Thai Journal of Mathematics
Volume 11 (2013) Number 3 : 683–695
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

Interval Valued Intuitionistic Fuzzy Ideals of Regular LA-Semigroups

Naveed Yaqoob

Department of Mathematics
Quaid-i-Azam University, Islamabad, Pakistan
e-mail : nayaqoob@ymail.com

Abstract : In this paper, the concept of interval valued intuitionistic fuzzy sets is applied to regular LA-semigroups and the characterizations of regular LA-semigroups in terms of interval valued intuitionistic fuzzy left ideal [right ideal, generalized bi-ideal and bi-ideal] are given.

Keywords : interval valued intuitionistic fuzzy sets; bi-ideals; interior ideals; interval valued intuitionistic fuzzy ideals.
2010 Mathematics Subject Classification : 20M10; 20M12; 03E72.

1 Introduction

A fuzzy subset μ of a set S is a function from S to the closed interval $[0, 1]$. This concept of a fuzzy set was introduced by Zadeh [1], in 1965. Since its inception, the theory has developed in many directions and found applications in a wide variety of fields. Kuroki [2] introduced the notion of fuzzy ideals and fuzzy bi-ideals in semigroups. Faisal et al. [3] applied fuzzy set theory on Γ-AG**-groupoids also see [4]. Zadeh [5] made an extension of the concept of a fuzzy subset by an interval valued fuzzy subset, i.e., a fuzzy subset with an interval valued membership function. Interval valued fuzzy subsets have many applications in several areas. Biswas [6] worked on Rosenfeld's fuzzy subgroups with interval valued membership function. In [7], Narayanan and Manikantan, introduced the concept of an interval valued fuzzy left (right, two-sided, interior, bi-) ideal generated by an interval valued fuzzy subset in semigroups.

Atanassov introduced the idea of defining a fuzzy set by ascribing a membership degree and a non-membership degree separately in such a way that sum of the two degrees must not exceed one. Such a pair was given the name of Intuitionistic Fuzzy Sets [8]. Atanassov et al. [9, 10] introduced the notion of interval valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval valued fuzzy sets. Several mathematicians applied the concept of interval valued intuitionistic fuzzy sets to algebraic structures. Akram and Dudek [11] introduced the notion of interval valued intuitionistic fuzzy lie ideals of lie algebras and in [12, 13] Akram et al. applied the theory of interval valued intuitionistic fuzzy sets to K-algebras. Yaqoob et al. [14] characterized left almost semigroups by their interval valued fuzzy ideals also see [15, 16]. The theory of interval valued intuitionistic (S,T)-fuzzy sets applied to different algebraic structures by Akram [17], Hedayati [18, 19], Lee et al. [20] and Zhan et al. [21].

This paper concerned the relationship between interval valued intuitionistic fuzzy sets and regular LA-semigroups. The left almost semigroup abbreviated as an LA-semigroup, was first introduced by Kazim and Naseerudin [22]. They generalized some useful results of semigroup theory. Every LA-semigroup satisfies the law $(ab)c = (cb)a$, and this law is known as left invertive law. Despite the fact, the structure is non-associative and non-commutative. It nevertheless possesses many interesting properties which we usually find in commutative and associative algebraic structures. It is a useful nonassociative structure with wide applications in theory of flocks. In this paper, the concept of interval valued intuitionistic fuzzy sets is applied to regular LA-semigroups.

2 Preliminaries and Basic Definitions

Let S be an LA-semigroup. From [23], a non-empty subset A of S is called an LA-subsemigroup of S if $ab \in A$ for all $a, b \in A$. A non-empty subset L of S is called a left ideal of S if $SL \subseteq L$ and a non-empty subset R of S is called a right ideal of S if $RS \subseteq R$. A non-empty subset I of S is called an ideal of S if I is both a left and a right ideal of S. A subset A of S is called an interior ideal of S if $(SA)S \subseteq A$. An LA-subsemigroup A of S is called a bi-ideal of S if $(AS)A \subseteq A$. A subset Q of S is called a quasi-ideal of S if $QS \cap SQ \subseteq Q$. An LA-semigroup S is said to be regular if for all $a \in S$, there exists $x \in S$ such that $a = (ax)a$.

In an LA-semigroup the medial law holds:

$$(ab)(cd) = (ac)(bd) \text{ for all } a, b, c, d \in S.$$

In an LA-semigroup S with left identity, the paramedial law holds:

$$(ab)(cd) = (dc)(ba) \text{ for all } a, b, c, d \in S.$$

If an LA-semigroup contain a left identity then the following law holds:

$$a(bc) = b(ac) \text{ for all } a, b, c \in S.$$

Now we will recall the concept of interval valued fuzzy sets. An interval number is $\overline{a} = [a^-, a^+]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$, i.e.,

$$D[0,1] = \{\overline{a} = [a^-, a^+] : a^- \leqslant a^+, \text{ for } a^-, a^+ \in I\}.$$

We define the operations "$\leqslant$", "$=$", "$<$", "min" and "max" in case of two elements in $D[0,1]$. We consider two elements $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$ in $D[0,1]$. Then

(1) $\overline{a} \leqslant \overline{b}$ if and only if $a^- \leqslant b^-$ and $a^+ \leqslant b^+$,

(2) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,

(3) $\overline{a} < \overline{b}$ if and only if $\overline{a} \leqslant \overline{b}$ and $\overline{a} \neq \overline{b}$,

(4) $\overline{a} \cap \overline{b} = \min\{\overline{a}, \overline{b}\} = \overline{a} \wedge \overline{b} = [a^- \wedge b^-,\ a^+ \wedge b^+]$,

(5) $\overline{a} \cup \overline{b} = \max\{\overline{a}, \overline{b}\} = \overline{a} \vee \overline{b} = [a^- \vee b^-,\ a^+ \vee b^+\}$.

