Homomorphisms and Derivations in Lie JC*-Algebras

Javad Shokri¹, Ali Ebadian and Rasoul Aghalary

Department of Mathematics, Urmia University
P.O.Box 165, Urmia, Iran
e-mail: j.shokri@urmia.ac.ir (J. Shokri)
a.ebadian@urmia.ac.ir (A. Ebadian)
r.aghalary@urmia.ac.ir (R. Aghalary)

Abstract: We investigate isomorphisms between JC*-algebras, homomorphisms between Lie JC*-algebras and derivations on Lie JC*-algebras associated with the functional inequality \[ |f(x + y + z) - f(x) - 2f(y)| \leq |f(x + y + z)|. \]

Keywords: Lie JC*-algebra homomorphism; Lie JC*-algebra derivation; JC*-algebra isomorphism; functional inequality.

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1 Introduction

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [1]). Let \( L(H) \) be the real vector space of all bounded self-adjoint linear operators on \( H \), interpreted as the (bounded) observables of the system. In 1932, Jordan observed that \( L(H) \) is the (nonassociative) algebra via the anticommutator product \( x \circ y := \frac{x + y + z}{2} \). A commutative algebra \( X \) with product \( x \circ y \) is called a Jordan algebra. A Jordan C*-subalgebra of a C*-algebra, endowed with the anticommutator product, is called a JC*-algebra.

A C*-algebra \( C \), endowed with the Lie product \( [x, y] = \frac{x + y}{2} \) on \( C \), is called a Lie C*-algebra. A C*-algebra \( C \), endowed with the Lie product \( [\cdot, \cdot] \) and the

¹Corresponding author.

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anticommutator product $\circ$, is called a Lie $JC^*$-algebra if $(C, \circ)$ is a $JC^*$-algebra and $(C, [,])$ is a Lie $C^*$-algebra (see [2, 3, 4]). During the last decades several Lie theory arguments related to functional equations and functional inequalities have been investigated by a number of mathematicians; cf. [5, 6, 7, 8] and references therein.

In this paper we study Lie $JC^*$-algebra homomorphisms in Lie $JC^*$-algebras. Our results generalize the $JC^*$-algebra isomorphisms posed by Park, An and Cui [9] in $JC^*$-algebras. Moreover, we present the Lie $JC^*$-algebras derivations on Lie $JC^*$-algebras associated by the following functional inequality

$$\left\| f\left(\frac{x-y}{2} + z\right) - f(x) - 2f(z) \right\| \leq \left\| f\left(\frac{x+y}{2} + z\right) \right\|.$$  \hspace{1cm} (1.1)

\section{Homomorphisms between Lie $JC^*$-algebras and Isomorphisms in $JC^*$-algebras}

At the first of this section we would like to investigate Lie $JC^*$-algebra homomorphisms between two Lie $JC^*$-algebras and then, as corollaries, result $JC^*$-algebra isomorphisms between two $JC^*$-algebras associated with the functional inequality (1.1). Throughout this section, assume that $A$ and $B$ are two Lie $JC^*$-algebras respectively with norm $\| \cdot \|_A$ and $\| \cdot \|_B$, and also assume that $X$ and $Y$ are two $JC^*$-algebra respectively with norm $\| \cdot \|_X$ and $\| \cdot \|_Y$. First we need the following proposition.

\textbf{Definition 2.1.} [10] A $\mathbb{C}$-linear mapping $H : A \to B$ is called a Lie $JC^*$-algebra homomorphism if $H$ satisfies

$$H(x \circ y) = H(x) \circ H(y),$$

$$H([x,y]) = [H(x), H(y)],$$

$$H(x^*) = H(x)^*$$

for all $x, y \in A$.

\textbf{Definition 2.2.} [10, 11] For two $JC^*$-algebras $A$ and $B$, a bijective $\mathbb{C}$-linear mapping $H : A \to B$ is called a Lie $JC^*$-algebra isomorphism if $H$ satisfies

$$H(x \circ y) = H(x) \circ H(y),$$

for all $x, y \in A$.

