Stability of Generalized Euler Differential Equations of First Order with Variable Coefficients

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1 Introduction

Let $X$ be a normed space over a scalar field $\mathbb{K}$, let $I$ be an open interval, and let $a_0,a_1,\ldots,a_{n-1}$ be fixed elements of $\mathbb{K}$. Consider the differential equation

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) + g(t) = 0, \quad \forall t \in I,$$

where $g(t)$ is a $n$ times continuously differentiable function $g : I \to X$ is given and the $n$ times continuously differentiable function $y : I \to X$ is unknown. As usual, equation (1.1) is said to be Hyers-Ulam stable if for any $n$ times continuously differentiable function $\tilde{y} : I \to X$ satisfying the inequality

$$\|\tilde{y}^{(n)}(t) + a_{n-1}\tilde{y}^{(n-1)}(t) + \cdots + a_1\tilde{y}'(t) + a_0\tilde{y}(t)\| \leq \varepsilon, \quad \forall t \in I,$$

for some constant $\varepsilon > 0$, there is a solution $y_0 : I \to X$ of equation (1.1) such that

$$\|y(t) - y_0(t)\| \leq K(\varepsilon), \quad \forall t \in I,$$

where $K(\varepsilon)$ is a function of $\varepsilon$ satisfying $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$. For more detailed definition of the Hyers-Ulam stability, we may refer to [1, 2, 3, 4, 5, 6, 7, 8].

Applications of Hyers-Ulam stability to certain types of ordinary differential equations were firstly investigated by Alsina and Ger [9]. They proved that if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leq \varepsilon$ for all $t \in I$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t) - f_0(t)| \leq 3\varepsilon$. Using the methods given in [9], Miura [10], Miura et al. [11], Miura et al. [12] and Takahasi et al. [13] proved that the differential equation $y'(t) = \lambda y(t)$ is Hyers-Ulam stable. In 2004, Jung [14] proved a similar result for the differential equation $\varphi(t)y'(t) = y(t)$. Further results for the nonhomogeneous linear differential equation of first order in the form of

$$y' + p(t)y + q(t) = 0,$$

have been investigated by Miura, Takahasi and Jung [15, 16, 17, 18]. In 2006, using matrix method, Jung [19] proved the Hyers-Ulam stability of first order linear differential equations with constant coefficients in the form of

$$\overrightarrow{y}'(t) = A \overrightarrow{y}(t) + \overrightarrow{b}(t),$$

where

$$\overrightarrow{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \overrightarrow{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

By adopting the idea of [19], Jung et al. [20] proved the Hyers-Ulam stability of Euler differential equations of first order in the form of

$$t \overrightarrow{y}'(t) = A \overrightarrow{y}(t) + \overrightarrow{b}(t).$$

(1.3)
In this paper we generally consider generalized Euler differential equations of first order with variable coefficients in the form of

\[ t \vec{y}'(t) = A(t) \vec{y}(t) + \vec{b}(t), \tag{1.4} \]

where

\[ \vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \vec{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix} \]

\[ A(t) = (a_{jk}(t))_{n \times n} \text{ and } a_{jk}(t) : \mathbb{R}^+ \to \mathbb{C}^n \text{ are continuous and uniformly bounded functions for all } j,k = 1, \ldots, n. \]

Following the idea of [20, 21] we prove the Hyers-Ulam stability of equation (1.4). Furthermore, our results can be applied to equation (1.3) so that the related results by Jung et al. [20] are generalized.