It is obvious that $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $\overline{0} = [0,0]$ as its least element and $\overline{1} = [1,1]$ as its greatest element.

Let X be an ordinary set. An interval valued fuzzy set B on X is defined as

$$B = \left\{ \left\langle x, [\mu_B^-(x), \mu_B^+(x)] \right\rangle : x \in X \right\},$$

where $\mu_B^-(x) \leqslant \mu_B^+(x)$, for all $x \in X$. Then the ordinary fuzzy sets $\mu_B^- : X \to [0,1]$ and $\mu_B^+ : X \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\widetilde{\mu}$, respectively. Let $\widetilde{\mu}_B(x) = [\mu_B^-(x), \mu_B^+(x)]$, then

$$B = \{\langle x, \widetilde{\mu}_B(x)\rangle : x \in X\},$$

where $\widetilde{\mu}_B : X \longrightarrow D[0,1]$.

A mapping $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle : X \longrightarrow D[0,1] \times D[0,1]$ is called an interval valued intuitionistic fuzzy set (IIF-set, in short) in X if $\mu_A^-(x) + \gamma_A^-(x) \leq 1$ and $\mu_A^+(x) + \gamma_A^+(x) \leq 1$ for all $x \in X$. Where the mappings

$$\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)] : X \longrightarrow D[0,1] \text{ and } \widetilde{\gamma}_A(x) = [\gamma_A^-(x), \gamma_A^+(x)] : X \longrightarrow D[0,1],$$

denote the degree of membership (namely $\widetilde{\mu}_A(x)$) and the degree of non-membership (namely $\widetilde{\gamma}_A(x)$) of each element $x \in X$. We define $\widetilde{0}(x) = [0,0]$ and $\widetilde{1}(x) = [1,1]$, for all $x \in X$.

3 Interval Valued Intuitionistic Fuzzy Ideals of Regular LA-semigroups

In this section, we characterize IIF-left (right, two-sided) ideals, IIF-generalized bi-ideals and IIF-bi-ideals of regular LA-semigroups.

Definition 3.1. Let S be an LA-semigroup. An IIF-subset $A=\langle\widetilde{\mu}_A,\widetilde{\gamma}_A\rangle$ of S is called an *IIF-LA-subsemigroup of S* if

$$\widetilde{\mu}_A(xy)\geq\widetilde{\mu}_A(x)\wedge\widetilde{\mu}_A(y)\ \text{ and }\ \widetilde{\gamma}_A(xy)\leq\widetilde{\gamma}_A(x)\vee\widetilde{\gamma}_A(y)\ \text{ for all } x,y\in S.$$

Definition 3.2. An IIF-subset $A=\langle\widetilde{\mu}_A,\widetilde{\gamma}_A\rangle$ of an LA-semigroup S is called an *IIF-left ideal of S* if

$$\widetilde{\mu}_A(xy)\geq\widetilde{\mu}_A(y)\ \text{ and }\ \widetilde{\gamma}_A(xy)\leq\widetilde{\gamma}_A(y)\ \text{ for all } x,y\in S.$$

Definition 3.3. An IIF-subset $A=\langle\widetilde{\mu}_A,\widetilde{\gamma}_A\rangle$ of an LA-semigroup S is called an *IIF-right ideal of S* if

$$\widetilde{\mu}_A(xy)\geq\widetilde{\mu}_A(x)\ \text{ and }\ \widetilde{\gamma}_A(xy)\leq\widetilde{\gamma}_A(x)\ \text{ for all } x,y\in S.$$

An IIF-subset $A=\langle\widetilde{\mu}_A,\widetilde{\gamma}_A\rangle$ of an LA-semigroup S is called an *IIF-two-sided ideal* or an *IIF-ideal of S* if it is both an IIF-left ideal and an IIF-right ideal of S.

Example 3.4. *Let $S=\{1,2,3,4\}$, the binary operation "$\cdot$" on S be defined as follows:*

$\cdot$	1	2	3	4
1	2	4	3	1
2	1	2	3	4
3	3	3	3	3
4	4	1	3	2

Clearly, $4=4\cdot(2\cdot 1)\neq(4\cdot 2)\cdot 1=2$. As S satisfies left invertive law so S is an LA-semigroup. We define IIF-subset as

$$A=\langle\widetilde{\mu}_A,\widetilde{\gamma}_A\rangle$$
$$=\left\{\left(\frac{1}{[0.1,0.2]},\frac{2}{[0.1,0.2]},\frac{3}{[0.5,0.6]},\frac{4}{[0.1,0.2]}\right),\left(\frac{1}{[0.4,0.5]},\frac{2}{[0.4,0.5]},\frac{3}{[0.1,0.3]},\frac{4}{[0.4,0.5]}\right)\right\}.$$

By routine calculations one can see that

$$\widetilde{\mu}_A(xy)\geq\widetilde{\mu}_A(y)\ \text{ and }\ \widetilde{\gamma}_A(xy)\leq\widetilde{\gamma}_A(y)\ \text{ for all } x,y\in S.$$

And

$$\widetilde{\mu}_A(xy)\geq\widetilde{\mu}_A(x)\ \text{ and }\ \widetilde{\gamma}_A(xy)\leq\widetilde{\gamma}_A(x)\ \text{ for all } x,y\in S.$$

Hence, $A=\langle\widetilde{\mu}_A,\widetilde{\gamma}_A\rangle$ is an IIF-ideal of S.