\textbf{Proposition 2.3.} Suppose $f : A \to B$ be a mapping such that

$$\| f\left(\frac{x-y}{2} + z\right) - f(x) - 2f(z) \|_B \leq \| f\left(\frac{x+y}{2} + z\right) \|_B$$  \hspace{1cm} (2.1)

for all $x, y, z \in A$. Then $f$ is Cauchy additive.
Proof. Assume that \( x = y = z = 0 \) in (2.1), we get
\[
\| -2f(0) \|_B \leq \| f(0) \|_B,
\]
so \( f(0) = 0 \).

Let \( y = x, z = -x \) in (2.1), it follows that
\[
\| f(-x) - f(x) - 2f(-x) \|_B = \| f(x) - f(-x) \|_B \leq \| f(0) \|_B = 0
\]
for all \( x \in A \). Hence \( f(-x) = -f(x) \) for all \( x \in A \).

Let us suppose \( x = 0, y = -2z \) in (2.1), we get
\[
\| f(2z) - 2f(z) \|_B \leq \| f(0) \|_B = 0
\]
for all \( z \in A \). Thus \( f(2z) = 2f(z) \) for all \( z \in A \).

Let \( z = -\frac{2y}{x+y} \) in (2.1), it follows that
\[
\| f(-y) - f(x) - 2f(-\frac{x+y}{2}) \|_B = \| f(y) - f(x) + f(x+y) \|_B \leq \| f(0) \|_B = 0
\]
for all \( x, y \in A \), which this proves that
\[
f(x+y) = f(x) + f(y)
\]
for all \( x, y \in A \) and so that \( f \) is Cauchy additive.

\( \square \)

**Theorem 2.1.** Suppose \( r \neq 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a mapping such that
\[
\| f\left(\frac{\mu x - y}{2} + z\right) - \mu f(x) - 2f(z) \|_B \leq \| f\left(\frac{\mu x + y}{2} + z\right) \|_B, \tag{2.2}
\]
\[
\| f([x, y]) - [f(x), f(y)] \|_B \leq \theta(\|x\|_A^2 + \|y\|_B^2), \tag{2.3}
\]
\[
\| f(x \circ y) - f(x) \circ f(y) \|_B \leq \theta(\|x\|_A^2 + \|y\|_B^2), \tag{2.4}
\]
\[
\| f(x^*) - f(x)^* \|_B \leq \theta(\|x\|_A^2 + \|x\|_A^2), \tag{2.5}
\]
for all \( \mu \in T^1 := \{ \lambda \in C : \|\lambda\| = 1 \} \) and all \( x, y, z \in A \). Then the mapping \( f : A \to B \) is a Lie \( JC^* \)-algebra homomorphism.

**Proof.** Assume \( r < 1 \).

Suppose \( \mu = 1 \) in (2.2), then by Proposition 2.3, implies the mapping \( f : A \to B \) is a Cauchy additive. So \( f(0) = 0 \). Assume \( y = -\mu x \) and \( z = 0 \) in (2.2), so that
\[
\| f(\mu x) - \mu f(x) \|_B \leq \| f(0) \|_B = 0
\]
for all \( x \in A \) and all \( \mu \in T^1 \). Therefore it is concluded that \( f(\mu x) = \mu f(x) \) for all \( x \in A \) and all \( \mu \in T^1 \). Now by Theorem 2.1 of [12], the mapping \( f \) is a \( C \)-linear. So one can consider \( f(x) = \lim_{n \to \infty} \frac{1}{2\pi} f(2^n x) \) for all \( x \in A \).
It follows from (2.3) that
\[ \| f([x, y]) - [f(x), f(y)] \|_B = \lim_{n \to \infty} \frac{1}{4^n} \| f(2^n x, 2^n y) - [f(2^n x), f(2^n y)] \|_B \]
\[ \leq \lim_{n \to \infty} \frac{4^n r \theta}{4^n} (\| x \|_A^r + \| y \|_A^r) = 0 \]
for all \( x, y \in \mathcal{A} \), which proves
\[ f([x, y]) = [f(x), f(y)], \]
for all \( x, y \in \mathcal{A} \).

