On the Diophantine Equation $4^x - p^y = 3z^2$
where $p$ is a Prime

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Abstract: We find all solutions to $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$ to complement the results found by Suvarnamani, et. al. in [1]. We also consider the two Diophantine equations $4^x - 7^y = 3z^2$ and $4^x - 19^y = 3z^2$ and show that these two equations have exactly two solutions $(x, y, z)$ in non-negative integers, i.e. $(x, y, z) \in \{(0, 0, 0), (1, 0, 1)\}$. In fact, the Diophantine equation $4^x - p^y = 3z^2$ has the two solutions $(0, 0, 0)$ and $(1, 0, 1)$ under some additional assumption on $p$. These results were all obtained using elementary methods and Mihăilescu’s Theorem. Finally, we end our paper with an open problem.

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1 Introduction

Recently, there have been an increasing interest in finding solutions to exponential Diophantine equations of the form $p^x + q^y = z^2$, see e.g. [2][11], and the references therein.

In [1], A. Suvarnamani, A. Singta, and S. Chotchaisthit showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solution in the set of non-negative integers. In fact, the Diophantine equation $4^x - 11^y = z^2$ also contain no solution in the set of non-negative integers (this set we denote by $\mathbb{N}_0$ throughout the paper) except possibly when $x = y = z = 0$, and the Diophantine
equation \(4^x - 7^y = z^2\) holds true in \(\mathbb{N}_0\) for \((x, y, z) = (0, 0, 0)\) and \((2, 1, 3)\) only.

The results found in [1] were obtained using Mihăilescu’s Theorem: \(a^x - b^y = 1\) has the unique solution \((a, b, x, y) = (3, 2, 2, 3)\) in positive integers for \(\min(a, b, x, y) > 1\). This remarkable result was first conjectured by E. Catalan in a one page note dated 1844 (see [12]) and was finally proven by P. Mihăilescu in 2002 (see [13]). B. Peker and S. I. Çenberci generalized the results found in [1] by considering the Diophantine equation \((4^n)^x + p^y = z^2\) where \(p\) is an odd prime, \(n \in \mathbb{N}\), and \(x, y, \) and \(z \in \mathbb{N}_0\) in [14]. On the other hand, in [15], the author and J. B. Bacani obtained all solutions to the Diophantine equation \(p^x + q^y = z^2\) where \(p\) and \(q\) are twin primes under some additional assumptions on \(p\) and \(q\). The paper [15] gives a correct set of solutions to \(p^x + q^y = z^2\) (under some assumptions on \(p\) and \(q\)) in contrary to the main result presented in [16].

Another type of Diophantine equations of great interest are those of the form \(a^x \pm b^y \pm c^z = w^n\). In [17] and [18], the authors studied exponential Diophantine equations of the form \(p^x \pm q^y \pm r^z = c\) where \(p, q, r\) are primes, \(x, y, \) and \(z \in \mathbb{N}_0\), and \(c\) an integer have been studied. Particularly, J. Leitner [17] solved the equation \(3^a + 5^b - 7^c = 1\) for \(a, b, c \in \mathbb{N}_0\) and the equation \(y^2 = 3^a + 2^b + 1\) for \(a, b \in \mathbb{N}_0\) and integer \(y\). R. Scott and R. Styer [18] studied, among other things, the Diophantine equation \(p^x \pm q^y \pm 2^z = 0\) for primes \(p\) and \(q\) and positive integers \(x, y, \) and \(z\). These authors used elementary methods to show that, with a few explicitly listed exceptions, there are at most two solutions \((x, y)\) to \(|p^x \pm q^y| = c\) (where \(c\) is a fixed positive integer) and at most two solutions \((x, y, z)\) to \(p^x \pm q^y \pm 2^z = 0\) in positive integers.

In an earlier paper, the author along with Bacani gave all solutions to the Diophantine equation \(3^x + 5^y + 7^z = w^2\) in response to an open problem posed by B. Sroysang in [19].