Definition 3.5. Let $A=\langle\widetilde{\mu}_A,\widetilde{\gamma}_A\rangle$ and $B=\langle\widetilde{\mu}_B,\widetilde{\gamma}_B\rangle$ be two IIF-subsets of an LA-semigroup S then $A\circ B$ is defined as

$$(\widetilde{\mu}_A\circ\widetilde{\mu}_B)(x)=\begin{cases}\bigvee_{x=yz}\{\widetilde{\mu}_A(y)\wedge\widetilde{\mu}_B(z)\} & \text{if } x=yz \text{ for some } y,z\in S\\ [0,0] & \text{otherwise.}\end{cases}$$

And

$$(\widetilde{\gamma}_A\circ\widetilde{\gamma}_B)(x)=\begin{cases}\bigwedge_{x=yz}\{\widetilde{\gamma}_A(y)\vee\widetilde{\gamma}_B(z)\} & \text{if } x=yz \text{ for some } y,z\in S\\ [1,1] & \text{otherwise.}\end{cases}$$

Note that an LA-semigroup S can be considered as an IIF-subset of itself and let

$$\begin{aligned}\beta &= \left\langle \widetilde{S_\beta}, \widetilde{\Theta_\beta} \right\rangle \\ &= \left\{ \left\langle x, \widetilde{S_\beta}(x), \widetilde{\Theta_\beta}(x) \right\rangle : \widetilde{S_\beta}(x) = [1,1] \text{ and } \widetilde{\Theta_\beta}(x) = [0,0], \text{ for all } x \text{ in } S \right\}\end{aligned}$$

be an IIF-subset and $\beta = \left\langle \widetilde{S_\beta}(x), \widetilde{\Theta_\beta}(x) \right\rangle$ will be carried out in operations with an IIF-subset $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ such that $\widetilde{S_\beta}$ and $\widetilde{\Theta_\beta}$ will be used in collaboration with $\widetilde{\mu}_A$ and $\widetilde{\gamma}_A$ respectively.

Let $IIF(S)$ denote the set of all IIF-subsets of an LA-semigroup S. Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ and $B = \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle$ be two IIF-subsets of an LA-semigroup S. The symbol $A \cap B$ will means the following

$$(\widetilde{\mu}_A \cap \widetilde{\mu}_B)(x) = \widetilde{\mu}_A(x) \wedge \widetilde{\mu}_B(x) \text{ for all } x \in S.$$

$$(\widetilde{\gamma}_A \cup \widetilde{\gamma}_B)(x) = \widetilde{\gamma}_A(x) \vee \widetilde{\gamma}_B(x) \text{ for all } x \in S.$$

The symbol $A \cup B$ will means the following

$$(\widetilde{\mu}_A \cup \widetilde{\mu}_B)(x) = \widetilde{\mu}_A(x) \vee \widetilde{\mu}_B(x) \text{ for all } x \in S.$$

$$(\widetilde{\gamma}_A \cap \widetilde{\gamma}_B)(x) = \widetilde{\gamma}_A(x) \wedge \widetilde{\gamma}_B(x), \text{ for all } x \in S.$$

Definition 3.6. Let S be an LA-semigroup and let $\emptyset \neq A \subseteq S$. Then interval valued intuitionistic characteristic function $\widetilde{\chi_A} = \langle \widetilde{\mu}_{\chi_A}, \widetilde{\gamma}_{\chi_A} \rangle$ of A is defined as

$$\widetilde{\mu}_{\chi_A} = \begin{cases} [1,1] & \text{if } x \in A \\ [0,0] & \text{if } x \notin A \end{cases} \quad \text{and} \quad \widetilde{\gamma}_{\chi_A} = \begin{cases} [0,0] & \text{if } x \in A \\ [1,1] & \text{if } x \notin A. \end{cases}$$

Lemma 3.7. *For any IIF-subset $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ of an LA-semigroup S, the following properties holds.*

(1) A is an LA-subsemigroup of S if and only if A is an IIF-LA-subsemigroup of S.

(2) A is a left (right, two-sided) ideal of S if and only if A is an IIF-left (right, two-sided) ideal of S.

Proof. Proof is straightforward. □

Lemma 3.8. *Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-subset of an LA-semigroup S. Then,*

(1) $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-LA-subsemigroup of S if and only if $\widetilde{\mu}_A \circ \widetilde{\mu}_A \subseteq \widetilde{\mu}_A$ and $\widetilde{\gamma}_A \circ \widetilde{\gamma}_A \supseteq \widetilde{\gamma}_A$.

(2) $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-left (resp. IIF-right) ideal of S if and only if $\widetilde{S} \circ \widetilde{\mu}_A \subseteq \widetilde{\mu}_A$ and $\widetilde{\Theta} \circ \widetilde{\gamma}_A \supseteq \widetilde{\gamma}_A$ (resp. $\widetilde{\mu}_A \circ \widetilde{S} \subseteq \widetilde{\mu}_A$ and $\widetilde{\gamma}_A \circ \widetilde{\Theta} \supseteq \widetilde{\gamma}_A$).

