac{q_k(x^i_k)}{\rho}\right)\right]^{p_k}.$$ 

Consider

$$M \left(\frac{q_k(x^i_k)}{2\rho}\right) = M \left(\frac{q_k(x^i_k + x_k - x_k^i)}{2\rho}\right) \leq M \left(\frac{q_k(x^i_k)}{\rho}\right) + M \left(\frac{q_k(x_k - x_k^i)}{\rho}\right)$$

Hence,

$$k^{-s} \left[M \left(\frac{q_k(x^i_k)}{2\rho}\right)\right]^{p_k} \leq k^{-s} \left[M \left(\frac{q_k(x^i_k)}{\rho}\right) + M \left(\frac{q_k(x^i_k - x_k)}{\rho}\right)\right]^{p_k} \leq C k^{-s} \left\{ \left[M \left(\frac{q_k(x^i_k)}{\rho}\right)\right]^{p_k} + \left[M \left(\frac{q_k(x_k - x_k^i)}{\rho}\right)\right]^{p_k} \right\} \leq C (1 + \delta_k^i) k^{-s} \left[M \left(\frac{q_k(x^i_k)}{\rho}\right)\right]^{p_k}.$$ 

Since $F$ is normal,

$$\left(k^{-s} \left[M \left(\frac{q_k(x^i_k)}{2\rho}\right)\right]^{p_k}\right) \in F,$$

that is, $x = (x_k) \in F(X_k, M, p, s)$. This step completes the proof. \hfill \Box

**Theorem 2.4** Let $M$ and $M_1$ be two Orlicz functions. If $M$ satisfies the $\Delta_2$-condition, then

$$F(X_k, M_1, p, s) \subseteq F(X_k, M \circ M_1, p, s).$$

**Proof.** Let $x \in F(X_k, M_1, p, s)$. Then

$$\left(k^{-s} \left[M_1 \left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}\right) \in F$$

for some $\rho > 0$. Since $M$ satisfies the $\Delta_2$-condition, we have

$$k^{-s} \left[M \left(M_1 \left(\frac{q_k(x_k)}{\rho}\right)\right)\right]^{p_k} \leq k^{-s} \left[KM_1 \left(\frac{q_k(x_k)}{\rho}\right) M(1)\right]^{p_k} \leq \max \left(1, [KM(1)]^h\right) k^{-s} \left[M_1 \left(\frac{q_k(x_k)}{\rho}\right)\right]^{p_k}. $$
Thus, we obtain by the normality of $F$ that $x \in F(X_k, M \circ M_1, p, s)$. \hfill \Box

**Theorem 2.5** Let $M_1$ and $M_2$ be two Orlicz functions. Then the following inclusions are hold for non-negative real numbers $s_1, s_2, s$:

(i) $F(X_k, M_1, p, s) \cap F(X_k, M_2, p, s) \subseteq F(X_k, M_1 + M_2, p, s)$,

(ii) If $\limsup_{t \to \infty} M_1(t)/M_2(t) < \infty$, then $F(X_k, M_2, p, s) \subseteq F(X_k, M_1, p, s)$,

(iii) If $s_1 \leq s_2$, then $F(X_k, M_1, p, s_1) \subseteq F(X_k, M_1, p, s)$,

(iv) If $F_1 \subseteq F_2$, then $F_1 (X_k, M_1, p, s) \subseteq F_2 (X_k, M_1, p, s)$.

**Proof.** (i) Let $x \in F(X_k, M_1, p, s) \cap F(X_k, M_2, p, s)$. Then there exist some $\rho_1, \rho_2 > 0$ such that

$$
\left( k^{-s} \left[ M_1 \left( \frac{q_k(x_k)}{\rho_1} \right) \right]^{p_k} \right), \left( k^{-s} \left[ M_2 \left( \frac{q_k(x_k)}{\rho_2} \right) \right]^{p_k} \right) \in F.
$$

Letting $\rho = \max(\rho_1, \rho_2)$, we get

$$
k^{-s} \left[ (M_1 + M_2) \left( \frac{q_k(x_k)}{\rho} \right) \right]^{p_k} \leq k^{-s} \left[ M_1 \left( \frac{q_k(x_k)}{\rho_1} \right) + M_2 \left( \frac{q_k(x_k)}{\rho_2} \right) \right]^{p_k}
\leq C \left\{ k^{-s} \left[ M_1 \left( \frac{q_k(x_k)}{\rho_1} \right) \right]^{p_k} + k^{-s} \left[ M_2 \left( \frac{q_k(x_k)}{\rho_2} \right) \right]^{p_k} \right\}.
$$

Since $F$ is a normal space, $x \in F(X_k, M_1 + M_2, p, s)$.