In this note, we verify our claim that \(4^x - 7^y = z^2\) contains no solution in \(\mathbb{N}_0\) except possibly when \(x = y = z = 0\), and the Diophantine equation \(4^x - 7^y = z^2\) has exactly two solutions \((x, y, z)\) in non-negative integers, i.e. \((x, y, z) \in \{(0, 0, 0), (2, 1, 3)\}\). We remark that our approach in proving these two claims can be applied to prove a general case of the problem. Also, we show that the two Diophantine equations \(4^x - 7^y = 3z^2\) and \(4^x - 19^y = 3z^2\) have exactly two solutions \((x, y, z)\) in \(\mathbb{N}_0\), i.e. \((x, y, z) \in \{(0, 0, 0), (1, 0, 1)\}\). Finally, we state and prove a generalization of these two previous results at the end of our paper. Most precisely, we show that \(4^x - p^y = 3z^2\) has the two solutions \((x, y, z) = (0, 0, 0)\) and \((1, 0, 1)\) in \(\mathbb{N}_0\) for prime \(p \equiv 3 \pmod{4}\).

2 Preliminaries

In this section we state some helpful results to prove our claims. First, it is known that the equation \(X^2 - dY^2 = 1\) has a solution in positive integers \(X\) and \(Y\) for all positive, nonsquare integers \(d\) (see e.g. [20] Theorem 1, pg. 9), and that if \(k\) is a perfect square, then the Pell Equation \(X^2 - dY^2 = k\) is solvable in integers for all positive, nonsquare integers \(d\) (cf. [20] Theorem 6, pg. 16)). In relation to
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Pell’s equation the following lemma was proved by K. Matthews in [21 Section 3].

**Lemma 2.1.** Let $N \geq 1$ be an odd integer, $D > 1$ and not a perfect square. Then, a necessary condition for the solvability of the equation $x^2 - Dy^2 = N$ with $\gcd(x, y) = 1$ is that the congruence $u^2 \equiv D \pmod{N}$ shall be soluble.

The following results shall be used to show that the title equation has no solution in positive integers $x, y, z$ for prime $p \equiv 3 \pmod{4}$.

**Theorem 2.2** ([22], Theorem 2.9, pg. 32). Let $p \equiv 3 \pmod{4}$ and $k = m^2n$ with $n$ square free. If $X^2 - pY^2 = k$ is solvable, then $n \equiv 1 \pmod{4}$.

**Corollary 2.3** ([22], Corollary 2.10, pg. 33). Let $k = m^2n$ with $n$ square free. If $p \equiv k \equiv 3 \pmod{4}$, then $X^2 - pY^2 = kl$ is not solvable.

**Corollary 2.4** ([22], Corollary 2.11, pg. 33). If $p \equiv k \equiv 3 \pmod{4}$ and $l \equiv 1 \pmod{4}$, then $X^2 - pY^2 = kl$ is not solvable.

Now we prove our results in the following section.

## 3 Main Results

We first consider the two Diophantine equations $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$ and later in this section we study the equations $4^x - 7^y = 3z^2$ and $4^x - 19^y = 3z^2$.

**Theorem 3.1.** The Diophantine equation $4^x - 7^y = z^2$ has exactly two solutions $(x, y, z)$ in $\mathbb{N}_0$, namely (the trivial solution) $(0, 0, 0)$ and $(2, 1, 3)$.

**Proof.** Evidently, the case when $z = 0$ will give us $(x, y, z) = (0, 0, 0)$, so we may assume that $z > 0$. For $z > 0$, we consider three cases.

