Modules which are Reduced over their Endomorphism Rings

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Abstract: Let $R$ be an arbitrary ring with identity and $M$ a right $R$-module with $S = \text{End}_R(M)$. The module $M$ is called reduced if for any $m \in M$ and $f \in S$, $fm = 0$ implies $fM \cap Sm = 0$. In this paper, we investigate properties of reduced modules and rigid modules.

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1 Introduction

Throughout this paper $R$ denotes an associative ring with identity. For a module $M$, $S = \text{End}_R(M)$ denotes the ring of right $R$-module endomorphisms of $M$. Then $M$ is a left $S$-module, right $R$-module and $(S, R)$-bimodule. In this work, for any rings $S$ and $R$ and any $(S, R)$-bimodule $M$, $r_M(.)$ and $l_M(.)$ denote the right annihilator of a subset of $M$ in $R$ and the left annihilator of a subset of $R$ in $M$, respectively. Similarly, $l_S(.)$ and $r_M(.)$ denote the left annihilator of a subset

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2 Reduced Modules

Let $M$ be an $R$-module with $S = \text{End}_R(M)$. Some properties of $R$-modules do not characterize the ring $R$, namely there are reduced $R$-modules but $R$ need not be reduced and there are abelian $R$-modules but $R$ is not an abelian ring. Because of that reduced, rigid, symmetric, semicommutative and Armendariz modules in terms of endomorphism rings $S$ are introduced by the present authors (see [8]). In this section we study properties of modules which are reduced over their endomorphism rings.

We start with the following proposition.

**Proposition 2.1.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. Consider the following conditions for $f \in S$.

1. $S(\text{Ker}f) \cap \text{Im}f = 0$.
2. Whenever $m \in M$, $fm = 0$ if and only if $\text{Im}f \cap Sm = 0$.

Then (1) $\Rightarrow$ (2). If $M$ is a semicommutative module, then (2) $\Rightarrow$ (1).

**Proof.** Clear.

Following the definition of Lee and Zhou [1], $M$ is a reduced module if and only if condition (2) of Proposition 2.1 holds for each $f \in S$. If $M$ is a reduced
module, then it is semicommutative and so condition (1) of Proposition also holds for each \( f \in S \).

As an illustration we state the following examples.

**Example 2.2.** Let \( p \) be any prime integer and \( M \) denote the \( \mathbb{Z} \)-module \( \left( \mathbb{Z}/\mathbb{Z}p \right) \oplus \mathbb{Q} \). Then \( S \) is isomorphic to the matrix ring \( \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in \mathbb{Z}_p, b \in \mathbb{Q} \right\} \). It is evident that \( M \) is a reduced module.

Note that every module need not be reduced.

**Example 2.3.** Let \( p \) be any prime integer and \( M = \mathbb{Z}(p^\infty) \) the Prüfer \( p \)-group as a \( \mathbb{Z} \)-module. Let \( \{v_i\} \) \((i = 1, 2, 3, \cdots)\) be elements in \( M \) which they satisfy the equalities \( pv_1 = 0 \), \( pv_i = v_{i-1} \) \((i = 2, 3, \cdots)\). By [9, page 54], \( S \) is isomorphic to the ring of \( p \)-adic integers \( \mathbb{A}(p) \). Define \( f \) as \( f(v_1) = 0 \) and \( f(v_i) = v_{i+1} \) for \((i = 2, 3, 4, \cdots)\). Let \( m = v_2 \). Then \( f(v_2) = v_1 \) and \( f^2(v_2) = 0 \). Hence \( M \) is not reduced.

**Lemma 2.4.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). If \( M \) is a reduced module, then \( S \) is a reduced ring.

**Proof.** It is clear from [8, Lemma 2.11] and [8, Proposition 2.14].

**Definition 2.5.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). The module \( M \) is called principally projective if for any \( m \in M \), \( \ell_S(m) = Se \) for some \( e^2 = e \in S \).

