Some Inequalities for the $q$-Gamma and the $q$-Polygamma Functions

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Abstract: In this paper, the complete monotonicity property for functions related to the $q$-gamma and the $q$-polygamma functions, where $q$ is a positive real number, is proved and exploited to establish some inequalities for the $q$-gamma and the $q$-polygamma functions.

Keywords: inequalities; $q$-gamma function; $q$-polygamma functions; completely monotonic function.

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1 Introduction

Batir [1] presented a sharp double inequality

$$x^e^{-x} \sqrt{2\pi(x+a)} < \Gamma(x+1) < x^e^{-x} \sqrt{2\pi(x+b)}$$ (1.1)

for the gamma function. He proved that the inequality (1.1) is valid for $x > 1$ with the best possible constant $a = \frac{1}{6}$ and $b = \frac{3}{2\pi} - 1$. His proof depended on strictly decreasing monotone of the function

$$g(x) = \frac{(\Gamma(x+1))^2}{2\pi x^2 e^{-2x}} - x, \quad x > 1$$ (1.2)

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and the inequality
\[
\sqrt{\pi}x^re^{-x}(8x^3 + 4x^2 + x + \frac{1}{100})^\frac{1}{r} < \Gamma(x + 1)
< \sqrt{\pi}x^re^{-x}(8x^3 + 4x^2 + x + \frac{1}{30})^\frac{1}{r}.
\]  
(1.3)

One of the important aims of this paper is to extend the double inequality (1.1) to the $q$-gamma function for all positive real numbers $x$ and $q$ under slightly different conditions by means of studying the complete monotonicity property for the function

\[
F_a(x; q) = \log \Gamma_q(x + 1) - x \log[x]_q - \frac{\text{Li}_2(1 - q^x)}{\log q} - \hat{C}_q - \frac{1}{2}(1 - a)H(q - 1) \log q - \frac{1}{2}\log[x + a]_q.
\]
(1.4)

where $H(\cdot)$ denotes the Heaviside step function, $[x]_q = (1 - q^x)/(1 - q)$, $\text{Li}_2(z)$ is the dilogarithm function defined for complex argument $z$ as

\[
\text{Li}_2(z) = - \int_0^z \frac{\log(1 - t)}{t} dt, \quad z \notin (1, \infty)
\]
(1.5)

$\Gamma_q(x)$ is the $q$-gamma function defined for all positive real variable $x$ as

\[
\Gamma_q(x) = (1 - q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}}, \quad 0 < q < 1,
\]
(1.6)

\[
= (q - 1)^{1-x} q^{\frac{x(x-1)}{2}} \prod_{n=0}^{\infty} \frac{1 - q^{-(n+1)}}{1 - q^{-(n+x)}}, \quad q > 1,
\]
(1.7)

\[
\hat{q} = \begin{cases} 
q & \text{if } 0 < q \leq 1 \\
q^{-1} & \text{if } q \geq 1
\end{cases}
\]

and

\[
\hat{C}_q = \frac{1}{2} \log(2\pi) + \frac{1}{2} \log \left( \frac{q - 1}{\log q} \right) - \frac{1}{24} \log q + \log \left( \sum_{m=-\infty}^{\infty} \frac{r^{m(6m+1)} - r^{(2m+1)(3m+1)}}{r^{m(6m+1)}} \right)
\]
(1.8)

where $r = \exp(4\pi^2/\log q)$.

From the previous definitions, for a positive $x$ and $q \geq 1$, we get

\[
\Gamma_q(x) = q^{\frac{x-1}{x-2}} \Gamma_{q^{-1}}(x).
\]
(1.9)

Many of the classical facts about the ordinary gamma function have been extended to the $q$-gamma function (see [3,6] and the references given therein). An
important fact for gamma function in applied mathematics as well as in probability is the Stirling formula that gives a pretty accurate idea about the size of gamma function. With the Euler-Maclaurin formula, Moak [5] obtained the following $q$-analogue of Stirling formula (see also [7])

$$
\log \Gamma_q(x) \sim \left(x - \frac{1}{2}\right) \log[x]_q + \frac{\text{Li}_2(1 - q^x)}{\log q} + \frac{1}{2} H(q - 1) \log q + C_q \\
+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{\log \hat{q}}{\hat{q}^x - 1}\right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x), \quad x \to \infty
$$