**Case 1.** $x = 0$. This case is trivial.

**Case 2.** $y = 0$. If $y = 0$, then we have $(2^x)^2 - z^2 = 1$ which is impossible due to Mihăilescu’s Theorem.

**Case 3.** $x, y > 0$. For this case we have $(2^x)^2 - z^2 = (2^x + z)(2^x - z) = 7^y$. It follows that $(2^x + z) + (2^x - z) = 2^{x+1} = 7\beta + 7\alpha$ for some $\alpha < \beta$, where $\alpha + \beta = y$. Hence, $2^{x+1} = 7\alpha(7^{\beta-\alpha} + 1)$. Thus, $\alpha = 0$ and $2^{x+1} - 7\beta = 1$, which is true when $x = 2$ and $y = 1$. These give us the value $z = 3$. Therefore, $(2, 1, 3)$ is a solution of $4^x - 7^y = z^2$. Now, if we assume $y > 1$, then we get $2^{x+1} - 7\beta = 1$ which has no solution because of Mihăilescu’s Theorem and this proves the theorem.

**Theorem 3.2.** The Diophantine equation $4^x - 11^y = z^2$ contains no solution in $\mathbb{N}_0$ except the trivial solution $x = y = z = 0$.

**Proof.** The theorem can be shown easily by utilizing Mihăilescu’s Theorem and is similar to the proof of the previous theorem. The case when $z = 0$ and $x = 0$ are both trivial. So we may assume without loss of generality that $\min(x, z) > 0$. If this is the case, then we have $(2^x)^2 - z^2 = (2^x + z)(2^x - z) = 11^y$. It follows that,
(2^x + z) + (2^x - z) = 2^{x+1} = 11\beta + 11\alpha \text{ for some } \alpha < \beta, \text{ where } \alpha + \beta = y. \text{ Hence, } 2^{x+1} = 11\alpha(11\beta - \alpha + 1). \text{ Thus, } \alpha = 0 \text{ and } 2^{x+1} - 11\beta = 1 \text{ and by Mihăilescu’s Theorem, we can now conclude that this Diophantine equation has no solution. The theorem is now proved.} 

In the following result we shall state and prove a more general case of Theorem 3.1 and Theorem 3.2.

**Theorem 3.3.** The Diophantine equation \( 4^x - p^y = z^2 \) has the set of all solutions \( \{(x, y, z)\} \) given by

\[
\{(x, y, z)\} = \{(0, 0, 0)\} \cup \{(q - 1, 1, 2^{q-1} - 1)\},
\]

for prime \( p = 2^q - 1 \) (with \( q \) also a prime). For \( p \equiv 3 \pmod{4} \) not of the form \( 2^q - 1 \), the Diophantine equation \( 4^x - p^y = z^2 \) has only the trivial solution \( (x, y, z) = (0, 0, 0) \).

**Proof.** Consider the Diophantine equation \( 4^x - p^y = z^2 \). We consider the following cases.

**Case 1.** \( x = 0 \). If \( x = 0 \), then \( 1 - z^2 = p^y \) which implies that \( z = y = 0 \) and \( p \) is any prime number.

**Case 2.** \( y = 0 \). If \( y = 0 \), then \( 2^{2x} - z^2 = 1 \) which is obviously impossible because of Mihăilescu’s Theorem.

**Case 3.** \( x, y > 0 \). If \( \min(x, y) > 0 \), then \( 4^x - p^y = z^2 \) is equivalent to \( (2^x + z)(2^x - z) = p^y \). Hence, \( 2^{x+1} = (2x + z) + (2^x - z) = p^y(p^{y-\alpha} - 1) \) for some integers \( \alpha \) and \( \beta \) such that \( \alpha + \beta = y \) and \( \beta > \alpha \geq 0 \). Therefore, \( \alpha = 0 \) and \( 2^{x+1} - p^y = 1 \) which has no solution for \( \min(x, y) > 1 \) by Mihăilescu’s Theorem.