(ii) We can find $K > 0$ such that $M_1(t)/M_2(t) \leq K$ for all $t \geq 0$, since $\limsup_{t \to \infty} M_1(t)/M_2(t) < \infty$. Let $x \in F(X_k, M_2, p, s)$. There exists a $\rho > 0$ such that

$$
\frac{M_1 \left( \frac{q_k(x_k)}{\rho} \right)}{M_2 \left( \frac{q_k(x_k)}{\rho} \right)} \leq K.
$$

Hence

$$
k^{-s} \left[ M_1 \left( \frac{q_k(x_k)}{\rho} \right) \right]^{p_k} \leq \max (1, K^h) k^{-s} \left[ M_2 \left( \frac{q_k(x_k)}{\rho} \right) \right]^{p_k}.
$$

Since $F$ is normal, $x \in F(X_k, M_1, p, s)$.

The proofs of the cases (iii) and (iv) are trivial. \hfill \Box

**Corollary 1** We have

(i) $F(X_k, p, s) \subseteq F(X_k, M, p, s)$ for any Orlicz function $M$ satisfying the $\Delta_2$-condition,

(ii) $F(X_k, M, p) \subseteq F(X_k, M, p, s)$ for any Orlicz function $M$. 
3 A Closed Subspace of $F(X_k, M, p, s)$

We define $[F(X_k, M, p, s)]$ by

$$[F(X_k, M, p, s)] = \left\{ x = (x_k) : x_k \in X_k \text{ for each } k \in \mathbb{N} \text{ and } \left( k^{-s} \left\{ M \left( \frac{q_k(x_k)}{\rho} \right) \right\}^{p_k} \right) \in F \text{ for every } \rho > 0 \right\}.$$ 

The space $[F(X_k, M, p, s)]$ is clearly a subspace of $F(X_k, M, p, s)$, and its topology is introduced by the paranorm of $F(X_k, M, p, s)$ given by (1.1).

**Theorem 3.1** $[F(X_k, M, p, s)]$ is a complete paranormed space with the paranorm given by (1.1) if $(X_k, q_k)$ is complete seminormed space for each $k \in \mathbb{N}$.

**Proof.** Since $F(X_k, M, p, s)$ is just shown that a complete paranormed space under the paranorm (1.1) and $[F(X_k, M, p, s)]$ is a subspace of $F(X_k, M, p, s)$, it is sufficient to show that it is closed. For this let us consider $(x^i) = (x^i_k) \in [F(X_k, M, p, s)]$ such that $g(x^i - x) \to 0$ as $i \to \infty$, where $x = (x_k) \in F(X_k, M, p, s)$.

So for given $\xi > 0$, we can choose an integer $i_0$ such that

$$g(x^i - x) < \frac{\xi}{2}, \forall i > i_0.$$ 

Consider

$$k^{-s} \left\{ M \left( \frac{q_k(x_k)}{\xi} \right) \right\}^{p_k} \leq k^{-s} \left\{ \left( \frac{1}{2} M \left( \frac{q_k(x^i_k - x_k)}{\xi/2} \right) \right) + \frac{1}{2} M \left( \frac{q_k(x^i_k)}{\xi/2} \right) \right\}^{p_k} \leq Ck^{-s} \left\{ \left( M \left( \frac{q_k(x^i_k - x_k)}{g(x^i - x)} \right) \right)^{p_k} + \left( M \left( \frac{q_k(x^i_k)}{g(x^i - x)} \right) \right)^{p_k} \right\}.$$ 

Since

$$k^{-s} \left\{ M \left( \frac{q_k(x^i_k - x_k)}{g(x^i - x)} \right) \right\}^{p_k}, k^{-s} \left\{ M \left( \frac{q_k(x^i_k)}{g(x^i - x)} \right) \right\}^{p_k} \in F$$

and $F$ is normal space,

$$k^{-s} \left\{ M \left( \frac{q_k(x_k)}{\xi} \right) \right\}^{p_k} \in F.$$ 

This implies $x = (x_k) \in [F(X_k, M, p, s)]$ which shows that $[F(X_k, M, p, s)]$ is complete.