It follows from (2.4) that
\[ \| f(x \circ y) - f(x) \circ f(y) \|_B = \lim_{n \to \infty} \frac{1}{4^n} \| f(2^n x \circ 2^n y) - f(2^n x) \circ f(2^n y) \|_B \]
\[ \leq \lim_{n \to \infty} \frac{4^n r \theta}{4^n} (\| x \|_A^r + \| y \|_A^r) = 0 \]
for all \( x, y \in \mathcal{A} \). Then we obtain
\[ f(x \circ y) = f(x) \circ f(y) \]
for all \( x, y \in \mathcal{A} \).

And also from (2.5) is concluded that
\[ \| f(x^*) - f(x) \|_B = \lim_{n \to \infty} \frac{1}{2^n} \left\| f\left(2^n x^* \right) - f\left(2^n x \right)^* \right\|_B \]
\[ \leq \lim_{n \to \infty} \frac{2^n r \theta}{2^n} (\| x \|_A^r + \| x \|_A^r) \]
for all \( x \in \mathcal{A} \). Thus we proved
\[ f(x^*) = f(x)^* \]
for all \( x \in \mathcal{A} \), which this completes the proof. Similarly, one can obtains the result for the case \( r > 1 \).

\[ \square \]

**Theorem 2.2.** Suppose \( r \neq 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : \mathcal{A} \to \mathcal{B} \) be a mapping satisfying (2.2) such that
\[ \| f([x, y]) - [f(x), f(y)] \|_B \leq \theta(\| x \|_A^r, \| y \|_B^r), \]  
\[ \| f(x \circ y) - f(x) \circ f(y) \|_B \leq \theta(\| x \|_A^r, \| y \|_B^r), \]
\[ \| f(x^*) - f(x)^* \|_B \leq \theta(\| x \|_A^r, \| x \|_A^r) \]
for all \( x, y, z \in \mathcal{A} \). Then the mapping \( f : \mathcal{A} \to \mathcal{B} \) is a Lie JC*-algebra homomorphism.
Proof. Assume $r > 1$.
By the same reasoning as in the proof of Theorem 2.1, the mapping $f$ is a $\mathbb{C}$-linear.
So one can consider $f(x) = \lim_{n \to \infty} 2^n f(x/2^n)$ for all $x \in A$.
It follows from (2.6) that
\[
\|f([x, y]) - [f(x), f(y)]\|_B = \lim_{n \to \infty} 4^n \|f([x/2^n, y/2^n]) - [f(x/2^n), f(y/2^n)]\|_B \\
\leq \lim_{n \to \infty} 4^n \theta \left(\|x\|^r_A \cdot \|y\|^r_A\right) = 0
\]
for all $x, y \in A$, which proves
\[
f([x, y]) = [f(x), f(y)],
\]
for all $x, y \in A$.
It follows from (2.7) that
\[
\|f(x \circ y) - f(x) \circ f(y)\|_B = \lim_{n \to \infty} 4^n \|f(x/2^n \circ y/2^n) - f(x/2^n) \circ f(y/2^n)\|_B \\
\leq \lim_{n \to \infty} 4^n \theta \left(\|x\|^r_A \cdot \|y\|^r_A\right) = 0
\]
for all $x, y \in A$. This implies
\[
f(x \circ y) = f(x) \circ f(y),
\]
for all $x, y \in A$.
And also from (2.8) is derived that
\[
\|f(x^*) - f(x)^*\|_B = \lim_{n \to \infty} 2^n \left\|f\left(\frac{x^*}{2^n}\right) - f\left(\frac{x}{2^n}\right)^*\right\|_B \\
\leq \lim_{n \to \infty} \frac{2^n \theta}{2^{nr}} \left(\|x\|^r_A \cdot \|x\|^r_A\right)
\]
for all $x \in A$, and this proves
\[
f(x^*) = f(x)^*
\]
for all $x \in A$. Therefore we conclude $f : A \to B$ is a Lie $JC^*$-algebra homomorphism. Similarly, one can obtains the result for the case $r < 1$.

Now we investigate $JC^*$-algebra isomorphisms in the remaining of this section as the results of above Theorems.