It is obvious that the module \( R \) is principally projective if and only if the ring \( R \) is left principally projective. It is straightforward that all Baer and quasi-Baer modules are principally projective. And every quasi-Baer module is principally quasi-Baer. There are principally projective modules which are not quasi-Baer or Baer (see [10, Example 8.2]).

**Example 2.6.** Let \( R \) be a Prüfer domain (a commutative ring with an identity, no zero divisors and all finitely generated ideals are projective) and \( M \) the right \( R \)-module \( R \oplus R \). By ([8], page 17), \( S \) is a \( 2 \times 2 \) matrix ring over \( R \) and it is a Baer ring. Hence \( M \) is Baer and so principally projective module.

Note that the endomorphism ring of a principally projective module may not be a right principally projective ring in general. For if \( M \) is a principally projective module and \( \varphi \in S \), then we have two cases. \( \text{Ker}\varphi = 0 \) or \( \text{Ker}\varphi \neq 0 \). If \( \text{Ker}\varphi = 0 \), then for any \( f \in rs(\varphi) \), \( \varphi f = 0 \) implies \( f = 0 \). Hence \( rs(\varphi) = 0 \). Assume that \( \text{Ker}\varphi \neq 0 \). There exists a nonzero \( m \in M \) such that \( \varphi m = 0 \). By hypothesis, \( \varphi \in \ell_S(m) = Se \) for some \( e^2 = e \in S \). In this case \( \varphi = \varphi e \) and so \( rs(\varphi) \leq (1 - e)S \). The following example shows that this inclusion is strict.

**Example 2.7.** Let \( Q \) be the ring and \( N \) the \( Q \)-module constructed by Osofsky in ([11]). Since \( Q \) is commutative, we can just as well think of \( N \) as a right \( Q \)-module. Let \( S = \text{End}_Q(N) \). It is easy to see that \( N \) is a principally projective
module. Identify $S$ with the ring \[
\begin{bmatrix}
Q & 0 \\
Q/I & Q/I
\end{bmatrix}
\] in the obvious way, and consider \[
\varphi = \begin{bmatrix}
0 & 0 \\
1 + I & 0
\end{bmatrix} \in S.
\] Then \[
r_S(\varphi) = \begin{bmatrix}
I & 0 \\
Q/I & Q/I
\end{bmatrix}.
\] This is not a direct summand of $S$ because $I$ is not a direct summand of $Q$. Therefore $S$ is not a right principally projective ring.

**Proposition 2.8.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $M$ is semicommutative, then we have the followings.

(1) $M$ is a Baer module if and only if $M$ is a quasi-Baer module.

(2) $M$ is a principally projective module if and only if $M$ is a principally quasi-Baer module.

**Proof.** Let $M$ be an $R$-module with $M$ semicommutative.

(1) The necessity is clear. By Theorem 2.14 of [12] and [2, Lemma 2.15], the sufficiency follows.

(2) The necessity follows from the proof of Lemma 2.15 of [12]. The sufficiency is clear from the semicommutativity. \qed

Recall that a ring $R$ is called abelian if every idempotent is central, that is, $ae = ea$ for any $e^2 = e, a \in R$. Abelian modules are introduced by Roos in [13] and studied by Goodearl and Boyle [14], Roman and Rizvi [15]. Following Roos [13], a module $M$ is called abelian if all idempotents of $S$ are central.

**Remark 2.9.** It is easy to show that if $M$ is a semicommutative module, then $S$ is an abelian ring. It follows from Theorem 2.14 of [12], every reduced module $M$ is semicommutative, and every semicommutative module $M$ is abelian. The converses hold if $M$ is a principally projective module. Note that for a prime integer $p$, the cyclic group $M$ of $p^2$ elements is a $\mathbb{Z}$-module for which $S = \mathbb{Z}_{p^2}$. The module $M$ is neither reduced nor principally projective although it is semicommutative.