**Proposition 1** $[F(X_k, M, p, s)]$ is an AK-space.

**Proof.** Let $x = (x_k) \in [F(X_k, M, p, s)]$. Therefore,

$$k^{-s} \left\{ M \left( \frac{q_k(x_k)}{\rho} \right) \right\}^{p_k} \in F.$$
for every $\rho > 0$. Since $(e_k)$ is a Schauder basis of $F$, for a given $\varepsilon \in (0, 1)$, we can find an arbitrary positive integer $m_0$ such that

$$
\left\| \sum_{k=m_0}^{\infty} k^{-s} \left[ M \left( \frac{q_k(x_k)}{\varepsilon} \right) \right]^{p_k} e_k \right\|_F < 1.
$$

(3.1)

Using the definition of the paranorm, we have

$$
g(x - x^{[m]}) = \inf \left\{ \xi^{p_n/H} > 0: \left\| \sum_{k=m+1}^{\infty} k^{-s} \left[ M \left( \frac{q_k(x_k)}{\xi} \right) \right]^{p_k} e_k \right\|_F^{1/H} \leq 1, n \in \mathbb{N} \right\},
$$

where $x^{[m]}$ denotes the $m$-th section of $x$. From this equality and (3.1), it is obvious that $g(x - x^{[m]}) < \varepsilon$ for all $m > m_0$.

Therefore $[F(X_k, M, p, s)]$ is an AK-space.

**Theorem 3.2** Let $(x^i) = (x^i_k)$ be a sequence of the elements of $[F(X_k, M, p, s)]$ and $x = (x_k) \in [F(X_k, M, p, s)]$. Then $x^i \rightarrow x$ in $[F(X_k, M, p, s)]$ iff

(i) $x^i_k \rightarrow x_k$ in $X_k$ for each $k \geq 1$,

(ii) $g(x^i) \rightarrow g(x)$ as $i \rightarrow \infty$.

**Proof.** The necessity part is obvious.

Sufficiency. Suppose that (i) and (ii) hold, and let $m$ be an arbitrary positive integer. Then

$$
g(x^i - x) \leq g(x^i - x^{[m]}) + g(x^{[m]} - x^{[m]}) + g(x^{[m]} - x),
$$

where $x^{i[m]}$, $x^{[m]}$ denote the $m$-th sections of $x^i$ and $x$, respectively. Letting $i \rightarrow \infty$, we get

$$
\limsup_{i \rightarrow \infty} g(x^i - x) \leq \limsup_{i \rightarrow \infty} g(x^i - x^{[m]}) + \limsup_{i \rightarrow \infty} g(x^{[m]} - x^{[m]}) + g(x^{[m]} - x)
\leq 2g(x^{[m]} - x).
$$

Since $m$ is arbitrary, letting $m \rightarrow \infty$, we get $\limsup_{i \rightarrow \infty} g(x^i - x) = 0$, i.e. $g(x^i - x) \rightarrow 0$ as $i \rightarrow \infty$.

**Theorem 3.3** $[F(X_k, M, p, s)]$ is separable if for each $k \in \mathbb{N}$, $X_k$ is.

**Proof.** Suppose $X_k$ is separable for each $k \in \mathbb{N}$. Then, there exists a countable dense subset $U_k$ of $X_k$. Let $Z$ denote the set of finite sequences $z = (z_k)$ where $z_k \in U_k$ for each $k \in \mathbb{N}$ and

$$
(z_k) = (z_1, z_2, \ldots, z_m, \theta_{m+1}, \theta_{m+2}, \ldots)
$$
for arbitrary \( m \in \mathbb{N} \). Obviously, \( Z \) is a countable subset of \([F(X_k, M, p, s)]\). We shall prove that \( Z \) is dense in \([F(X_k, M, p, s)]\). Let \( x \in [F(X_k, M, p, s)] \). Since \([F(X_k, M, p, s)]\) is an AK-space, \( g \left( x - x^m \right) \to 0 \) as \( m \to \infty \). So for a given \( \varepsilon > 0 \), there exists an integer \( m_1 > 1 \) such that
\[
g \left( x - x^m \right) < \varepsilon/2 \text{ for all } m \geq m_1.
\]
If we take \( m = m_1 \), then
\[
g \left( x - x^{m_1} \right) < \varepsilon/2.
\]
Let us choose \( y = (y_k) = (y_1, y_2, \ldots, y_{m_1}, \theta_{m_1+1}, \theta_{m_1+2}, \ldots) \in Z \) such that
\[
q_k \left( x^{m_1}_k - y_k \right) < \frac{\varepsilon}{2M(1)m_1 \|e_k\|_F} \text{ for each } k \in \mathbb{N}.
\]
Now
\[
g \left( x - y \right) = g \left( x - x^{m_1} + x^{m_1} - y \right) \\
\leq g \left( x - x^{m_1} \right) + g \left( x^{m_1} - y \right) < \varepsilon.
\]
This implies that \( Z \) is dense in \([F(X_k, M, p, s)]\). Hence \([F(X_k, M, p, s)]\) is separable.

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**References**


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M. K. Özdemir and İ. Solak
Department of Mathematics
Inonu University
44280 Malatya, Turkey.
e-mail : kozdemir@inonu.edu.tr