Strong Convergence Theorems for Fixed Points of Nonexpansive Semigroups

Sun Young Cho† and Shin Min Kang‡,

†Department of Mathematics, Gyeongsang National University, Jinju 660-701, Republic of Korea
e-mail : ooly61@yahoo.co.kr
‡Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea
e-mail : smkang@gnu.ac.kr

Abstract : Let \( H \) be a real Hilbert space and \( F = \{ T(t) : t \geq 0 \} \) be a strongly continuous semigroup of nonexpansive mappings on \( H \) with a common fixed point. Let \( f : H \to H \) be an \( \alpha \)-contraction and \( A : H \to H \) be a strongly positive linear bounded self-adjoint operator with the coefficient \( \bar{\gamma} > 0 \). Assume that \( 0 < \gamma < \frac{\bar{\gamma}}{\alpha} \).

The implicit iterative scheme is given as follows:

\[
x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 1.
\]

It is proved that the sequence \( \{ x_n \} \) generated in the above iterative process converges strongly to a common fixed point \( p \in \bigcap_{t \geq 0} F(T(t)) \), where \( F(T(t)) \) denotes the fixed point of the nonexpansive mapping \( T(t) \). The point \( p \) also solves the variational inequality \( \langle (\gamma f - A)p, x - p \rangle \leq 0, \forall x \in \bigcap_{t \geq 0} F(T(t)) \).

Keywords : Nonexpansive mapping; Common fixed point; Variational inequality; Iterative method.

2010 Mathematics Subject Classification : 47H09; 47HJ25.

1 Introduction and Preliminaries

Throughout this paper, we denote by \( \mathbb{R}^+ \) the set of nonnegative real numbers.

Let \( H \) be a real Hilbert space, \( C \) be a nonempty closed convex subset of \( H \) and \( T \)
be a nonlinear mapping. Recall that $T : C \to C$ is said to be nonexpansive if
$$
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.
$$
A point $x \in C$ is said to be a fixed point of $T$ provided $Tx = x$. Denote by $F(T)$ the set of all fixed points of $T$, that is, $F(T) = \{x \in C : Tx = x\}$.

Let $F = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on a closed convex subset $C$ of a Hilbert space $H$, i.e.,

(a) for each $t \in \mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on $C$;
(b) $T(0)x = x$ for all $x \in C$;
(c) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(d) for each $x \in H$, the mapping $T(\cdot)s$ from $\mathbb{R}^+$ into $C$ is continuous.

We denote by $F$ the set of common fixed points of $F$, that is,
$$
F = \{x \in C : T(t)x = x, \ t > 0\} = \bigcap_{t > 0} F(T(t)).
$$
We know that $F$ is nonempty if $C$ is bounded (see [1]). In [2], Shioji and Takahashi proved the following theorem.

**Theorem ST.** Let $C$ be a closed convex subset of a Hilbert space $H$. Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$ such that $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $\lim_{n \to \infty} \alpha_n = 0$, $t_n \geq 0$ and $\lim_{n \to \infty} t_n = \infty$. Fix $u \in C$ and define a sequence $\{x_n\}$ in $C$ by
$$
x_n = \alpha_n u + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds, \quad \forall n \geq 1.
$$
Then $\{x_n\}$ converges strongly to the element of $F$ nearest to $u$.

Recently, Suzuki [3] improved the results of Shinoji and Takahashi [2] and proved the following theorem.

**Theorem S.** Let $C$ be a closed convex subset of a Hilbert space $H$. Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on $C$ such that $\bigcap_{t \geq 0} F(T(t)) \neq \emptyset$. Let $\alpha_n$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0$. Fix $u \in C$ and define a sequence $\{x_n\}$ in $C$ by
$$
x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1.
$$
Then $\{x_n\}$ converges strongly to the element of $F$ nearest to $u$. 
Recall that an operator $A$ is said to be \textit{strongly positive} on $H$ if there exists a constant $\gamma > 0$ such that
\[
\langle Ax, x \rangle \geq \gamma \|x\|^2, \quad \forall x \in H.
\]
Recall also that a self mapping $f$ is said to be an $\alpha$-\textit{contraction} on $H$ if there exists a constant $\alpha \in (0, 1)$ such that
\[
\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H.
\]
Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see \cite{4–8} and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping $T$ on a real Hilbert space $H$:
\[
\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.1}
\]
where $A$ is a linear bounded self-adjoint operator and $b$ is a given point in $H$.