**Proposition 2.10.** Let $M$ be a uniform $R$-module with $S = \text{End}_R(M)$. If $M$ is a reduced module, then $S$ is a domain.

**Proof.** For $f, g \in S$, suppose $fg = 0$ with $f \neq 0$. We show that $g = 0$. For any $m \in M$, $fgmR = 0$ and so $fM \cap SgmR = 0$. By hypothesis $fM = 0$ or $SgmR = 0$. Then $Sgm = 0$ and so $gm = 0$ for all $m \in M$. Hence $g = 0$. \qed

**Lemma 2.11.** [16, Lemma 1.9] Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules $M_1, M_2$. Then $M_1$ is a fully invariant submodule of $M$ if and only if $\text{Hom}(M_1, M_2) = 0$.

We observe in Example 8.7 that the direct sum of reduced modules need not be reduced. Note the following fact.

**Proposition 2.12.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$. Let $M = M_1 \oplus M_2$ be a decomposition of $M$ where $M_1$ and $M_2$ are fully invariant submodules of $M$ with $S_1 = \text{End}_R(M_1)$ and $S_2 = \text{End}_R(M_2)$.

(1) If $M_1$ and $M_2$ are reduced over $S$, then $M$ is reduced.

(2) If $M_1$ and $M_2$ are reduced over $S_1$ and $S_2$ respectively, then $M$ is reduced.
Proof. (1) Let \( f \in S, \ m \in M \) and \( fm = 0 \). There exist \( m_1 \in M_1 \) and \( m_2 \in M_2 \) such that \( m = m_1 + m_2 \). Hence \( fm_1 + fm_2 = 0 \). Since \( M_1 \) and \( M_2 \) are fully invariant submodules of \( M \), \( fm_1 = 0 \) and \( fm_2 = 0 \) by Lemma 2.11. So 
\[ fM_1 \cap Sm_1 = 0 \text{ and } fM_2 \cap Sm_2 = 0. \] 
Then \( x = fm' = gm \) for some \( m' \in M \) and \( g \in S \). For \( m' \in M \) there exist \( m_1 \in M_1 \) and \( m_2 \in M_2 \) such that \( m' = m_1 + m_2 \). So 
\[ fm_1' - gm_1 = gm_2 - fm_2' \in M_1 \cap M_2 = 0. \] 
It follows that 
\[ fm_1 = gm_1 = 0 \text{ and } fm_2 = gm_2 = 0. \] 
Therefore \( x = 0 \).

(2) Let \( f \in S, \ m \in M \) and \( fm = 0 \). There exist \( m_1 \in M_1 \) and \( m_2 \in M_2 \) such that \( m = m_1 + m_2 \). Hence \( fm_1 + fm_2 = 0 \). Since \( M_1 \) and \( M_2 \) are fully invariant submodules of \( M \), \( fm_1 = 0 \) and \( fm_2 = 0 \). Let the restrictions of \( f \) to \( M_1 \) and \( M_2 \) be denoted by the same \( f \). Then 
\[ fM_1 \cap S_1m_1 = 0 \text{ and } fM_2 \cap S_2m_2 = 0. \]
Let \( x \in fM \cap Sm \). Then \( x = fm' = gm \) for some \( m' \in M \) and \( g \in S \). For \( m' \in M \), there exist \( m_1 \in M_1 \) and \( m_2 \in M_2 \) such that \( m' = m_1 + m_2 \). So 
\[ fm_1 + fm_2 = gm_1 + gm_2 \]. It follows that 
\[ fm_1 = gm_1 = 0 \text{ and } fm_2 = gm_2 = 0. \] 
Therefore \( x = 0 \). \( \square \)

**Corollary 2.13.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). Let \( M = M_1 \oplus M_2 \) where \( M_1 \) and \( M_2 \) are submodules of \( M \) with \( S_1 = \text{End}_R(M_1) \) and \( S_2 = \text{End}_R(M_2) \). If \( M \) is semicommutative, then we have the following.