### 2 Main Results

Throughout this paper, let \((\mathbb{C}^n, \|\cdot\|)\) be a complex normed space and let \(\mathbb{C}^{n \times n}\) be a vector space consisting of all \((n \times n)\) complex matrices. Define the vector norm \(\|\cdot\|\) as \(\|\vec{x}\| = \max\{ |x_1|, |x_2|, \ldots, |x_n| \}\) for all \(\vec{x} \in \mathbb{C}^n\). Then it is easy to see that \((\mathbb{C}^n, \|\cdot\|)\) is a Banach space and the matrix norm being subject to the vector norm \(\|\cdot\|\) can be obtained as

\[ \|A\| = \sup_{\|\vec{x}\|=1} \|A\vec{x}\| = \max_{1 \leq j \leq n} \sum_{k=1}^{n} |a_{jk}|, \quad \forall A := (a_{jk})_{n \times n} \in \mathbb{C}^{n \times n}. \]

**Definition 2.1 (cf. [22]).** Let \(A(t)\) be a piecewise continuous \(n \times n\) matrix valued function defined on an interval \(J = (-\infty, \infty)\). The linear differential equation

\[ \vec{y}'(t) = A(t) \vec{y}(t) \tag{2.1} \]

is said to have an exponential dichotomy on \(J\) if there are projections \(P(t)\), for all \(t \in J\), and positive constants \(K_1, K_2, \alpha_1, \alpha_2\) such that

\[ Y(t)Y^{-1}(s)P(s) = P(t)Y(t)Y^{-1}(s), \quad \forall t, s \in J, \tag{2.2} \]

\[ \|Y(t)Y^{-1}(s)P(s)\| \leq K_1 e^{-\alpha_1(t-s)}, \quad \forall t, s \in J, \quad t \geq s, \tag{2.3} \]

and

\[ \|Y(t)Y^{-1}(s)(I - P(s))\| \leq K_2 e^{-\alpha_2(s-t)}, \quad \forall t, s \in J, \quad t \leq s. \tag{2.4} \]

Here \(Y(t)\) is any fundamental matrix for equation (2.1). Note that \(K_1, K_2\) are called constants and \(\alpha_1, \alpha_2\) exponents associated with the dichotomy.
Now, we give our main result as follows:

**Theorem 2.1.** Suppose that the linear differential equation \( \ddot{y}(t) = A(e^t) \dot{y}(t) \) has an exponential dichotomy on \( J \) with projections \( \tilde{p}(s) \), constants \( K_1, K_2 \) and exponents \( \bar{\alpha}_1, \bar{\alpha}_2 \). If \( \dot{y}(t) : \mathbb{R}^+ \to \mathbb{C}^n \) is a continuously differentiable vector function satisfying the differential inequality

\[
\|t \dot{y}'(t) - A(t) \dot{y}(t) - \vec{b}(t)\| \leq \varepsilon,
\]  

(2.5)

for all \( t \in \mathbb{R}^+ \), for some \( \varepsilon > 0 \), where \( \vec{b}(t) : \mathbb{R}^+ \to \mathbb{C}^n \) is a continuous vector function, \( A(t) = (a_{jk}(t))_{n \times n} \), \( a_{jk}(t) : \mathbb{R}^+ \to \mathbb{C}^n \) are continuous and uniformly bounded functions, then there exists a unique solution \( \bar{y}_0(t) : \mathbb{R}^+ \to \mathbb{C}^n \) of (1.4) and a positive constant \( L \) such that

\[
\| \dot{y}(t) - \bar{y}_0(t) \| \leq \widetilde{L} \varepsilon,
\]

(2.6)

for all \( t \in \mathbb{R}^+ \), where \( \widetilde{L} = K_1 \bar{\alpha}_1^{-1} + K_2 \bar{\alpha}_2^{-1} \).

Before providing the proof of this theorem, we first present the following Lemma 2.2 (for a proof see [21]).