(1.10)

where $B_k, k \in \mathbb{N}$ are the Bernoulli numbers and $P_k$ is a polynomial of degree $k$ satisfying

$$P_k(z) = (z - z^2)P'_{k-1}(z) + (kz + 1)P_{k-1}(z), \quad P_0 = P_{-1} = 1, \quad k \in \mathbb{N}. \quad (1.11)$$

It is easy to see that

$$\lim_{q \to 1} C_q = C_1 = \frac{1}{2} \log(2\pi), \quad \lim_{q \to 1} \frac{\text{Li}_2(1 - q^x)}{\log q} = -x \quad \text{and} \quad P_k(1) = (k + 1)! \quad (1.12)$$

and so (1.4) when letting $q \to 1$, tends to the ordinary Stirling formula [2]

$$
\log \Gamma(x) \sim \left(x - \frac{1}{2}\right) \log x - x + \frac{1}{2} \log(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k - 1)} \frac{1}{x^{2k-1}}, \quad x \to \infty.
$$

(1.13)

An important function related to $q$-gamma function is the so-called $q$-digamma function $\psi_q(x)$ which defined as the logarithmic derivative of the $q$-gamma function

$$\psi_q(x) = \frac{d}{dx}(\log \Gamma_q(x)) = \frac{\Gamma'_q(x)}{\Gamma_q(x)} \quad (1.14)$$

The $q$-digamma function $\psi_q(x)$ appeared in the work of Krattenthaler and Srivastava [8], when they studied the summations for basic hypergeometric series. Some of its properties have been presented and proved in their work. Also, in their work, they proved that $\psi_q(x)$ tends to the digamma function $\psi(x)$ when letting $q \to 1$. Some inequalities involve the $q$-gamma function and some of its related functions ($q$-beta, $q$-digamma and $q$-polygamma functions) have been introduced in [9–21]. For more details on the $q$-digamma function (see [22] and the references given therein).

For all positive real variable $x$, (1.6) gives

$$
\psi_q(x) = -\log(1 - q) + \log q \sum_{k=1}^{\infty} \frac{q^{xk}}{1 - q^k}, \quad 0 < q < 1 \quad (1.15)
$$
and (1.7) gives
\[ \psi_q(x) = -\log(q - 1) + \log q \left( x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-xk}}{1 - q^{-k}} \right), \quad q > 1. \] (1.16)

An important recursive formula that we need, obtained in [11] as
\[ \psi_q(x + 1) = \psi_q(x) - \frac{q^x \log q}{1 - q^x}, \quad q > 0; \quad x > 0. \] (1.17)

2 The Complete Monotone Functions

In this section, the complete monotonicity property for the function \( F_a(x; q) \) mentioned in (1.4) is proved by means of studying the complete monotonicity of its derivative with respect to \( x \). As a consequence of these results, some inequalities for the \( q \)-gamma and the \( q \)-polygamma functions are established.

**Theorem 2.1.** Let \( x \) and \( q \) be positive real. Then, the function
\[ G_a(x; q) = \psi_q(x + 1) - \log[x]q + \frac{1}{2} \frac{q^{x+a} \log q}{1 - q^{x+a}} \] (2.1)
is strictly completely monotonic on \((0, \infty)\) if \( a \geq \sqrt{259 - 7} \approx 0.216511 \ldots \); and the function \(-G_b(x; q)\) is strictly completely monotonic on \((-b, \infty)\) if \( b \leq 0 \).