1. If \( M_1 \) and \( M_2 \) are reduced over \( S \), then \( M \) is reduced.
2. If \( M_1 \) and \( M_2 \) are reduced over \( S_1 \) and \( S_2 \) respectively, then \( M \) is reduced.

**Proof.** Let \( M \) be a semicommutative module. It is enough to show that every direct summand \( N \) of \( M \) is fully invariant. We write \( M = N \oplus L \). Let \( \pi \) denote the natural projection of \( M \) onto \( N \). From \( \pi(1 - \pi) = 0 \) and \( (1 - \pi)\pi = 0 \) we have 
\[ \pi y(1 - \pi) = 0 \text{ and } (1 - \pi)y\pi = 0 \]
for each \( g \in S \). Then \( \pi \) is a central idempotent in \( S \). Hence 
\[ g(N) = g(\pi(M)) = \pi(g(M)) \leq N. \] 
This completes the proof. \( \square \)

We end this section with some observations relating to being \( M \) an reduced module and \( S \) an reduced ring. Recall that a module \( M \) is called \( n \)-epiretractable \([7]\) if every \( n \)-generated submodule of \( M \) is a homomorphic image of \( M \).

**Theorem 2.1.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). Then the following hold.

1. If \( M \) is a 1-epiretractable module and \( S \) is a reduced ring, then \( M \) is reduced.
2. If \( M \) is a principally projective module and \( S \) is a reduced ring, then \( M \) is reduced.

**Proof.** (1) Let \( fm = 0 \) for \( f \in S \) and \( m \in M \). Since \( M \) is 1-epiretractable, there exists \( g \in S \) such that \( gM = mR \). We have \( fgM = 0 \) and \( fg = gf = 0 \) since \( S \) is reduced. Let \( fm' = hm \in fM \cap Sm \) where \( m' \in M, h \in S \). Then 
\[ gfm' = ghm = 0 \text{ and } ghmR = 0. \] 
This implies \( ghM = 0 \), i.e., \( gh = 0 \). Therefore 
\[ gh = hq = 0. \] 
Now by assumption, there exists \( m_1 \in M \) such that 
\[ m = gm_1. \] 
Then 
\[ fm' = hm = hgm_1 = 0. \] 
Hence \( M \) is reduced.

(2) Let \( fm = 0 \) for \( f \in S \) and \( m \in M \), and \( fm' = gm \in fM \cap Sm \). Since \( fm = 0 \in mR \), we may find an idempotent \( e \) in \( S \) such that \( f \in l_S(mR) = Se \).
By hypothesis, \( e \) is central in \( S \). So \( s = fe = ef, \) \( cm = 0 \). Then \( fm' = gem = gem = 0 \). Hence \( fM \cap Sm = 0 \). Thus \( M \) is reduced.

**Theorem 2.2.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). If \( M \) is a reduced module, then the following hold.

1. Assume that for every submodule \( N \) of \( M \) there exist \( e^2 = e \in S \) and \( f \in S \) such that \( N \subseteq eM \) and \( f(N) = eM \). Then \( M \) is a Baer module.
2. If every fully invariant submodule is a direct summand of \( M \), then \( M \) is a Baer module.
3. If \( M \) is a uniform module, then each nonzero element of \( S \) is a monomorphism.