**Lemma 2.2 (cf. [21]).** Suppose that the linear differential equation (2.7) has an exponential dichotomy on \( J \) with projections \( P(t) \), constants \( K_1, K_2 \) and exponents \( \alpha_1, \alpha_2 \). If \( \bar{y}(t) : \mathbb{R} \to \mathbb{C}^n \) is a continuously differentiable vector function satisfying the differential inequality

\[
\| \bar{y}'(t) - A(t) \bar{y}(t) - \vec{b}(t)\| \leq \varepsilon,
\]

(2.7)

for all \( t \in \mathbb{R} \) and for some \( \varepsilon > 0 \), where \( \vec{b}(t) : \mathbb{R} \to \mathbb{C}^n \) is a continuous vector function, \( A(t) = (a_{jk}(t))_{n \times n} \), \( a_{jk}(t) : \mathbb{R} \to \mathbb{C}^n \) are continuous and uniformly bounded functions, then there exists a unique solution \( \bar{y}_0(t) : \mathbb{R} \to \mathbb{C}^n \) of the linear differential equation \( \bar{y}'(t) = A(t) \bar{y}(t) + \vec{b}(t) \) and a positive constant \( L \) such that

\[
\| \bar{y}(t) - \bar{y}_0(t) \| \leq L \varepsilon,
\]

(2.8)

where \( L = K_1 \alpha_1^{-1} + K_2 \alpha_2^{-1} \).

**Proof. (Proof of the Theorem 2.1).** Let \( t = e^t \) and \( \bar{z} : \mathbb{R} \to \mathbb{C}^n \) be given by \( \bar{z}(\tau) = \bar{y}(e^\tau) \). Then

\[
\bar{z}'(\tau) = \frac{d \bar{z}(\tau)}{d \tau} = e^\tau \frac{d \bar{y}}{d t}(e^\tau) = t \dot{y}(t)
\]

and

\[
\bar{z}''(\tau) - A(e^\tau) \bar{z}(\tau) - \vec{b}(e^\tau) = t \ddot{y}(t) - A(t) \dot{y}(t) - \vec{b}(t).
\]
From the assumption (2.5), we obtain
\[ \|z'(\tau) - A(e^\tau)z(\tau) - b(e^\tau)\| \leq \varepsilon, \]
for all \( \tau \in \mathbb{R} \) and for some \( \varepsilon > 0 \).

By the assumption of the Lemma 2.2 and this theorem, there exists a differential vector function \( z_0(t) : \mathbb{R} \to \mathbb{C}^n \) such that
\[ z'(\tau) = A(e^\tau)z(\tau) + b(e^\tau), \]
and
\[ \|z(\tau) - z_0(\tau)\| \leq \tilde{L} \varepsilon, \quad \forall \tau \in \mathbb{R}. \]

Then the function \( y(t) = z_0(\ln t) \) satisfies
\[ y'(t) = \frac{1}{t} [A(e^{\ln t})z_0(\ln t) + b(e^{\ln t})] = \frac{1}{t} [A(t)y_0(\ln t) + b(t)], \]
i.e.,
\[ ty'(t) = A(t)y(t) + b(t) \]
with
\[ \|y(t) - y_0(t)\| = \|z(\ln t) - z_0(\ln t)\| \leq \tilde{L} \varepsilon \]
for all \( t \in \mathbb{R}^+ \). This completes the proof of the Theorem.

To give a corollary of Theorem 2.1, we need the following Lemmas 2.3 and 2.4, which were proved in [21]:

**Lemma 2.3 (cf. [21]).** Suppose that \( A \in \mathbb{C}^{n \times n} \) is a nonsingular matrix whose eigenvalues have nonzero real parts. Then the homogeneous differential equation \( \ddot{y}(t) = A \dot{y}(t) \) has an exponential dichotomy on \( J \).