Proof. Proof is straightforward. □

Lemma 3.9. *Let S be an LA-semigroup. Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ and $B = \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle$ be any two IIF-two-sided ideals of S, then $A \circ B = A \cap B$.*

Proof. Suppose that $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ and $B = \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle$ be any two IIF-two-sided ideals of a regular LA-semigroup S, then by using Lemma 3.8, we have $\widetilde{\mu}_A \circ \widetilde{\mu}_B \subseteq \widetilde{\mu}_A \cap \widetilde{\mu}_B$ and $\widetilde{\gamma}_A \circ \widetilde{\gamma}_B \supseteq \widetilde{\gamma}_A \cup \widetilde{\gamma}_B$, which shows that $A \circ B \subseteq A \cap B$. Let $a \in S$, then there exists $x \in S$ such that $a = (ax)a$ and therefore, we have

$$\begin{aligned}(\widetilde{\mu}_A \circ \widetilde{\mu}_B)(a) &= \bigvee_{a=(ax)a} \{\widetilde{\mu}_A(ax) \wedge \widetilde{\mu}_B(a)\} \geq \widetilde{\mu}_A(ax) \wedge \widetilde{\mu}_B(a) \\ &\geq \widetilde{\mu}_A(a) \wedge \widetilde{\mu}_B(a) = (\widetilde{\mu}_A \cap \widetilde{\mu}_B)(a).\end{aligned}$$

And

$$\begin{aligned}(\widetilde{\gamma}_A \circ \widetilde{\gamma}_B)(a) &= \bigwedge_{a=(ax)a} \{\widetilde{\gamma}_A(ax) \vee \widetilde{\gamma}_B(a)\} \leq \widetilde{\gamma}_A(ax) \vee \widetilde{\gamma}_B(a) \\ &\leq \widetilde{\gamma}_A(a) \vee \widetilde{\gamma}_B(a) = (\widetilde{\gamma}_A \cup \widetilde{\gamma}_B)(a).\end{aligned}$$

Thus, we get that $\widetilde{\mu}_A \circ \widetilde{\mu}_B \supseteq \widetilde{\mu}_A \cap \widetilde{\mu}_B$ and $\widetilde{\gamma}_A \circ \widetilde{\gamma}_B \subseteq \widetilde{\gamma}_A \cup \widetilde{\gamma}_B$. This implies that $A \circ B \supseteq A \cap B$. Hence $A \circ B = A \cap B$. □

Example 3.10. *Let $S = \{1, 2, 3, 4\}$, the binary operation "$\cdot$" on S be defined as follows:*

$\cdot$	1	2	3	4
1	4	4	2	4
2	4	4	1	4
3	1	2	3	4
4	4	4	4	4

Clearly, $1 = 2 \cdot (3 \cdot 3) \neq (2 \cdot 3) \cdot 3 = 2$. As S satisfies left invertive law so S is an LA-semigroup. We define IIF-subsets as

$$\begin{aligned}A &= \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle \\ &= \left\{ \left(\frac{1}{[0.3, 0.4]}, \frac{2}{[0.3, 0.4]}, \frac{3}{[0.1, 0.2]}, \frac{4}{[0.4, 0.5]} \right), \right. \\ &\qquad \left. \left(\frac{1}{[0.4, 0.45]}, \frac{2}{[0.4, 0.45]}, \frac{3}{[0.5, 0.6]}, \frac{4}{[0.1, 0.2]} \right) \right\},\end{aligned}$$

and

$$\begin{aligned}B &= \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle \\ &= \left\{ \left(\frac{1}{[0.5, 0.55]}, \frac{2}{[0.5, 0.55]}, \frac{3}{[0.4, 0.5]}, \frac{4}{[0.6, 0.7]} \right), \right. \\ &\qquad \left. \left(\frac{1}{[0.2, 0.3]}, \frac{2}{[0.2, 0.3]}, \frac{3}{[0.37, 0.4]}, \frac{4}{[0.1, 0.12]} \right) \right\}.\end{aligned}$$

Clearly, $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ *and* $B = \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle$ *are IIF-two-sided ideals of* S*. Now*

$$(\widetilde{\mu}_A \circ \widetilde{\mu}_B)(a) = \{[0.1, 0.2],\ [0.3, 0.4],\ [0.4, 0.5]\} = (\widetilde{\mu}_A \cap \widetilde{\mu}_B)(a)$$

for all $a \in S$*. Similarly, we can check that* $(\widetilde{\gamma}_A \circ \widetilde{\gamma}_B)(a) = (\widetilde{\gamma}_A \cap \widetilde{\gamma}_B)(a)$*, for all* $a \in S$*. But note that* S *is not regular, because* $2 \in S$ *is not regular.*

Theorem 3.11. *Let* S *be an LA-semigroup with left identity and let* $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ *be any IIF-subset of* S*, then* S *is regular if* $A(x) = A(x^2)$ *holds for all* $x \in S$*.*