**Proof.**

1. Let \( N \) be a submodule of \( M \). Then there exist an idempotent homomorphism \( e \in S \) and \( f \in S \) such that \( N \subseteq eM \) and \( f(N) = eM \). We prove that \( fS(N) = S(1-e) \). It is trivial that \( S(1-e) \leq lS(N) \) since \( N \subseteq eM \). Let \( g \in lS(N) \). By hypothesis \( gN = 0 \) implies \( gfN = 0 \). Then \( gfN = geM = 0 \), and so \( ge = 0 \). Hence \( g = g(1-e) \in S(1-e) \). So \( lS(N) \subseteq S(1-e) \). This completes the proof.
2. Since \( M \) is a reduced module, if \( fm = 0 \) where \( f \in S \), then for all \( g \in S \), \( fgm \in fM \cap Sm = 0 \). This implies that for all \( f \in S \), \( Kerf \) is a fully invariant submodule of \( M \). Let \( I \) be an ideal of \( S \). Since \( r_M(I) = \cap_{f \in I} Kerf \) and all the \( Kerf \) are fully invariant submodules of \( M \), \( r_M(I) \) is a fully invariant submodule of \( M \). So it is a direct summand of \( M \) and therefore \( M \) is a Baer module.
3. Let \( fm = 0 \) where \( f \in S, m \in M \). Then \( fmR = 0 \). By hypothesis, \( fM \cap SmR = 0 \) and so \( fM = 0 \) or \( SmR = 0 \). Hence \( f = 0 \) or \( m = 0 \).

3 Rigid Modules

Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). Rigid \( R \)-modules are introduced and studied in [18] and [19] by the present authors. Recently, rigid modules over their endomorphism rings are studied in [8]. In this section we continue to investigate further properties of a rigid module over its endomorphism ring as a generalization of a reduced module over its endomorphism ring and relations between reduced, semicommutative and \( K \)-co(non)singular modules.

We mention the following obvious proposition.

**Proposition 3.1.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). For any \( f \in S \), the following are equivalent.

1. \( Kerf \cap Imf = 0 \).
2. For \( m \in M \), \( f^2m = 0 \) if and only if \( fm = 0 \).

A module \( M \) is called rigid if it satisfies Proposition 3.1 for every \( f \in S \). By [8] Lemma 2.20, if \( M \) is a rigid module, then \( S \) is a reduced ring and therefore abelian.

Rickart modules provide a generalization of a right principally projective ring to the general module theoretic setting. It is clear that every Baer module is a Rickart module while the converse is not true. For example, \( \mathcal{Z}(R) \) is Rickart but not Baer as a \( Z \)-module.
Proposition 3.2. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $M$ is a reduced module, then $M$ is a rigid module. The converse holds if $M$ satisfies one of the following conditions.

1. $M$ is a semicommutative module.
2. $M$ is a principally projective module.
3. $M$ is a Rickart module.

Proof. For any $f \in S$, $S(\text{Ker} f) \cap \text{Im} f = 0$ by hypothesis. Since $\text{Ker} f \cap \text{Im} f \subset S(\text{Ker} f) \cap \text{Im} f$, $\text{Ker} f \cap \text{Im} f = 0$. By Proposition 3.1, $M$ is a rigid module.

Conversely, (1) Assume that $M$ is a rigid and semicommutative module. Let $f \in S$ and $m \in M$ with $fm = 0$. Let $fm' = gm \in fM \cap Sm$. We multiply it by $f$ from the left and we have $f^2m' = fgm$. Since $M$ is semicommutative and $fm = 0$, $f^2m' = fgm = 0$. By hypothesis $fm' = 0$.

(2) Let $M$ be a rigid and principally projective module. Assume that $fm = 0$ for $f \in S$ and $m \in M$. Then there exists $e^2 = e \in S$ such that $l_S(mR) = Se$. Since $e$ is central in $S$, $fe = ef = f$ and $eg = gf$ for each $g \in S$ and $em = 0$. Let $fm' = gm \in fM \cap Sm$. Multiply $fm' = gm$ by $e$ from the left to obtain $efm' = fm' = gem = 0$. Therefore $M$ is a reduced module.

(3) Let $M$ be a Rickart and rigid module. Assume that $fm = 0$ for $f \in S$ and $m \in M$. Then there exists $e^2 = e \in S$ such that $r_M(f) = eM$. Since $e$ is central in $S$, $fe = ef = 0$ and $m = em$. Let $fm' = gm \in fM \cap Sm$. We multiply $fm' = gm$ from the left by $e$ to obtain $efm' = fem' = egm = gem = gm = 0$. Therefore $M$ is a reduced module. \qed

There are semicommutative modules which are neither rigid nor principally projective.