**Proof.** This Lemma was proved in [21], however we prove it again for completeness and convenience. Assume that \( A \) has \( d \) distinct eigenvalues \( \lambda_\mu \) with algebraic multiplicity \( n_\mu \) and geometric multiplicity \( m_\mu \), where \( \mu \in \{1, 2, \cdots, d\} \) and denote by \( \text{Re}(\lambda_\mu) \) their real part. Choose a nonsingular matrix \( N \in \mathbb{C}^{n \times n} \) such that \( J = N^{-1}AN \), where \( J \) is the Jordan form matrix of the form

\[ J = \begin{pmatrix}
J_{11} & & & \\
& \ddots & & \\
& & J_{\mu\nu} & \\
& & & J_{d \times m_\mu}
\end{pmatrix}, \quad J_{\mu\nu} = \begin{pmatrix}
\lambda_\mu & 1 & \cdots & 0 \\
0 & \lambda_\mu & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_\mu
\end{pmatrix}, \]

where \( J_{11}, \ldots, J_{d \times m_\mu} \) are Jordan blocks corresponding to the eigenvalues \( \lambda_1, \ldots, \lambda_d \).
and the Jordan block \( J_{\mu \nu} \) is an \( \nu \times \nu \) matrix for each \( \mu \in \{1, 2, \cdots, d\} \) and \( \nu \in \{1, 2, \cdots, m_\mu\} \), and \( \sum_{\nu=1}^{m_\nu} \nu = n_\mu \) for any \( \mu \in \{1, 2, \cdots, d\} \).

Note that \( e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots \), where \( I \) denotes the unite matrix in \( \mathbb{C}^{n \times n} \). Then the fundamental matrix solution \( X(t) \) for the differential equation \( \dot{y}(t) = A y(t) \) can be expressed in the form

\[
X(t) = e^{At} = Ne^{Jt}N^{-1}.
\]

If we set

\[
e^{Jt} = \begin{pmatrix}
e^{J_{11}t} & & \cdots & & \\
& \ddots & & \cdots & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & e^{J_{mm}t}
\end{pmatrix}
\]

\[
e^{J_{\mu \nu}t} = e^{\lambda_\mu t}
\]

Choose a diagonal matrix \( P = \text{diag}(P_{11}, \cdots, P_{\mu \nu}, \cdots, P_{dm_\mu}) \), where the Jordan block \( P_{\mu \nu} \) is an \( \nu \times \nu \) zero matrix or unite matrix corresponding with \( \text{Re}(\lambda_\mu) < 0 \) or \( \text{Re}(\lambda_\mu) > 0 \) for each \( \mu \in \{1, 2, \cdots, d\} \) and \( \nu \in \{1, 2, \cdots, m_\mu\} \).

Let \( Q := NPN^{-1} \), then

\[
QX(t)X^{-1}(s) = QNe^{J(t-s)}N^{-1} = NPN^{-1}Ne^{J(t-s)}N^{-1} = Ne^{J(t-s)}PN^{-1} = Ne^{J(t-s)}N^{-1}PN^{-1}
\]

\[
= X(t)X^{-1}(s)Q, \quad \forall t, s \in J,
\]

and for all \( t \geq s \), it follows that

\[
\|X(t)X^{-1}(s)Q\| \leq \|Ne^{J(t-s)}N^{-1}PN^{-1}\| \leq \|N\|\|N^{-1}\|\|e^{J(t-s)}P\|
\]

\[
\leq \|N\|\|N^{-1}\|\max_{\text{Re}(\lambda_\mu) < 0} e^{\lambda_\mu(t-s)} \max_{\text{Re}(\lambda_\mu) < 0} \sum_{i=0}^{m_\mu} \frac{(t-s)^i}{i!}. \quad (2.11)
\]

For a sufficiently small nonnegative number \( \delta \), we know that

\[
M_1 := e^{-\delta(t-s)} \max_{\text{Re}(\lambda_\mu) < 0} \sum_{i=0}^{m_\mu} \frac{(t-s)^i}{i!}
\]

is bounded for all \( t \geq s \). Let \( \lambda^- := \max_{\text{Re}(\lambda_\mu) < 0} \{\lambda_\mu\} + \delta \), then by (2.11), we have