For \( y = 1 \), we get \( p = 2^{x+1} - 1 \). Note that \( 2^{x+1} - 1 \) is a prime if and only if \( x + 1 \) is also a prime. Thus, we get a family of solutions to \( 4^x - p^y = z^2 \) given by \( (x, y, z) = \{(q - 1, 1, 2^{q-1} - 1) \mid q \text{ is a prime}\} \) for \( p = 2^q - 1 \). On the other hand, if \( p \equiv 1 \pmod{4} \) not of the form \( 2^q - 1 \) (with \( y = 1 \)), then we get \(-1 \equiv 1 \pmod{4} \) and this a clear contradiction. Thus, we only have the trivial solution \( (0, 0, 0) \) to \( 4^x - p^y = z^2 \) for \( p \equiv 3 \pmod{4} \). Now, conclusion follows.

**Remark 3.4.** Theorem 3.1 (respectively, Theorem 3.2) agrees with Theorem 3.3 since \( 7 = 2^3 - 1 \) (respectively, \( 11 \equiv 3 \pmod{4} \)).

**Theorem 3.5.** The Diophantine equation \( 4^x - 7^y = 3z^2 \) has exactly two solutions \((x, y, z) \in \mathbb{N}_0 \). In particular, the solutions are \((0, 0, 0)\) and \((1, 0, 1)\).

**Proof.** Evidently, for the case when \( z = 0 \) we get \((x, y, z) = (0, 0, 0)\). So we let \( z > 0 \) and consider the following three cases.

**Case 1.** \( x = 0 \). This case is trivial.

**Case 2.** \( y = 0 \). If \( y = 0 \), then we have \( 4^x - 3z^2 = 1 \). It can be seen easily that the equation holds true for \( x = z = 1 \). Here we get \((x, y, z) = (1, 0, 1)\). Now, it remains for us to show that there is no solution to \((2^x)^2 - 3z^2 = 1\) other than \((x, y, z) = (1, 0, 1)\) for \( y = 0 \). Note that \((2^x)^2 - 1 = (2^x + 1)(2^x - 1) = 3z^2\).
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So, $2^x + 1 = 3z$ and $2^x - 1 = z$. This claim is easily verified as follows: if $x$ is odd, then there exists an integer $k > 0$ such that $2^x + 1 = 3k$. So $(2^x)^2 - 1 = (2^x + 1)(2^x - 1) = 3k(3k - 2)$. Hence, we have $3k - 2 = 2^x - 1$ but this is impossible since $3k - 2 \neq k$ for $k \neq 1$. Indeed, $2^x + 1 = 3z$ and $2^x - 1 = z$. It follows that, $z = 1$ and $2^x - 1 = z$. The latter equation is true only when $x = 1$. Thus, the case when $y = 0$ implies a unique solution $(x, y, z) = (1, 0, 1)$ to $4^x - 7^y = 3z^2$.

**Case 3.** $x, y > 0$. Suppose $4^x - 7^y = 3z^2$ has a solution in $\mathbb{N}$ for $\min(x, y) > 0$. We rewrite the equation into $(2^x)^2 - 3z^2 = 7^y$. First, suppose that $y$ is even, say $y = 2m$ for some $m \in \mathbb{N}$. Then, by [20, Theorem 6, pg. 16], we could find $X_n = 2^n s_n, Z_n = z_n$, and $Y_n = 7^m n$ such that $X_n^2 - 3Z_n^2 = Y_n^2$. Furthermore, by [20, Theorem 1, pg. 9], there is a solution $(u, v)$ such that $u^2 - 3v^2 = 1$. Let $u = 2^x j$ and $v = z_j$ where $(2^x j)^2 - 3z_j^2 = 1$. Multiplying $(2^x j)^2 - 3z_j^2 = 1$ by $7^{2m j}$ both sides with $m_j > 0$, we get $(2^x j 7^{m j})^2 - 3(z_j 7^{m j})^2 = (7^{m j})^2$. This is impossible since every solution $X_n$ is a power of two. It follows that $y$ is odd. Suppose now that $y$ is odd. Then, we have $(2^x)^2 - 3z^2 = 7(49^m)$. Let $p = 3$ and $k = 7$ in Corollary 2.4. Obviously, $7 \equiv 3 \pmod{4}$. Since 49 is of the form $4t + 1$ (with $t = 12$), then $49^m$ is of the form $4t' + 1$ for some $t' \in \mathbb{N}$. Thus, by Corollary 2.4, $(2^x)^2 - 3z^2 = 7^y$ for odd $y$ is not solvable. This completes the proof of the theorem.