Example 3.3. Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$$

and the right $R$-module

$$M = \left\{ \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $f \in S$ and $f \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ c & d \end{bmatrix}$. Multiplying the latter by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $f \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$. For any $\begin{bmatrix} 0 & a \\ a & b \end{bmatrix} \in M$, $f \begin{bmatrix} 0 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & ac \\ ac & ad + bc \end{bmatrix}$.

Similarly, let $g \in S$ and $g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c' \\ c' & d' \end{bmatrix}$. Then $g \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c' \end{bmatrix}$. 

For any \([0\ a\ a\ b]\in M\), \(g\left[\begin{array}{c}0\
\a\
\a\
\a\
\b\end{array}\right]=\left[\begin{array}{c}0\ ac'\ ac\ ad'\ adc\ adc'\ bcc\ end{array}\right]\). Then it is easy to check that for any \([0\ a\ a\ b]\in M\),

\[fg\left[\begin{array}{c}0\
\a\
\a\
\a\
\b\end{array}\right]=f\left[\begin{array}{c}0\ ac'\ ac\ ad'\ adc\ adc'\ bcc\ end{array}\right]=\left[\begin{array}{c}0\ ac'\ ac\ ad'\ adc\ adc'\ bcc\ end{array}\right]\]

and

\[gf\left[\begin{array}{c}0\
\a\
\a\
\a\
\b\end{array}\right]=g\left[\begin{array}{c}0\ ac'\ ac\ ad'\ adc\ adc'\ bcc\ end{array}\right]=\left[\begin{array}{c}0\ ac'\ ac\ ad'\ adc\ adc'\ bcc\ end{array}\right].\]

Hence \(fg=gf\) for all \(f,\ g\in S\). Therefore \(S\) is commutative and so \(M\) is semicommutative. Define \(f\in S\) by \(f\left[\begin{array}{c}0\
\a\
\a\
\a\
\b\end{array}\right]=\left[\begin{array}{c}0\ 0\ 0\ 0\ end{array}\right]\) where \([0\ a\ a\ b]\in M\). Then \(f\left[\begin{array}{c}0\
1\
1\
1\end{array}\right]=\left[\begin{array}{c}0\ 0\ 0\ 0\ end{array}\right]\) and \(f^2\left[\begin{array}{c}0\
1\
1\
1\end{array}\right]=0\). Hence \(M\) is not rigid. Let \(m=\left[\begin{array}{c}0\ 0\ 0\ 0\ end{array}\right]\), then \(l_S(m)\neq 0\) since the endomorphism \(f\) defined preceding belongs to \(l_S(m)\). \(M\) is indecomposable as a right \(R\)-module, therefore \(S\) does not have any idempotents other than zero and identity. Hence \(l_S(m)\) can not be generated by an idempotent as a left ideal of \(S\).

An \(R\)-module \(M\) is called \textit{Hopfian} provided every surjective endomorphism of \(M\) is an isomorphism. For example, every Noetherian module is Hopfian (see [9, Lemma 11.6]).

**Theorem 3.1.** Let \(T\) be a ring and \(M\) a left \(T\)-module. If \(t\in T\) satisfies \(M=tM\) and \(M\) is rigid over \(T\), then \(tm=0\) implies \(m=0\) for any \(m\in M\).

**Proof.** Let \(m\in M\) with \(tm=0\). Since \(M=tM\), there exists \(u\in M\) such that \(m=tu\). Then \(0=tm=t^2u\). It implies \(tu=0\) by hypothesis. Hence \(m=0\). \(\square\)