**Proof.** When \( 0 < q < 1 \), (1.15), Taylor series for logarithm function and binomial theorem give
\[ G_a(x; q) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{q^{xk}}{k(1 - q^k)} f(a, y), \quad y = q^k \]
where
\[ f(a, y) = 2y \log y + 2(1 - y) + y^a (1 - y) \log y. \]

It is obvious that the function \( a \mapsto f(a, y) \) is increasing on \( \mathbb{R} \) for all \( 0 < y < 1 \). When \( a = 0 \), the function \( f(a, y) \) can be rewritten after simple calculations in the form
\[ f(0, y) = -y \sum_{n=3}^{\infty} \frac{\log^n(1/y)}{n!} (n - 2) < 0 \]
which means that \( f(a, y) < 0 \) if \( a \leq 0 \) for all \( 0 < y < 1 \).

When \( a > 0 \), the function \( f(a, y) \) can be rewritten in the form
\[ f(a, y) = 2y^{a+1} \log y e^{a \log(1/y)} + 2y^{a+1} e^{a \log(1/y)} (e^{a \log(1/y)} - 1) + y^{a+1} \log y (e^{a \log(1/y)} - 1). \]
Using the series expansion of the exponential function and Cauchy product rule would yield

\[
   f(a, y) = y^{a+1} \sum_{n=3}^{\infty} \frac{\log^n(1/y)}{n!} \left( 2 \sum_{r=0}^{n-1} \binom{n}{r} a^r - 2na^{n-1} - n \right)
\]

\[
   = y^{a+1} \sum_{n=3}^{\infty} \frac{\log^n(1/y)}{n!} \left( 2 \sum_{r=0}^{n-2} \binom{n}{r} a^r - n \right)
\]

\[
   = y^{a+1} \sum_{n=3}^{\infty} \frac{\log^n(1/y)}{n!} g_n(a).
\]

It is obvious that \( g_n(a) \) is a polynomial of \( a \) with degree \( n - 2 \) and all its roots depend on \( n \). According to Descartes’ rules of sign, the polynomial \( g_n(a) \) has only one positive root, say \( a(n) \), depends on \( n \) for all \( n \geq 3 \). It is very difficult to determine this root due to that \( n \) has infinite values start with \( n = 3 \) and thus we suffice to identify a suitable upper bound for \( a(n) \) to be close as much as possible to the highest root of \( g_n(a) \). Let us now rewrite \( g_n(a) \) to be

\[
   g_n(a) = 2 \sum_{r=3}^{n-2} \binom{n}{r} a^r + n(n-1)a^2 + 2na + 2 - n.
\]

It is clear that the quadratic polynomial \( n(n-1)a^2 + 2na + 2 - n \) has only one positive root depends on \( n \) at

\[
   a(n) = \frac{-n + \sqrt{n^4 - 2n^2 + 2n}}{n(n-1)}, \quad \text{for all} \quad n \geq 3.
\]

Therefore, the polynomial \( g_n(a) \) is greater than zero if \( a \geq a(n) \) for all \( n \geq 3 \) and so is the function \( f(a, y) \). By differentiating \( a(n) \), we get \( a'(n) > 0 \) if \( 3 \leq n \leq 6 \) and \( a'(n) < 0 \) if \( n \geq 7 \) and thus \( a(n) \) is decreasing for all \( n \geq 7 \) which reveals that

\[
   a(n) \leq a(7) = \frac{\sqrt{259} - 7}{42} \approx 0.216511... = \alpha, \quad \text{for all} \quad n \geq 7.
\]

Since \( a(n) \leq a(6) = (2\sqrt{39} - 6)/30 \approx 0.216333... < \alpha \) for all \( 3 \leq n \leq 6 \), then \( a(n) \leq \alpha \) for all \( n \geq 3 \) which leads to \( g_n(a) > 0 \) for all \( a > \alpha \) with \( n \geq 3 \) and so is the function \( f(a, y) \). In view of the previous results, we conclude that \( G_a(x; q) < 0 \) if \( a \leq 0 \) and \( G_a(x; q) > 0 \) if \( a > \alpha \).