**Remark 3.6.** The conclusion in Case 3 of the previous theorem can also be shown using Lemma 2.1. That is, if we rewrite $4^x - 7^y = 3z^2$ into $(X)^2 - 3z^2 = 7^y$ where $X = 2x$, then, by Lemma 2.1, this equation is soluble if and only if there is a natural number $u$ such that $u^2 \equiv 3 \pmod{7^y}$. Note that Lemma 2.1 applies to $(X)^2 - 3z^2 = 7^y$ since $z^2 \equiv -3z^2 \equiv 7^y$ (mod 7) implies that $z$ must be odd. Indeed, we have $\gcd(X, z) = 1$. Now, the equivalence relation $u^2 \equiv 3 \pmod{7^y}$ is soluble provided $u^2 \equiv 3 \pmod{7}$ has a solution. So we must find a solution to $u^2 \equiv 3 \pmod{7}$. But, $3^3 \equiv 6 \pmod{7}$. Hence, by Euler’s Criterion, 3 is a quadratic nonresidue of 7. Thus, $u^2 \equiv 3 \pmod{7^y}$ is insoluble. Here we conclude that $(X)^2 - 3z^2 = (2^x)^2 - 3z^2 = 7^y$ has no solution in $\mathbb{N}_0$.

**Theorem 3.7.** The Diophantine equation $4^x - 19^y = 3z^2$ has exactly two solutions $(x, y, z)$ in $\mathbb{N}_0$, i.e. $(x, y, z) \in \{(0, 0, 0), (1, 0, 1)\}$.

**Proof.** The proof is similar to the previous theorem. The case when $z = 0$ and $x = 0$ are equivalent and are both trivial. We only consider the following two cases.

**Case 1.** $y = 0$. If $y = 0$, then we have $4^x - 3z^2 = 1$ which has the unique solution $(x, y, z) = (1, 0, 1)$ by Case 2 of Theorem 3.5.

**Case 2.** $x, y > 0$. For $\min(x, y) > 0$, the Diophantine equation $4^x - 19^y = 3z^2$ is equivalent to $(2^x)^2 - 3z^2 = 19^y$. First, suppose that $y$ is even, say $y = 2m$ for some $m \in \mathbb{N}$. Then, we could find $X_n = 2^n s_n, Z_n = z_n$, and $Y_n = 19^m n$ such that $X_n^2 - 3Z_n^2 = Y_n^2$. Moreover there is a solution $(u, v)$ such that $u^2 - 3v^2 = 1$. Choose $u = 2^x j$ and $v = z_j$ where $(2^x j)^2 - 3z_j^2 = 1$. Multiplying $(2^x j)^2 - 3z_j^2 = 1$ by $19^{2m j}$ both sides (with $m_j > 0$), we obtain $(2^x j 19^{m j})^2 - 3(z_j 19^{m j})^2 = (19^{m j})^2$. This is clearly a contradiction since every solution $X_n$ is a power of two. So $y$ must be
odd. Now, suppose that \( y \) is odd. Then, \((2^r)^2 - 3z^2 = 7(361^m)\). Let \( p = 3 \) and \( k = 19 \). Obviously, \( 19 \equiv 3 \pmod{4} \). Since 361 is of the form \( 4t + 1 \) (with \( t = 90 \)), then 361 is of the form \( 4t' + 1 \) for some \( t' \in \mathbb{N} \). Therefore, by Corollary 2.4, the Diophantine equation \((2^r)^2 - 3z^2 = 19^p\) for odd \( y \) is not solvable. This proves the theorem.

\[ \Box \]

**Remark 3.8.** Similar to what we remarked for Case 3 of Theorem 3.5, the conclusion obtained in Case 2 of Theorem 3.7 can be shown using Lemma 2.1 running along the same inductive line of argument in Remark 3.6.