**Corollary 3.4.** Let \(M\) be an \(R\)-module with \(S=\text{End}_R(M)\). If \(M\) is rigid, then \(M_R\) is Hopfian.

**Proof.** It is clear from Theorem 3.1. \(\square\)

A right \(R\)-module \(M\) is said to be \textit{nonsingular} if for any \(m\in M\), \(mE=0\) for an essential right ideal \(E\) of \(R\) implies \(m=0\), and \(M\) is called \textit{cononsingular} if each submodule \(N\) of \(M\) with \(r_R(N)=\{r\in R\mid Nr=0\}\neq 0\) is essential in \(M\). In [4], a module \(M\) is said to be \textit{\(K\)-nonsingular} if for every \(\varphi\in S\), \(\text{Ker}\varphi\) is essential in \(M\) implies \(\varphi=0\). Also the module \(M\) is said to be \textit{\(K\)-cononsingular} if for every submodule \(N\) of \(M\), \(\varphi N\neq 0\) for all \(0\neq \varphi\in S\) implies \(N\) is essential in \(M\).

**Proposition 3.5.** Let \(M\) be an \(R\)-module with \(S=\text{End}_R(M)\). If \(M\) is a rigid module, then \(M\) is a \(K\)-nonsingular module.
Proof. Let \( f \in S \). Assume that \( \text{Ker} f \) is an essential submodule of \( M \). Since \( M \) is rigid, \( \text{Ker} f \cap \text{Im} f = 0 \). Then \( \text{Im} f = 0 \) and so \( f = 0 \). Hence \( M \) is \( \mathcal{K} \)-nonsingular.

**Corollary 3.6.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). If \( M \) is a reduced module, then \( M \) is \( \mathcal{K} \)-nonsingular.

Example 3.7 shows that the converse statement of Corollary 3.6 need not be true in general. There exists a \( \mathcal{K} \)-nonsingular module which is neither reduced nor \( \mathcal{K} \)-cononsingular.

**Example 3.7.** Let \( M \) denote the \( \mathbb{Z} \)-module \( \mathbb{Z} \oplus \mathbb{Q} \). We show that for any \( f \in S \) with \( \text{Ker} f \) essential in \( M \) we have \( f = 0 \). Since \( S \) is isomorphic to the ring

\[
\left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \mid a \in \mathbb{Z}, b, c \in \mathbb{Q} \right\},
\]

we may assume \( S \) as this ring. We write the elements of \( S \) as matrices and the elements of \( \mathbb{Z} \oplus \mathbb{Q} \) as \( 2 \times 1 \) columns. Let

\[
f = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in S
\]

and \( m = \begin{bmatrix} n \\ q \end{bmatrix}, a, n \in \mathbb{Z} \) and \( b, c \in \mathbb{Q} \) with \( fm = 0 \). Then we have \( an = 0 \), \( bn + cq = 0 \). Assume that \( \text{Ker} f \) is essential in \( M \). Then \( \text{Ker} f \cap (\mathbb{Z} \oplus 0) \neq 0 \).

There exists \( m \in \text{Ker} f \) such that \( n \) is nonzero and \( an = 0 \) and \( bn = 0 \). Hence \( a = b = 0 \). Similarly, \( \text{Ker} f \cap ((0) \oplus \mathbb{Q}) \neq 0 \). We may find \( m' = \begin{bmatrix} 0 \\ q' \end{bmatrix} \in \text{Ker} f \) such that \( q' \) is nonzero. So \( cq' = 0 \) and then \( c = 0 \). It follows \( f = 0 \) and \( M \) is \( \mathcal{K} \)-nonsingular.

Let \( f = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \in S \) and \( m = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). Then \( fm = 0 \). Let

\[
g = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \in S \text{ and } m' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Then \( fm' = gm \in fM \cap Sm \neq 0 \). Therefore \( M \) is not reduced. Let \( N = (1, 1/2)\mathbb{Z} + (1, 1/3)\mathbb{Z} \). Then \( N \) is not essential in \( M \). If \( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in l_S(N) \), then \( \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \\ 1/3 \end{bmatrix} = 0 \) implies \( a = 0 \) and \( b + c/2 = 0, b + c/3 = 0 \). It follows that \( a = 0, b = 0 \) and \( c = 0 \). Hence \( M \) is not \( \mathcal{K} \)-cononsingular.