Proof. Assume that S be an LA-semigroup with left identity. Clearly x^2S is a subset of S and therefore its characteristic function $\widetilde{\chi_{x^2S}} = \left\langle \widetilde{\mu}_{\chi_{x^2S}}, \widetilde{\gamma}_{\chi_{x^2S}} \right\rangle$ is an IIF-subset of S. Let $x \in S$, then by given assumption $\widetilde{\mu}_{\chi_{x^2S}}(x) = \widetilde{\mu}_{\chi_{x^2S}}(x^2)$ and $\widetilde{\gamma}_{\chi_{x^2S}}(x) = \widetilde{\gamma}_{\chi_{x^2S}}(x^2)$ holds for all $x \in S$. As $x^2 \in x^2S$, because by using paramedial law, we have

$$x^2S = (xx)(SS) = (SS)(xx) = Sx^2.$$

Therefore, $\widetilde{\mu}_{\chi_{x^2S}}(x^2) = [1, 1]$ and $\widetilde{\gamma}_{\chi_{x^2S}}(x^2) = [0, 0]$, which implies that $x \in x^2S$. Now by using left invertive and paramedial laws, we have

$$\begin{aligned} x \in x^2S &= (xx)(SS) = ((SS)x)x \subseteq ((SS)(x^2S))x = ((SS)((xx)S))x \\ &= ((SS)((Sx)x))x = ((Sx)(Sx))x = ((xS)(xS))x = (x((xS)S))x \subseteq (xS)x. \end{aligned}$$

Thus, S is regular. □

An IIF-subset $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ of an LA-semigroup S is said to be idempotent if $\widetilde{\mu}_A \circ \widetilde{\mu}_A = \widetilde{\mu}_A$ and $\widetilde{\gamma}_A \circ \widetilde{\gamma}_A = \widetilde{\gamma}_A$, that is, $A \circ A = A$.

Lemma 3.12. *Every IIF-two-sided ideal* $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ *of a regular LA-semigroup is idempotent.*

Proof. Let S be a regular LA-semigroup and let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-two-sided ideal of S. Now for $a \in S$ there exists $x \in S$ such that $a = (ax)a$ and therefore, we have

$$\begin{aligned} (\widetilde{\mu}_A \circ \widetilde{\mu}_A)(a) &= \bigvee_{a=(ax)a} \{\widetilde{\mu}_A(ax) \wedge \widetilde{\mu}_A(a)\} \geq \widetilde{\mu}_A(ax) \wedge \widetilde{\mu}_A(a) \\ &\geq \widetilde{\mu}_A(a) \wedge \widetilde{\mu}_A(a) = \widetilde{\mu}_A(a). \end{aligned}$$

And

$$\begin{aligned} (\widetilde{\gamma}_A \circ \widetilde{\gamma}_A)(a) &= \bigwedge_{a=(ax)a} \{\widetilde{\gamma}_A(ax) \vee \widetilde{\gamma}_A(a)\} \leq \widetilde{\gamma}_A(ax) \vee \widetilde{\gamma}_A(a) \\ &\leq \widetilde{\gamma}_A(a) \vee \widetilde{\gamma}_A(a) = \widetilde{\gamma}_A(a). \end{aligned}$$

This implies that $\widetilde{\mu}_A \circ \widetilde{\mu}_A \supseteq \widetilde{\mu}_A$ and $\widetilde{\gamma}_A \circ \widetilde{\gamma}_A \subseteq \widetilde{\gamma}_A$. Now by using Lemma 3.8, we get $\widetilde{\mu}_A \circ \widetilde{\mu}_A = \widetilde{\mu}_A$ and $\widetilde{\gamma}_A \circ \widetilde{\gamma}_A = \widetilde{\gamma}_A$. Hence, $A \circ A = A$. □

Lemma 3.13. *In a regular LA-semigroup S, $A \circ \beta = A$ and $\beta \circ A = A$ holds for every IIF-two-sided ideal $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ of S, where $\beta = \left\langle \widetilde{S_\beta}, \widetilde{\Theta_\beta} \right\rangle$.*

Proof. Let S be a regular LA-semigroup and let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-two-sided ideal of S. Now for $a \in S$ there exists $x \in S$ such that $a = (ax)a$, therefore

$$\begin{aligned}\left(\widetilde{\mu}_A \circ \widetilde{S_\beta}\right)(a) &= \bigvee_{a=(ax)a} \left\{\widetilde{\mu}_A(ax) \wedge \widetilde{S_\beta}(a)\right\} \geq \widetilde{\mu}_A(ax) \wedge \widetilde{S_\beta}(a) \\ &\geq \widetilde{\mu}_A(a) \wedge [1,1] = \widetilde{\mu}_A(a).\end{aligned}$$

And

$$\begin{aligned}\left(\widetilde{\gamma}_A \circ \widetilde{\Theta_\beta}\right)(a) &= \bigwedge_{a=(ax)a} \left\{\widetilde{\gamma}_A(ax) \vee \widetilde{\Theta_\beta}(a)\right\} \leq \widetilde{\gamma}_A(ax) \vee \widetilde{\Theta_\beta}(a) \\ &\leq \widetilde{\gamma}_A(a) \vee [0,0] = \widetilde{\gamma}_A(a).\end{aligned}$$

This implies that $\widetilde{\mu}_A \circ \widetilde{S_\beta} \supseteq \widetilde{\mu}_A$ and $\widetilde{\gamma}_A \circ \widetilde{\Theta_\beta} \subseteq \widetilde{\gamma}_A$. Now by using Lemma 3.8, we get $\widetilde{\mu}_A \circ \widetilde{S_\beta} = \widetilde{\mu}_A$ and $\widetilde{\gamma}_A \circ \widetilde{\Theta_\beta} = \widetilde{\gamma}_A$. Hence, $A \circ \beta = A$. Now we can prove $\beta \circ A = A$ in a similar way. □