In \cite{3}, it is proved that the sequence $\{x_n\}$ defined by the following iterative method:
\[
x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad \forall n \geq 0,
\]
converges strongly to the unique solution of the minimization problem (1.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Marino and Xu \cite{5} studied the following continuous scheme:
\[
x_t = t\gamma f(x_t) + (I - tA)Tx_t,
\]
where $f$ is an $\alpha$-contraction on $H$, $A$ is a strongly positive linear bounded self-adjoint operator and $\gamma > 0$ is a constant. They showed that $\{x_t\}$ converges strongly to a fixed point $\bar{x}$ of $T$. Also, in \cite{5}, they introduced a general iterative scheme by the viscosity approximation method which first was considered by Moudafi \cite{9}:
\[
x_0 \in H, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0
\]
and proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:
\[
\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T),
\]
which is the optimality condition for the minimization problem:
\[
\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),
\]
where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for any $x \in H$).
In this paper, motivated by the recent work announced in [2, 3, 5, 10–14], we consider the following implicit iterative scheme:

\[ x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 1, \]

where \( \gamma > 0 \) is a constant, \( f \) is an \( \alpha \)-contraction on \( H \), \( A \) is a strongly positive linear bounded self-adjoint operator on \( H \) and prove that the sequence \( \{x_n\} \) generated in the above iterative process converge strongly to a common fixed point \( p \in F \). Also, we show that the point \( p \) solves the variational inequality:

\[ \langle (\gamma f - A)p, x - p \rangle \leq 0, \quad \forall x \in F. \]

The results presented in this paper mainly improve and extend the corresponding results announced in Shioji and Takahashi [2], Suzuki [3] and Xu [14].

In order to prove our main result, we need the following concepts and lemmas.

Recall that a space \( X \) satisfies Opial’s condition ([15]) if, for each sequence \( \{x_n\} \) in \( X \) which converges weakly to point \( x \in X \),

\[ \liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in X \quad (y \neq x). \]

**Lemma 1.1** ([5]). Assume that \( A \) is a strongly positive linear bounded self-adjoint operator on a Hilbert space \( H \) with the coefficient \( \bar{\gamma} > 0 \) and \( 0 < \rho \leq \|A\|^{-1} \). Then \( \|I - \rho A\| \leq 1 - \rho \bar{\gamma} \).

**Lemma 1.2.** Let \( H \) be a Hilbert space, \( C \) be a closed convex subset of \( H \), \( f : H \to H \) be an \( \alpha \)-contraction and \( A \) be a strongly positive linear bounded operator with the coefficient \( \bar{\gamma} > 0 \). Then, for any \( 0 < \gamma < \frac{\bar{\gamma}}{\alpha} \), we see that

\[ \langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma \alpha)\|x - y\|^2, \quad \forall x, y \in H. \quad (1.2) \]

That is, \( A - \gamma f \) is strongly monotone with the coefficient \( \bar{\gamma} - \alpha \gamma \).

**Proof.** From the definition of strongly positive linear bounded operators, we see that

\[ \langle x - y, A(x - y) \rangle \geq \bar{\gamma}\|x - y\|^2. \]

On the other hand, we have

\[ \langle x - y, \gamma f x - \gamma f y \rangle \leq \gamma \alpha\|x - y\|^2. \]

It follows that

\[ \langle x - y, (A - \gamma f)x - (A - \gamma f)y \rangle = \langle x - y, A(x - y) \rangle - \langle x - y, \gamma f x - \gamma f y \rangle \geq (\bar{\gamma} - \gamma \alpha)\|x - y\|^2, \quad \forall x, y \in H. \]

This completes the proof. \( \square \)

**Remark 1.3.** Taking \( \gamma = 1 \) and \( A = I \) (: the identity mapping), we have the following inequality:

\[ \langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \alpha)\|x - y\|^2, \quad \forall x, y \in H. \]

Furthermore, if \( f \) is a nonexpansive mapping in (1.2), then we have

\[ \langle x - y, (I - f)x - (I - f)y \rangle \geq 0, \quad \forall x, y \in H. \quad (1.3) \]
2 Main Results

Now, we are ready to give our main results.