The proof of Theorem 3.2 is clear from Rizvi and Roman [3] Theorem 2.12]. We give a proof for the sake of completeness.

**Theorem 3.2.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). If \( M \) is a rigid and extending module, then it is Baer and \( \mathcal{K} \)-cononsingular.

Proof. If \( M \) is a rigid module, from Proposition 3.5 \( M \) is a \( \mathcal{K} \)-nonsingular module. Since a \( \mathcal{K} \)-nonsingular and extending module is a Baer module by [4] Theorem 2.12], \( M \) is Baer. Let \( N \) be a submodule of \( M \) with \( l_S(N) = 0 \). We claim \( N \) is essential in \( M \). We may find a direct summand \( K \) of \( M \) so that \( N \) is an essential submodule of \( K \). Let \( M = K \oplus L \) and \( \pi_L \) denote the canonical projection from \( M \) onto \( L \). Then \( \pi_L(N) = 0 \). Hence \( \pi_L \in l_S(N) \). Thus \( \pi_L = 0 \) and so \( L = 0 \), \( M = K \) and \( N \) is essential in \( M \).
Corollary 3.8. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $M$ is a reduced and extending module, then $M$ is Baer and $K$-cononsingular.

Corollary 3.9. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $M$ is a rigid and extending module, then $M$ is a Rickart module.

Proof. It is clear from Theorem 3.2 since Baer modules are Rickart modules. \hfill \Box

Corollary 3.10. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $M$ is a reduced and extending module, then $M$ is a Baer module.

In the following result we give the relations between principally projective modules, reduced modules, semicommutative modules, abelian modules and rigid modules.

Theorem 3.3. Let $M$ be an $R$-module with $S = \text{End}_R(M)$. If $M$ is a principally projective module, then the following conditions are equivalent.
1. $M$ is a reduced module.
2. $M$ is a semicommutative module.
3. $M$ is an abelian module.
4. $M$ is a rigid module.
5. $S$ is a reduced ring.

Proof. (1) ⇔ (2) Clear from Lemma 3.2.
(2) ⇒ (3) Clear from Remark 2.9.
(3) ⇒ (2) Let $f \in S$, $m \in M$ with $fm = 0$. There exists $e^2 = e \in S$ such that $l_S(m) = Se$. Then $f = ef = fe$, $em = 0$ and $e$ is central in $S$. So $0 = em = Sem = fSem = feSm = fSm$. Hence $M$ is semicommutative.
(3) ⇒ (4) Let $f^2m = 0$ for $f \in S$, $m \in M$. For some $e^2 = e \in S$ we have $f \in l_S(fm) = Se$. Then $fe = f$ and $efm = 0$. By hypothesis, $efm = fem$. Hence $0 = efm = fem = fm$. So $M$ is rigid.
(4) ⇒ (3) Let $e^2 = e \in S$. For any $f \in S$, $(ef - efe)^2m = 0$ for all $m \in M$ since $(ef - efe)^2 = 0$. We have $(ef - efe)m = 0$ for all $m \in M$ by hypothesis. Hence $ef = efe = 0$. Similarly, $(fe - efe)^2m = 0$ for all $m \in M$ implies $fe - efe = 0$. It follows that $ef = fe = efe$ and so $S$ is abelian, therefore $M$ is abelian.
(1) ⇒ (5) It follows from Lemma 2.4.
(5) ⇒ (1) Let $f \in S$ and $m \in M$ with $fm = 0$. Assume that $fm = 0$. There exists $e^2 = e \in S$ such that $f \in l_S(m) = Se$. Then $em = 0$, $f = fe$. By hypothesis, $e$ is a central idempotent in $S$. Hence $f = fe = ef$. Let $fm' = gm \in fM \cap Sm$. Then $fm' = efm' = egm = gem = 0$. It follows that $fM \cap Sm = 0$ and (1) holds. \hfill \Box
References


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