We note that \( 4^x - 7^y = 3z^2 \) and \( 4^x - 19^y = 3z^2 \) are of the form \( 4^x - p^y = 3z^2 \) where \( p \equiv 3 \pmod{4} \). This Diophantine equation has in fact the two solutions \((0,0,0)\) and \((1,0,1)\) in \( \mathbb{N}_0 \), and this result is the content of our last and final theorem.

**Theorem 3.9.** Let \( p \equiv 3 \pmod{4} \) be a prime. Then, the Diophantine equation \( 4^x - p^y = 3z^2 \) has exactly two solutions \((x,y,z)\) in \( \mathbb{N}_0 \), i.e. \((x,y,z) \in \{(0,0,0),(1,0,1)\}\).

**Proof.** Let \( p \equiv 3 \pmod{4} \) be a prime and consider the Diophantine equation \( 4^x - p^y = 3z^2 \) where \( x, y, \) and \( z \) are non-negative integers. We first treat the case when \( \min(x,y,z) = 0 \). If \( x = 0 \), then we have \( 1 - p^y = 3z^2 \). Note that \( p^y \equiv 1 \pmod{4} \) when \( y \) is even and \( p^y \equiv -1 \pmod{4} \) when \( y \) is odd. Also, note that \( z \) is odd. Hence, \( 1 - p^y \equiv 0,2 \pmod{4} \) whereas \( 3z^2 \equiv 3 \pmod{4} \). Therefore, \( 4^x - p^y = 3z^2 \) has no solution for \( x = 0 \).

If \( y = 0 \), then we get \( 4^x - 1 = 3z^2 \) which has the unique solution \((x,y,z) = (1,0,1)\) by Case 2 of Theorem 3.5.

If \( z = 0 \), then it immediately follows that \( x = y = 0 \). Here we get \((x,y,z) = (0,0,0)\).

Now suppose \( \min(x,y,z) > 0 \). Note that the equivalence relation \( 4^x - p^y \equiv 3z^2 \equiv -1 \pmod{4} \) implies that \( y \) and \( z \) are both odd. If \( y \) is odd, then \((2^r)^2 - 3z^2 = p(p^{2m})\). But, \( p = 4t + 3 \) for some \( t \in \mathbb{N}_0 \), hence \( p^2 \) is of the form \( 4t' + 1 \) for some \( t' \in \mathbb{N} \). Then, \( p^{2m} \equiv 1 \pmod{4} \). By virtue of Corollary 2.4 we conclude that \( (2^r)^2 - 3z^2 = p^y \) is not solvable. This proves the theorem. \[ \Box \]

4 Summary

In this work, we have exhibited all solutions to the Diophantine equation \( 4^x - p^y = z^2 \) in the set of non-negative integers for prime number \( p \). Also, we have given all solutions to the Diophantine equation \( 4^x - p^y = 3z^2 \) under the assumption that \( p \equiv 3 \pmod{4} \). With this restriction on \( p \), the case when \( p \equiv 1 \pmod{4} \) remains open and we leave this to the interested reader. Also, we leave the set of all solutions of the Diophantine equation \( 4^x - p^y = dz^2 \) in \( \mathbb{N}_0 \) (where \( d \) is an integer) as an open problem. It is worth mentioning that the two Diophantine equations \( 2^x + 3y^2 = 4z^2 \) and \( 2^x + 7y^2 = 4z^2 \) were already been studied by the author.
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in [23] in which a general solution to $2^x + dy^2 = 4z$ in non-negative integers (with $d = (2k - 1)/9$ and $k \equiv 0 \pmod{6}$) was also also presented.

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**References**


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