Lemma 3.14. *Every IIF-right ideal of an LA-semigroup S with left identity is an IIF-left ideal of S.*

Proof. Proof is straightforward. □

Theorem 3.15. *If $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-two-sided ideal of a regular LA-semigroup S with left identity, then $A(ab) = A(ba)$ holds for all a, b in S.*

Proof. Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-two-sided ideal of a regular LA-semigroup S with left identity and let $a, b \in S$, then $a = (ax)a$ and $b = (by)b$ for some $x, y \in S$. Now by using medial and paramedial laws, we have

$$\begin{aligned}\widetilde{\mu}_A(ab) &= \widetilde{\mu}_A(((ax)a)((by)b)) = \widetilde{\mu}_A(((ax)(by))(ab)) \\ &= \widetilde{\mu}_A((ba)((by)(ax))) \geq \widetilde{\mu}_A(ba),\end{aligned}$$

$$\begin{aligned}\widetilde{\mu}_A(ba) &= \widetilde{\mu}_A(((by)b)((ax)a)) = \widetilde{\mu}_A(((by)(ax))(ba)) \\ &= \widetilde{\mu}_A((ab)((ax)(by))) \geq \widetilde{\mu}_A(ab).\end{aligned}$$

Which shows that $\widetilde{\mu}_A(ab) = \widetilde{\mu}_A(ba)$ holds for all a, b in S and similarly $\widetilde{\gamma}_A(ab) = \widetilde{\gamma}_A(ba)$ holds for all a, b in S. Thus, $A(ab) = A(ba)$ holds for all a, b in S. □

Theorem 3.16. *Let S be a regular LA-semigroup with left identity, then $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-left ideal of S if and only if $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-bi-ideal of S.*

Proof. Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-left ideal of S and let $w, x, y \in S$, then by using left invertive law, we have

$$\widetilde{\mu}_A((xw)y) = \widetilde{\mu}_A(((yw)x)) \geq \widetilde{\mu}_A(x) \geq \widetilde{\mu}_A(x) \wedge \widetilde{\mu}_A(y).$$

And

$$\widetilde{\gamma}_A((xw)y) = \widetilde{\gamma}_A(((yw)x)) \geq \widetilde{\gamma}_A(x) \geq \widetilde{\gamma}_A(x) \wedge \widetilde{\gamma}_A(y).$$

Thus, $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-bi-ideal of S.

Conversely, let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-bi-ideal of S and let $x, y \in S$, then there exists $z \in S$ such that $y = (yz)y$. Now

$$\begin{aligned}\widetilde{\mu}_A(xy) &= \widetilde{\mu}_A(x((yz)y)) = \widetilde{\mu}_A((yz)(xy)) = \widetilde{\mu}_A((yx)(zy)) = \widetilde{\mu}_A(((zy)x)y) \\ &= \widetilde{\mu}_A(((zy)(ex))y) = \widetilde{\mu}_A(((xe)(yz))y) = \widetilde{\mu}_A((y((xe)z))y) \geq \widetilde{\mu}_A(y).\end{aligned}$$

Similarly, $\widetilde{\gamma}_A(xy) \leq \widetilde{\gamma}_A(y)$ and therefore $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-left ideal of S. □

An element a of an LA-semigroup S is called an intra-regular if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called an intra-regular if every element of S is an intra-regular.

Lemma 3.17. *For an intra-regular LA-semigroup S with left identity, the following holds.*

(1) Every IIF-right ideal of S is an IIF-semiprime.

(2) Every IIF-left ideal of S is an IIF-semiprime.

(3) Every IIF-two-sided ideal of S is an IIF-semiprime.

Proof. (1). Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-right ideal of an intra-regular LA-semigroup S with left identity and let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using paramedial law, we have

$$\widetilde{\mu}_A(a) = \widetilde{\mu}_A((xa^2)y) = \widetilde{\mu}_A((xa^2)(ey)) = \widetilde{\mu}_A((ye)(a^2x)) = \widetilde{\mu}_A(a^2((ye)x)) \geq \widetilde{\mu}_A(a^2),$$

and similarly

$$\widetilde{\gamma}_A(a) = \widetilde{\gamma}_A((xa^2)y) = \widetilde{\gamma}_A((xa^2)(ey)) = \widetilde{\gamma}_A((ye)(a^2x)) = \widetilde{\gamma}_A(a^2((ye)x)) \leq \widetilde{\gamma}_A(a^2).$$

Thus, $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-semiprime.

(2). Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-left ideal of an intra-regular LA-semigroup S with left identity and let $a \in S$, then there exist $x, y \in S$ such that $a = (xa^2)y$. Now by using left invertive law and paramedial law, we have

$$\begin{aligned}\widetilde{\mu}_A(a) &= \widetilde{\mu}_A((xa^2)y) = \widetilde{\mu}_A((x(aa))y) = \widetilde{\mu}_A((a(xa))y) = \widetilde{\mu}_A((((xa^2)y)(xa))y) \\ &= \widetilde{\mu}_A(((ax)(y(xa^2)))y) = \widetilde{\mu}_A(((ax)(y((ex)(aa))))y) \\ &= \widetilde{\mu}_A(((ax)(y(a^2(xe))))y) = \widetilde{\mu}_A(((ax)(a^2(y(xe))))y) \\ &= \widetilde{\mu}_A(a^2((ax)(y(xe)))y) = \widetilde{\mu}_A((y((y(xe))(ax)))a^2) \geq \widetilde{\mu}_A(a^2).\end{aligned}$$

Similarly, we can show that $\widetilde{\gamma}_A(a) \leq \widetilde{\gamma}_A(a^2)$ and therefore $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-semiprime.