**Theorem 2.1.** Let $H$ be a real Hilbert space and $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on $H$ such that $F \neq \emptyset$. Let $f : H \to H$ be an $\alpha$-contraction and $A : H \to H$ a strongly positive linear bounded self-adjoint operator with the coefficient $\gamma > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0$. Define a sequence $\{x_n\}$ in the following manner:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad \forall n \geq 1. \quad (2.1)$$

Then $\{x_n\}$ converges strongly to $p \in F$ which solves the following variational inequality:

$$\langle (A - \gamma f)p, p - x \rangle \leq 0, \quad \forall x \in F. \quad (2.2)$$

**Proof.** First, we show that the fixed point equation (2.1) is well defined. For any $n \geq 1$, define a mapping $T_n$ as follows:

$$T_nx = \alpha_n \gamma f(x) + (I - \alpha_n A)T(t_n)x.$$

It follows that

$$\|T_nx - T_ny\| = \|\alpha_n \gamma (f(x) - f(y)) + (I - \alpha_n A)(T(t_n)x - T(t_n)y)\|
\leq \alpha_n \gamma \|x - y\| + (1 - \alpha_n \gamma)\|x - y\|
= (1 - \alpha_n (\gamma - \alpha \gamma))\|x - y\|, \quad \forall x, y \in H.$$

Hence $T_n$ has a unique fixed point $x_n$, which uniquely solves the fixed point equation

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n.$$

The uniqueness of the solution of the variational inequality (2.2) is a consequence of the strong monotonicity of $A - \gamma f$. Suppose that $p, q \in F$ are solutions to (2.2). It follows that

$$\langle (A - \gamma f)p, p - q \rangle \leq 0 \quad (2.3)$$

and

$$\langle (A - \gamma f)q, q - p \rangle \leq 0. \quad (2.4)$$

Adding up (2.3) and (2.4), one obtains that

$$\langle (A - \gamma f)p - (A - \gamma f)q, p - q \rangle \leq 0, \quad \forall x \in F.$$

From Lemma 1.2, one sees that $p = q$. Next, we use $p$ to denote the unique solution of the variational inequality (2.2). Observing $\lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0$, we may assume, without loss of generality, that $\alpha_n < \|A\|^{-1}$ for all $n \geq 1$. From Lemma 1.1, we know that, if $0 < \alpha_n \leq \|A\|^{-1}$, then $\|I - \alpha_n A\| \leq 1 - \alpha_n \gamma$. \hfill \blacksquare
Next, we show that \( \{x_n\} \) is bounded. Fixing \( x \in F \), we have
\[
\|x_n - x\|^2 = (\alpha_n(\gamma f(x_n) - Ax) + (I - \alpha_n A)(T(t_n)x_n - x), x_n - x)
\]
\[
= \alpha_n \gamma \langle f(x_n) - f(x), x_n - x \rangle + \alpha_n \langle f(x) - Ax, x_n - x \rangle
\]
\[
+ \langle (I - \alpha_n A)(T(t_n)x_n - x), x_n - x \rangle,
\]
from which it follows that
\[
\|x_n - x\|^2 \leq \frac{1}{\gamma - \alpha \gamma} \langle \gamma f(x) - Ax, x_n - x \rangle.
\]  \text{(2.5)}
That is,
\[
\|x_n - x\| \leq \frac{1}{\gamma - \alpha \gamma} \| \gamma f(x) - Ax \|.
\]
This implies that \( \{x_n\} \) is bounded. Let \( \{x_{n_j}\} \) be an arbitrary subsequence of \( \{x_n\} \). Then there exists a subsequence \( \{x_{n_{j_i}}\} \) of \( \{x_{n_j}\} \) which converges weakly to a point \( p \).