**Corollary 2.4.** Suppose $r \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : \mathcal{X} \to \mathcal{Y}$ be a bijective mapping satisfying (2.2) such that
\[
\|f(x \circ y) - f(x) \circ f(y)\|_Y \leq \theta(\|x\|_{\mathcal{X}}^r + \|y\|_{\mathcal{Y}}^r) \tag{2.9}
\]
for all $x, y, z \in \mathcal{X}$. Then the mapping $f : \mathcal{X} \to \mathcal{Y}$ is a $JC^*$-algebra isomorphism.
Proof. Assume $r > 1$.
Similarly in the proof of Theorem 2.1, the mapping $f$ is a $C$-linear. So one can consider $f(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in \mathcal{X}$.

It follows from (2.9) that
\[
\| f(x \circ y) - f(x) \circ f(y) \|_Y = \lim_{n \to \infty} 4^n \| f\left(\frac{x}{2^n} \circ \frac{y}{2^n}\right) - f\left(\frac{x}{2^n}\right) \circ f\left(\frac{y}{2^n}\right) \|_Y \\
\leq \lim_{n \to \infty} \frac{4^n}{4^{nr}} (\|x\|_X^r + \|y\|_X^r) = 0
\]
for all $x, y \in \mathcal{X}$. Thus
\[
f(x \circ y) = f(x) \circ f(y)
\]
for all $x, y \in \mathcal{X}$. Hence the mapping $f$ is a $JC^*$-algebra isomorphism, as desired. Similarly, one can obtains the result for the case $r < 1$.

Corollary 2.5. Suppose $r \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : \mathcal{X} \to \mathcal{Y}$ be a bijective mapping satisfying (2.2) such that
\[
\| f(x \circ y) - f(x) \circ f(y) \|_Y \leq \theta (\|x\|_X^r \cdot \|y\|_X^r)
\]
for all $x, y \in \mathcal{X}$. Then the mapping $f : \mathcal{X} \to \mathcal{Y}$ is a $JC^*$-algebra isomorphism.

Proof. Assume $r < 1$.
Similarly in the proof of Theorem 2.1, the mapping $f$ is a $C$-linear. So one can consider $f(x) = \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in \mathcal{X}$.

It follows from (2.10) that
\[
\| f(x \circ y) - f(x) \circ f(y) \|_Y = \lim_{n \to \infty} \frac{1}{4^n} \| f(2^n x \circ 2^n y) - f(2^n x) \circ f(2^n y) \| \\
\leq \lim_{n \to \infty} \frac{4^n \theta}{4^{nr}} (\|x\|_X^r + \|y\|_X^r) = 0
\]
for all $x, y \in \mathcal{X}$. Thus
\[
f(x \circ y) = f(x) \circ f(y)
\]
for all $x, y \in \mathcal{X}$, which this completes the proof of this case. And by same reasons, we obtain the result for the case $r > 1$.

3 Derivations on Lie $JC^*$-algebras

In this section, we are going to investigate Lie $JC^*$-algebra derivations on Lie $JC^*$-algebras associated with the functional inequality (1.1). Throughout this section, assume that $A$ is a Lie $JC^*$-algebra with norm $\|\|$.
Definition 3.1. [10] A $\mathbb{C}$-linear mapping $D : A \rightarrow A$ is called a Lie JC$^*$-algebra derivation if $D$ satisfies

$$
D(x \circ y) = (Dx) \circ y + x \circ (Dy),
$$
$$
D([x, y]) = [Dx, y] + [x, Dy],
$$
$$
D(x^*) = D(x)^*
$$

for all $x, y \in A$.

Theorem 3.1. Suppose $r \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : A \rightarrow A$ be a mapping satisfying (2.2) and (2.5) such that

$$
\|f([x, y]) - [f(x), y] - [x, f(y)]\| \leq \theta(\|x\|^{2r} + \|y\|^{2r}),
$$

$$
\|f(x \circ y) - f(x) \circ y - x \circ f(y)\| \leq \theta(\|x\|^{2r} + \|y\|^{2r}),
$$

for all $x, y \in A$. Then the mapping $f : A \rightarrow A$ is a Lie JC$^*$-algebra derivation.