(3). It can be followed from (1) and (2). □

Lemma 3.18. *For any non-empty IIF-subsets $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ and $B = \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle$ of an LA-semigroup S, then $\chi_A \circ \chi_B = \chi_{AB}$.*

Proof. Proof is straightforward. □

Theorem 3.19. *For an LA-semigroup S with left identity, the following conditions are equivalent.*

(1) S is an intra-regular.

(2) $R \cap L = RL$, R is any right ideal and L is any left ideal of S such that R is semiprime.

(3) $A \cap B = A \circ B$, $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is any IIF-right ideal and $B = \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle$ is any IIF-left ideal of S such that $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-semiprime.

Proof. (1) $\Rightarrow$ (3) : Assume that S is an intra-regular LA-semigroup. Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be any IIF-right ideal and $B = \langle \widetilde{\mu}_B, \widetilde{\gamma}_B \rangle$ be any IIF-left ideal of S. Now for $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using left invertive law and paramedial law, we have

$$\begin{aligned} a &= (x(aa))y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\ &= (y((yx)(a^2x)))a = (y(a^2((yx)x)))a = (a^2(y((yx)x)))a. \end{aligned}$$

Therefore,

$$\begin{aligned} (\widetilde{\mu}_A \circ \widetilde{\mu}_B)(a) &= \bigvee\nolimits_{a=(a^2(y((yx)x)))a} \{\widetilde{\mu}_A(a^2(y((yx)x))) \wedge \widetilde{\mu}_B(a)\} \\ &\geq \widetilde{\mu}_A(a) \wedge \widetilde{\mu}_B(a) = (\widetilde{\mu}_A \cap \widetilde{\mu}_B)(a), \end{aligned}$$

and

$$\begin{aligned} (\widetilde{\gamma}_A \circ \widetilde{\gamma}_B)(a) &= \bigwedge\nolimits_{a=(a^2(y((yx)x)))a} \{\widetilde{\gamma}_A(a^2(y((yx)x))) \vee \widetilde{\gamma}_B(a)\} \\ &\leq \widetilde{\gamma}_A(a) \vee \widetilde{\gamma}_B(a) = (\widetilde{\gamma}_A \cup \widetilde{\gamma}_B)(a). \end{aligned}$$

Which implies that $A \cap B \supseteq A \circ B$ and by using Lemma 3.8, $A \circ B \subseteq A \cap B$, therefore $A \cap B = A \circ B$.

(3) $\Rightarrow$ (2) : Let R be any right ideal and L be any left ideal of an LA-semigroup S, then by Lemma 3.7, $\widetilde{\chi_R} = \langle \widetilde{\mu}_{\chi_R}, \widetilde{\gamma}_{\chi_R} \rangle$ and $\widetilde{\chi_L} = \langle \widetilde{\mu}_{\chi_L}, \widetilde{\gamma}_{\chi_L} \rangle$ are IIF-right and IIF-left ideals of S respectively. As $RL \subseteq R \cap L$ is obvious therefore let $a \in R \cap L$, then $a \in R$ and $a \in L$. Now by using Lemma 3.18 and given assumption, we have

$$\widetilde{\mu}_{\chi_{RL}}(a) = (\widetilde{\mu}_{\chi_R} \circ \widetilde{\mu}_{\chi_L})(a) = (\widetilde{\mu}_{\chi_R} \cap \widetilde{\mu}_{\chi_L})(a) = \widetilde{\mu}_{\chi_R}(a) \wedge \widetilde{\mu}_{\chi_L}(a) = [1, 1].$$

Similarly,

$$\widetilde{\gamma}_{\chi_{RL}}(a) = (\widetilde{\gamma}_{\chi_R} \circ \widetilde{\gamma}_{\chi_L})(a) = (\widetilde{\gamma}_{\chi_R} \cup \widetilde{\gamma}_{\chi_L})(a) = \widetilde{\gamma}_{\chi_R}(a) \vee \widetilde{\gamma}_{\chi_L}(a) = [0,0].$$

Which implies that $a \in RL$ and therefore $R \cap L = RL$. Therefore, R is semiprime.

$(2) \Rightarrow (1)$: Let S be an LA-semigroup, then clearly Sa is a left ideal of S such that $a \in Sa$ and a^2S is a right ideal of S such that $a^2 \in a^2S$. Since by assumption, a^2S is semiprime, therefore $a \in a^2S$. Now by using left invertive law and paramedial law, we have

$$\begin{aligned} a \in a^2S \cap Sa &= (a^2S)(Sa) = (aS)(Sa^2) = ((Sa^2)S)a = ((Sa^2)(SS))a \\ &= ((SS)(a^2S))a = (a^2((SS)S))a \subseteq (a^2S)S = (SS)(aa) = a^2S \\ &= (aa)S = (Sa)a \subseteq (Sa)(a^2S) = ((a^2S)a)S = ((aS)a^2)S \subseteq (Sa^2)S. \end{aligned}$$

Which shows that S is an intra-regular LA-semigroup. □

Theorem 3.20. *Let S be an intra-regular LA-semigroup with left identity and let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-subset, then the following conditions are equivalent.*

(1) $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-bi-ideal of S.

(2) $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-generalized bi-ideal of S.