\[
\|X(t)X^{-1}(s)Q\| \leq \|N\|\|N^{-1}\| M_1 e^{(t-s)\lambda^-}, \quad t \geq s. \quad (2.12)
\]
Similarly for all $t \leq s$ and $\delta > 0$, we have
\[
\|X(t)X^{-1}(s)(I - Q)\| = \|Ne^{J(t-s)}N^{-1}(I - NPN^{-1})\| \\
= \|Ne^{J(t-s)}(I - P)N^{-1}\| \\
\leq \|N\|\|N^{-1}\|M_2e^{(t-s)\lambda^+},
\] (2.13)
where
\[
M_2 := e^{-\delta(s-t)}\max_{\text{Re}(\lambda_\mu) > 0} \sum_{i=0}^{m_\mu} \frac{(s-t)^i}{i!} \quad \text{and} \quad \lambda^+ := \max_{\text{Re}(\lambda_\mu) > 0} \{\lambda_\mu\} - \delta.
\]
Thus, by (2.10), (2.11), (2.12) and (2.13), we know that $\vec{y}'(t) = A\vec{y}(t)$ has an exponential dichotomy on $J$ with projections $P(t) \equiv Q$, constants $\tilde{K}_1 = \|N\|\|N^{-1}\|M_1$, $\tilde{K}_2 = \|N\|\|N^{-1}\|M_2$ and exponents $\tilde{\alpha}_1 = -\lambda^-$, $\tilde{\alpha}_2 = \lambda^+$. This completes the proof of the Lemma. 

**Lemma 2.4** (cf. [21]). Suppose that $A \in \mathbb{C}^{n \times n}$ is a nonsingular matrix whose eigenvalues have nonzero real parts. If $\vec{y}(t) : \mathbb{R} \rightarrow \mathbb{C}^n$ is a continuously differentiable vector function satisfying the differential inequality
\[
\|\vec{y}'(t) - A\vec{y}(t) - \vec{b}(t)\| \leq \varepsilon,
\] (2.14)
for all $t \in \mathbb{R}$, for some $\varepsilon > 0$, where $\vec{b}(t) : \mathbb{R} \rightarrow \mathbb{C}^n$ is a continuous vector function, then there exists a unique solution $\vec{y}_0(t) : \mathbb{R} \rightarrow \mathbb{C}^n$ of (1.3) and a positive constant $\tilde{M}$ such that
\[
\|\vec{y}(t) - \vec{y}_0(t)\| \leq \tilde{M}\varepsilon,
\] (2.15)
for all $t \in \mathbb{R}$.

**Corollary 2.5.** Suppose that $A \in \mathbb{C}^{n \times n}$ is a nonsingular matrix whose eigenvalues have nonzero real parts. If $\vec{y}(t) : \mathbb{R}^+ \rightarrow \mathbb{C}^n$ is a continuously differentiable vector function satisfying the differential inequality
\[
\|t\vec{y}'(t) - A\vec{y}(t) - \vec{b}(t)\| \leq \varepsilon,
\] (2.16)
for all $t \in \mathbb{R}^+$, for some $\varepsilon > 0$, where $\vec{b}(t) : \mathbb{R}^+ \rightarrow \mathbb{C}^n$ is a continuous vector function, then there exists a unique solution $\vec{y}_0^+(t) : \mathbb{R}^+ \rightarrow \mathbb{C}^n$ of (1.3) and a positive constant $\tilde{M}$ such that
\[
\|\vec{y}(t) - \vec{y}_0^+(t)\| \leq \tilde{M}\varepsilon,
\] (2.17)
for all $t \in \mathbb{R}^+$.