When \( q \geq 1 \), [11.9] and [2.1] give

\[
   G_a(x; q) = \psi_{q^{-1}}(x + 1) - \log[x]_{q^{-1}} + \frac{1}{2} \frac{q^{-(x+a)} \log q^{-1}}{1 - q^{-(x+a)}} = G_a(x; q^{-1}).
\]

This completes the proof.
Corollary 2.2. Let \( x \) and \( q \) be positive real with \( x > \{0, -b\} \). Then, the double inequalities
\[
\log[x]_q + \frac{q^x \log q}{1 - q^x} - \frac{1}{2} \frac{q^{x+a} \log q}{1 - q^{x+a}} < \psi_q(x) < \log[x]_q + \frac{q^x \log q}{1 - q^x} - \frac{1}{2} \frac{q^{x+b} \log q}{1 - q^{x+b}}
\]
hold true for all \( a \geq \sqrt{\frac{259 - 7}{42}} \simeq 0.216511... \) and \( b \leq 0 \).
Moreover, for all positive integer \( n \), the class of inequalities
\[
(-1)^n \left( \frac{\log q}{1 - q^x} \right)^{n+1} q^x P_{n-1}(q^x) - (-1)^n \left( \frac{\log q}{1 - q^x} \right)^n q^x P_{n-2}(q^x)
\]
\[
- \frac{1}{2} \left( \frac{\log q}{1 - q^x} \right)^{n+1} q^{x+a} P_{n-1}(q^{x+a}) < (-1)^n \psi_q^{(n)}(x)
\]
\[
- (-1)^n \frac{1}{2} \left( \frac{\log q}{1 - q^x} \right)^{n+1} q^{x+b} P_{n-1}(q^{x+b})
\]
is valid for all \( a \geq \sqrt{\frac{259 - 7}{42}} \simeq 0.216511... \) and \( b \leq 0 \), where \( P_n \) is the polynomial mentioned in Section 1.

Proof. Theorem 2.1 tells that \( G_b(x, q) < 0 < G_a(x, q) \) which is equivalent to (2.2) with using the identity (1.17), and
\[
(-1)^n C_b^{(n)}(x, q) < 0 < (-1)^n C_a^{(n)}(x, q), \quad n \in \mathbb{N}
\]
which is equivalent to (2.3) with using the identities (1.17) and
\[
\frac{d^n}{dx^n} \left[ \frac{q^x \log q}{1 - q^x} \right] = \left( \frac{\log q}{1 - q^x} \right)^{n+1} q^x P_{n-1}(q^x), \quad n = 0, 1, 2, \cdots
\]
which was proved by Moak [5]. \( \square \)

Remark 2.3. When letting \( q \to 1 \), we have the two sided-inequality
\[
\log x - \frac{1}{x} + \frac{1}{2(x + a)} < \psi(x) < \log x - \frac{1}{2x}, \quad x > 0
\]
holds for \( a = \sqrt{\frac{259 - 7}{42}} \simeq 0.216511... \). The left hand side refines the inequality
\[
\log x - \frac{1}{x} < \psi(x) < \log x - \frac{1}{2x}
\]
which obtained by Anderson and Qiu [23] for all \( x > 0 \).
Theorem 2.4. Let \( x \) and \( q \) be positive real. Then, the function \( F_a(x; q) \) defined as in (1.4) is strictly completely monotonic on \((-a, \infty)\) if \( a \leq 0 \) and is strictly increasing on \((0, \infty)\) if \( a \geq \sqrt{\frac{259}{42}} - \frac{7}{42} \simeq 0.216511...\).