Proof. $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$: Let $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ be an IIF-generalized bi-ideal of an intra-regular LA-semigroup S and let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using left invertive law and paramedial law, we have

$$\begin{aligned} \widetilde{\mu}_A(ab) &= \widetilde{\mu}_A(((x(aa))y)b) = \widetilde{\mu}_A((((ea)(xa))y)b) = \widetilde{\mu}_A((((ax)(ae))y)b) \\ &= \widetilde{\mu}_A(((a((ax)e))(ey))b) = \widetilde{\mu}_A(((ye)(((ax)e)a))b) \\ &= \widetilde{\mu}_A(((ye)((ae)(ax)))b) = \widetilde{\mu}_A(((ye)(a((ae)x)))b) \\ &= \widetilde{\mu}_A((a((ye)((ae)x)))b) \geq \widetilde{\mu}_A(a) \wedge \widetilde{\mu}_A(b). \end{aligned}$$

Similarly, we can show that $\widetilde{\gamma}_A(ab) \leq \widetilde{\gamma}_A(a) \vee \widetilde{\gamma}_A(b)$ and therefore $A = \langle \widetilde{\mu}_A, \widetilde{\gamma}_A \rangle$ is an IIF-bi-ideal of S. □

4 Conclusion

In the structural theory of fuzzy algebraic systems, fuzzy ideals with special properties always play an important role. In this paper, we applied the interval valued intuitionistic fuzzy set theory to left almost semigroups and characterized regular LA-semigroups by their interval valued intuitionistic fuzzy ideals.

In our future study of interval valued intuitionistic fuzzy structure of LA-semigroups, may be the following topics should be considered:

1. To characterize other classes of LA-semigroups by the properties of these interval valued intuitionistic fuzzy ideals.

2. To characterize intra-regular LA-semigroups by the properties of these interval valued intuitionistic fuzzy ideals.
3. To characterize regular and intra-regular LA-semigroups by the properties of interval valued intuitionistic fuzzy soft ideals.

References

[1] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965) 338–353.

[2] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5 (1981) 203–215.

[3] Faisal, N. Yaqoob, K. Hila, On fuzzy $(2, 2)$-regular ordered Γ-AG**-groupoids, U.P.B. Sci. Bull. Series A 74 (2) (2012) 87–104.

[4] Faisal, N. Yaqoob, A. Ghareeb, Left regular AG-groupoids in terms of fuzzy interior ideals, Afr. Math., (In press).

[5] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, Inform. Sci. 8 (1975) 199–249.

[6] R. Biswas, Rosenfeld's fuzzy subgroups with interval valued membership functions, Fuzzy Sets and Systems 63 (1994) 87–90.

[7] A. Narayanan, T. Manikantan, Interval valued fuzzy ideals generated by an interval valued fuzzy subsets in semigroups, J. Appl. Math. Comput. 20 (2006) 455–464.

[8] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.

[9] K.T. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989) 343–349.

[10] K.T. Atanassov, Operators over interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 64 (1994) 159–174.

[11] M. Akram, W.A. Dudek, Interval valued intuitionistic fuzzy Lie ideals of Lie algebras, World Appl. Sci. J. 7 (2009) 812–819.

[12] M. Akram, K.H. Dar, B.L. Meng, K.P. Shum, Interval valued intuitionistic fuzzy ideals of K-algebras, WSEAS Trans. Math. 7 (9) (2008) 559–568.

[13] M. Akram, K.H. Dar, K.P. Shum, Interval valued (α, β)-fuzzy K-algebras, Appl. Soft Comput. 11 (1) (2011) 1213–1222.

[14] N. Yaqoob, R. Chinram, A. Ghareeb, M. Aslam, Left almost semigroups characterized by their interval valued fuzzy ideals, Afr. Math., (In press).

[15] A. Khan, Faisal, W. Khan, N. Yaqoob, Ordered LA-semigroups in terms of interval valued fuzzy ideals, J. Adv. Res. Pure Math., (In press).

[16] N. Yaqoob, Bipolar-valued fuzzy ideals in LA-semigroups, J. Adv. Stud. Topol. 3 (1) (2012) 60–71.

[17] M. Akram, Intuitionistic (S,T)-fuzzy Lie ideals of Lie algebras, Quasigroups Related Systems 15 (2007) 201–218.

[18] H. Hedayati, Interval valued intuitionistic (S,T)-fuzzy substructures in semirings, Int. Math. Forum 4 (6) (2009) 293–301.

[19] H. Hedayati, Intuitionistic (S,T)-fuzzy $(1,2)$-ideals of semigroups with interval valued membership functions, Int. J. Fuzzy Syst. 14 (1) (2012) 154–159.

[20] J.G. Lee, K.H. Kim, Interval valued intuitionistic (S^*,T^*)-fuzzy ideals of subtraction algebras, Int. Math. Forum 4 (46) (2009) 2281–2291.

[21] J. Zhan, W.A. Dudek, Interval valued intuitionistic (S,T)-fuzzy Hv-submodules, Acta Math. Sinica, English Series 22 (2006) 963–970.

[22] M.A. Kazim, M. Naseerudin, On almost-semigroup, Aligarh Bull. Math. 2 (1972) 1–7.

[23] M. Khan and N. Ahmad, Characterizations of left almost semigroups by their ideals, J. Adv. Res. Pure Math. 2 (2010) 61–73.

(Received 6 June 2011)
(Accepted 10 July 2012)