**Proof.** The proof of Corollary 2.5 is similar to the proof of Theorem 2.1 and by Lemmas 2.3 and 2.4 the differential equation (1.3) has the Hyers-Ulam stability with $\tilde{M} = \|N\|\|N^{-1}\|((\frac{1}{\lambda^+} + \frac{1}{\lambda^-})$. This completes the proof of the corollary.
3 Some Examples

Example 3.1. Consider a system of generalized Euler differential equations of first order with variable coefficients in the following form
\[
\overrightarrow{y}'(t) = A(t) \overrightarrow{y}(t) + \overrightarrow{b}(t),
\]  
(3.1)

where
\[
\overrightarrow{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} 2 + \cos \ln t & 4 \\ 0 & -2 + \cos \ln t \end{pmatrix}, \quad \overrightarrow{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.
\]

By Theorem 2.1, putting \( t = e^\tau \), we obtain
\[
A(e^\tau) = \begin{pmatrix} 2 + \cos \tau & 4 \\ 0 & -2 + \cos \tau \end{pmatrix}.
\]

Following some computation, we have
\[
Y(\tau) = \begin{pmatrix} e^{2\tau + \sin \tau} & e^{-2\tau + \sin \tau} - e^{2\tau + \sin \tau} \\ 0 & e^{-2\tau + \sin \tau} \end{pmatrix},
\]

and
\[
Y(\tau)Y^{-1}(s) = \begin{pmatrix} e^{2(\tau - s) + \sin \tau - \sin s} & e^{-2(\tau - s) + \sin \tau - \sin s} - e^{2(\tau - s) + \sin \tau - \sin s} \\ 0 & e^{-2(\tau - s) + \sin \tau - \sin s} \end{pmatrix}.
\]

Choosing \( P(\tau) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \), we can verify that
\[
P(\tau)Y(\tau)Y^{-1}(s) = Y(\tau)Y^{-1}(s)P(s) = \begin{pmatrix} 0 & e^{-2(\tau - s) + \sin \tau - \sin s} \\ 0 & e^{-2(\tau - s) + \sin \tau - \sin s} \end{pmatrix},
\]

and
\[
\|Y(\tau)Y^{-1}(s)P(s)\| \leq e^{2(\tau - s)}, \quad \tau \geq s,
\]

\[
\|Y(\tau)Y^{-1}(s)(I - P(s))\| \leq 2e^{2(\tau - s)}, \quad \tau \leq s.
\]

Thus the differential equation \( \overrightarrow{y}'(\tau) = A(e^\tau) \overrightarrow{y}(\tau) \) has an exponential dichotomy on \( J \) with \( K_1 = e^2, K_2 = 2e^2, \alpha_1^{-1} = \alpha_2^{-1} = 2. \) By Theorem 2.1, equation (3.1) satisfies the Hyers-Ulam stability with \( L = K_1 \alpha_1^{-1} + K_2 \alpha_2^{-1} = \frac{3}{2}e^2. \)

Example 3.2. Consider a system of Euler differential equations of first order in the following form
\[
\overrightarrow{y}'(t) = A \overrightarrow{y}(t) + \overrightarrow{b}(t),
\]  
(3.2)
where
\[
\vec{y}(t) = \begin{pmatrix}
y_1(t) \\
y_2(t) \\
y_3(t)
\end{pmatrix}, \quad A = \begin{pmatrix}
-3 & -4 & 2 \\
-3 & -5 & 3 \\
-7 & -10 & 6
\end{pmatrix}, \quad \vec{b}(t) = \begin{pmatrix}
b_1(t) \\
b_2(t) \\
b_3(t)
\end{pmatrix}.
\]

Since the matrix \( A \) has three eigenvalues \(-1, 1, \) and \(-2\), there exist nonsingular matrices
\[
N = \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
1 & 2 & 3
\end{pmatrix}, \quad N^{-1} = \begin{pmatrix}
-1 & -4 & 2 \\
-1 & -1 & 1 \\
1 & 2 & -1
\end{pmatrix}, \quad J = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix},
\]
such that \( J = N^{-1}AN \). According to Corollary 2.5, equation (3.2) satisfies the Hyers-Ulam stability with \( \|N\|\|N^{-1}\|(\frac{M_1}{2} + \frac{M_2}{2}) = 84 \).

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**References**


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