Next, we show that \( p \in F \). In fact, put \( z_j = x_{n_{j_i}}, \gamma_j = \alpha_{n_{j_i}} \) and \( s_j = t_{n_{j_i}} \) for all \( j \geq 1 \). Fix \( t > 0 \). Noticing that
\[
\|z_j - T(t)p\| \leq \sum_{k=0}^{[\frac{s_j}{t}] - 1} \|T((k + 1)s_j)z_j - T(k s_j)z_j\|
\]
\[
+ \left\| T \left( \left[ \frac{t}{s_j} \right] s_j \right) z_j - T \left( \left[ \frac{t}{s_j} \right] s_j \right) p \right\| + \left\| T \left( \left[ \frac{t}{s_j} \right] s_j \right) p - T(t)p \right\|
\]
\[
\leq \left[ \frac{t}{s_j} \right] \|T(s_j)z_j - z_j\| + \|z_j - p\| + \left\| T \left( t - \left[ \frac{t}{s_j} \right] s_j \right) p - p \right\|
\]
\[
\leq \frac{\gamma_j t}{s_j} \|AT(s_j)z_j - \gamma f(z_j)\| + \|z_j - p\|
\]
\[
+ \max\{\|T(s)p - p\| : 0 \leq s \leq s_j\}, \quad \forall j \geq 1,
\]
we have
\[
\liminf_{j \to \infty} \|z_j - T(t)p\| \leq \liminf_{j \to \infty} \|z_j - p\|.
\]
From Opial’s condition, we have \( T(t)p = p \). It follows that \( p \in F \). In the inequality (2.5), replacing \( p \) with \( x \), we have
\[
\|z_j - p\|^2 \leq \frac{1}{\gamma - \alpha \gamma} \langle \gamma f(p) - Ap, z_j - p \rangle.
\]  \text{(2.6)}
Taking the limit as \( j \to \infty \) in (2.6), we obtain that \( \lim_{j \to \infty} \|z_j - p\| = 0 \). Since the subsequence \( \{x_{n_j}\} \) is arbitrary, it follows that \( \{x_n\} \) converges strongly to \( p \).

Finally, we prove that \( p \in F \) is a solution of the variational inequality (2.2). From (2.1), we see that
\[
(A - \gamma f)x_n = -\frac{1}{\alpha_n}(I - \alpha_n A)(x_n - T(t_n)x_n).
\]
In view of (1.3), we see that

\[
\langle (A - \gamma f)x_n, x_n - x \rangle = -\frac{1}{\alpha_n} \langle (I - \alpha_n A)(x_n - T(t_n)x_n), x_n - x \rangle \\
= -\frac{1}{\alpha_n} \langle (I - T(t_n))x_n - (I - T(t_n))x, x_n - x \rangle \\
+ \langle A(I - T(t_n))x_n, x_n - x \rangle \\
\leq \langle A(I - T(t_n))x_n, x_n - x \rangle \\
= \langle A(\alpha_n \gamma f(x_n) - \alpha_n AT(t_n)x_n), x_n - x \rangle \\
= \alpha_n \langle A(\gamma f(x_n) - AT(t_n)x_n), x_n - x \rangle.
\]

It follows that

\[
\langle (A - \gamma f)p, p - x \rangle \leq 0, \quad \forall x \in F.
\]

That is, \( p \in F \) is the unique solution to the variational inequality (2.2). This completes the proof. \( \square \)

Taking \( \gamma = 1 \) and \( A = I \) in Theorem 2.1, we have the following result.

**Corollary 2.2.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \). Let \( \{T(t) : t \geq 0\} \) be a strongly continuous semigroup of nonexpansive mappings of \( C \) into itself such that \( F \neq \emptyset \) and \( f : C \to C \) be an \( \alpha \)-contraction. Let \( \{\alpha_n\} \) and \( \{t_n\} \) be sequences of real numbers satisfying \( 0 < \alpha_n < 1, \ t_n > 0 \) and \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0 \). Define a sequence \( \{x_n\} \) in the following manner:

\[
x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad \forall n \geq 1. \tag{2.7}
\]

Then \( \{x_n\} \) converges strongly to \( p \in F \) which solves the following variational inequality:

\[
\langle (f - I)p, x - p \rangle \leq 0, \quad \forall x \in F. \tag{2.8}
\]

**Remark 2.3.** If \( f(x) = u \in C \), a fixed point, for all \( x \in C \), then Corollary 2.2 is reduced to Suzuki’s results [3]. Corollary 2.2 also can be viewed as an improvement of the corresponding results in Shioji and Takahashi [2].

**Remark 2.4.** It is of interest to improve Theorem 2.1 to some Banach space.

**References**


(Received 23 April 2011)
(Accepted 20 May 2011)