Proof. The function \( F_a(x; q) \) defined as in (1.4) can be read as

\[
F_a(x; q) = \mu_q(x) + \nu_a(x; q)
\]

where

\[
\mu_q(x) = \log \Gamma_q(x) - \left(x - \frac{1}{2}\right) \log[x]_q - \frac{\text{Li}_2(1 - q^x)}{\log q} - C_q - \frac{1}{2} H(q - 1) \log q
\]

and

\[
\nu_a(x; q) = \frac{1}{2} \left(\log(1 - q^x) - \log(1 - q^{x+a}) + aH(q - 1) \log q\right).
\]

Using Moak formula (1.10) yields \( \lim_{x \to \infty} \mu_q(x) = 0 \) for all \( q > 0 \). Obviously, \( \lim_{x \to \infty} \nu_a(x; q) = 0 \) if \( 0 < q < 1 \) and when \( q > 1 \), we get

\[
\lim_{x \to \infty} \nu_a(x; q) = \frac{1}{2} \left(x \log q + \log(1 - q^{-x}) - (x+a) \log q - \log(1 - q^{-(x+a)}) + a \log q\right) = 0.
\]

These conclude that \( \lim_{x \to \infty} F_a(x; q) = 0 \) for all \( q > 0 \) and consequently from Theorem 2.1 we obtain the proof of this theorem.

Corollary 2.5. Let \( x \) and \( q \) be positive real. Then, the \( q \)-gamma function holds the two-sided inequalities

\[
\left|x\right| q^{\frac{1}{2}(1-b)H(q-1)} S_q \sqrt{2\pi [x+b]_q \exp\left(\frac{\text{Li}_2(1-q^x)}{\log q}\right)} < \Gamma_q(x+1)
\]

\[
< \left|x\right| q^{\frac{1}{2}(1-a)H(q-1)} S_q \sqrt{2\pi [x+a]_q \exp\left(\frac{\text{Li}_2(1-q^x)}{\log q}\right)}, \quad x > \max\{0, -b\}
\]

and the one-sided inequalities

\[
\Gamma_q(x+1) \geq \left|x\right|^\frac{1}{2} \sqrt{\frac{1 - q^{x+a}}{1 - q^a}} \exp\left(\frac{\text{Li}_2(1-q^x)}{\log q}\right), \quad x > 0
\]

for all \( a \geq \sqrt{\frac{259}{42}} - \frac{7}{42} \simeq 0.216511... \) and \( b \leq 0 \) with the best possible constants \( a = \sqrt{\frac{259}{42}} \) and \( b = 0 \), where

\[
S_q = q^{\frac{1}{2}\pi} \sqrt{\log q} \sum_{m=-\infty}^{\infty} \left(p^m(6m+1) - r^{(2m+1)(3m+1)}\right).
\]
Proof. The monotonicity properties of the function $F_a(x; q)$ in Theorem 2.4 give $F_a(x; q) < 0 < F_b(x; q)$ which is equivalent (2.9) and

$$F_a(x; q) > F_a(0; q) = -C_q - \frac{1}{2} H(q - 1) \log q - \frac{1}{2} \log[a]_q$$

which is equivalent (2.10).

Remark 2.6. When letting $q \to 1$, (2.9) tends to

$$x^x e^{-x} \sqrt{2\pi(x + b)} < \Gamma(x + 1) < x^x e^{-x} \sqrt{2\pi(x + a)} \quad (2.11)$$

which is valid for all $x > 0$ with the best possible constants $a = \sqrt{\frac{259 - 7}{42}}$ and $b = 0$. Although the values of the constants $a, b$ in (1.1) are better than here but we extend the values of $x$ to start with zero. Also, when letting $q \to 1$, (2.10) tends to

$$\Gamma(x + 1) \geq x^x e^{-x} \sqrt{\frac{x + a}{a}}, \quad x > 0 \quad (2.12)$$

with the best possible constants $a = \sqrt{\frac{259 - 7}{42}}$. The inequality (2.12) for the gamma function appears to be a new.

3 Conclusion

In this paper, the Moak formula (1.10) is used to prove the complete monotonicity property for the function $F_a(x; q)$ for all positive real $q$. The inequalities (2.2), (2.3), (2.9) and (2.10) come as an application of the complete monotonicity property for the function $F_a(x; q)$ and $G_a(x; q)$. When letting $q \to 1$, these inequalities give the inequalities (2.4), (2.5), (2.11) and (2.12). Some of them are considered refinement of existing inequalities and (2.12) is shown to be new.

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References


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