Proof. Assume $r > 1$. By the same reasoning as in the proof of Theorem 2.1, the mapping $f$ is a $\mathbb{C}$-linear. So we can consider $f(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$ for all $x \in A$.

It follows from (3.1) that

$$
\|f([x, y]) - [f(x), y] - [x, f(y)]\| = \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f([2^n x, 2^n y]) - [f(2^n x), 2^n y] - [2^n x, f(2^n y)]\|
$$

$$
\leq \lim_{n \rightarrow \infty} \frac{4^n \theta}{4^n} (\|x\|^{2r} + \|y\|^{2r}) = 0
$$

for all $x, y \in A$. Therefore we obtain

$$
f([x, y]) = [f(x), y] + [x, f(y)]
$$

for all $x, y \in A$.

It follows from (3.2) that

$$
\|f(x \circ y) - f(x) \circ y - x \circ f(y)\| = \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f(2^n x \circ 2^n y) - f(2^n x) \circ 2^n y - 2^n x \circ f(2^n y)\|
$$

$$
\leq \lim_{n \rightarrow \infty} \frac{4^n \theta}{4^n} (\|x\|^{2r} + \|y\|^{2r}) = 0
$$

for all $x, y \in A$. Then

$$
f(x \circ y) = f(x) \circ y + x \circ f(y)
$$

for all $x, y \in A$. And from (2.5) by the same explanation in the proof of Theorem 2.1 we derive that $f(x^*) = f(x)^*$ for all $x \in A$. Therefore we conclude $f : A \rightarrow A$ is a Lie JC$^*$-algebra derivation. Similarly, by the same arguments, we can obtain the result for the case $r < 1$. ☐
Theorem 3.2. Suppose $r \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying (2.2) and (2.8) such that

$$
\|f([x, y]) - [f(x), y] - [x, f(y)]\| \leq \theta(\|x\|^{r}, \|y\|^{r}),
$$

(3.3)

$$
\|f(x \circ y) - f(x) \circ y - x \circ f(y)\| \leq \theta(\|x\|^{r}, \|y\|^{r}),
$$

(3.4)

for all $x, y \in \mathcal{A}$. Then the mapping $f : \mathcal{A} \to \mathcal{A}$ is a Lie JC*-algebra derivation.

Proof. Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.1, the mapping $f$ is a $\mathbb{C}$-linear. So we can assume $f(x) = \lim_{n \to \infty} 2^{n}f\left(\frac{x}{2^{n}}\right)$ for all $x \in \mathcal{A}$.

It follows from (3.3) that

$$
\|f([x, y]) - [f(x), y] - [x, f(y)]\| = \lim_{n \to \infty} 4^{n}\|f\left(\frac{x}{2^{n}}, \frac{y}{2^{n}}\right) - [f(x), y] - [x, f(y)]\|
$$

$$
\leq \lim_{n \to \infty} 4^{n} \frac{\theta}{4^{nr}} (\|x\|^{r}, \|y\|^{r}) = 0
$$

for all $x, y \in \mathcal{A}$. Hence

$$
f([x, y]) = [f(x), y] + [x, f(y)]
$$

for all $x, y \in \mathcal{A}$.

It follows from (3.4) that

$$
\|f(x \circ y) - f(x) \circ y - x \circ f(y)\| = \lim_{n \to \infty} 4^{n}\|f\left(\frac{x}{2^{n}} \circ \frac{y}{2^{n}}\right) - f(x) \circ y - x \circ f(y)\|
$$

$$
\leq \lim_{n \to \infty} 4^{n} \frac{\theta}{4^{nr}} (\|x\|^{r}, \|y\|^{r}) = 0
$$

for all $x, y \in \mathcal{A}$. Therefore

$$
f(x \circ y) = f(x) \circ y + x \circ f(y)
$$

for all $x, y \in \mathcal{A}$.

And from (2.8) by the same explanation in the proof of Theorem 2.2 it is obtained that $f(x^{*}) = f(x)^{*}$ for all $x \in \mathcal{A}$. Therefore we conclude $f : \mathcal{A} \to \mathcal{A}$ is a Lie JC*-algebra derivation. Similarly, one can obtains the result for the case $r < 1$.

\[\square\]

References


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