On Intra-Regular Ordered $\star$-Semigroups

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Abstract : Nordahl and Scheiblich [1] considered a unary operation $\star$ on semigroups and introduced the concept of regularity on $\star$-semigroups. In this paper we impose this operation on ordered semigroups under the assumption of order preserving, i.e. if $a \geq b$ then $a^\star \geq b^\star$. Then we can characterize intra-regular ordered $\star$-semigroups. Indeed since $\star$ can be considered to be the identity mapping particularly, the results in this paper can be considered to be the extensions of some properties in ordered semigroups [2-5].

Keywords : ordered $\star$-semigroup; intra-regular ordered $\star$-semigroup; left (right) ideal; filter.

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1 Introduction

Szász [6] has shown that the ideals of a semigroup $S$ are prime if and only if $S$ is intra-regular and any two ideals are comparable. He also proved that an ideal of a semigroup $S$ is prime if and only if it is both weakly prime and semiprime; and that in commutative semigroups the prime and weakly prime ideals coincide. Ordered semigroups in which the ideals are prime, weakly prime have been considered by Kehayopulu [2, 3]. Above results, which Szász presented in semigroups, are also true in case of ordered semigroups [4]. Furthermore a characterization for intra-regular ordered semigroups was done [4].

In this paper we will present analogous results on ordered $\star$-semigroups. It will be seen that the ideals requires virtually no changes from that in ordered semigroups. However in order to guarantee ideals being able to be ideals after operated
by $\star$, it is necessary to assume the operator $\star$ preserves ordering. Section 2 will characterize ordered $\star$-semigroups in which all ideals are (weakly) prime. The final section is devoted to construct the concept of filters and creates a characterization on intra-regular ordered $\star$-semigroups in terms of the least filter.

An ordered semigroup $S$ is a partial ordering set at the same semigroup such that for any $a, b, x \in S, a \leq b$ implies $xa \leq xb$ and $ax \leq bx$. An ordered semigroup $S$ with a unary operation $\star : S \rightarrow S$ is called an ordered $\star$-semigroup if it satisfies $(x^\star)^\star = x$ and $(xy)^\star = y^\star x^\star$ for any $x, y \in S$. Such a unary operation $\star$ is called an involution [1]. If for any $a, b$ with $a \geq b$, we have $a^\star \geq b^\star$, then $\star$ is called an order preserving involution.

Example 1.1. Let $S = \{a, b, c, d, e\}$ be an ordered semigroup. The multiplication "\cdot", the order "\leq" and the corresponding Hasse diagram are given below [4]. Define the involution $\star$ by $a^\star = e$ (hence $e^\star = a$), $b^\star = c$ and $d^\star = d$. It is easy to check that $S$ is an ordered $\star$-semigroup with order preserving involution $\star$.

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, d), (d, c), (d, d), (e, c), (e, e)\}$$

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2 Characterization of Ordered $\star$-Semigroups in which all Ideals are (Weakly) Prime

Many of the deepest properties of ordered $\star$-semigroups depend on ideals. We shall introduce the basic concepts and derive some crucial important properties. Then we will permit us to characterize ordered $\star$-semigroups.

Let $S$ be an ordered $\star$-semigroup. For $H \subseteq S$, we denote $(H) := \{t \in S \mid t \leq h$ for some $h \in H\}$. If $H = \{a\}$, we write $(a)$ instead of $(\{a\})$ for convenience (cf. [5]). A non-empty subset $L$ (resp. $R$) of $S$ is called a left (resp. right) ideal of $S$ if (1) $SL \subseteq L$ (resp. $RS \subseteq R$), and (2) $a \in L$ (resp. $R$), $S \ni b \leq a$ implies $b \in L$ (resp. $R$). $I$ is called an ideal of $S$ if it is both a left and a right ideal of $S$ (cf. [5]). We denote by $L(a)$, $R(a)$ and $I(a)$ the left ideal, right ideal and the ideal of $S$, respectively, generated by $a$. Clearly $L(a) = (a \cup Sa)$, $R(a) = (a \cup aS)$, $I(a) = (a \cup Sa \cup aS \cup SaS)$ (cf. [2, 3]).

Let $S$ be an ordered $\star$-semigroup with order preserving involution $\star$. We will see that $L^\star$ is a right ideal for any left ideal $L$ of $S$, and $R^\star$ is a left ideal for any right ideal $R$ of $S$.

Proposition 2.1 (cf. [5, Lemma 1]). Let $S$ be an ordered $\star$-semigroup.
Let $S$ be an ordered semigroup and $T \subseteq S$. $T$ is called prime if $AB \subseteq T$, then $A^* \subseteq T$ or $B^* \subseteq T$.

Equivalence definition: if $ab \in T$, then $a^* \in T$ or $b^* \in T$.

**Definition 2.4.** Let $S$ be an ordered semigroup and $T \subseteq S$. $T$ is called weakly prime if for any ideals $A, B$ of $S$ such that $AB \subseteq T$ we have $A^* \subseteq T$ or $B^* \subseteq T$.

**Definition 2.5.** Let $S$ be an ordered semigroup. A subset $T$ of $S$ is called semiprime if $AA \subseteq T$, then $A^* \subseteq T$.

Equivalence definition: if $aa \in T$, then $a^* \in T$.

**Theorem 2.6.** Let $S$ be an ordered semigroup with order preserving involution $\ast$. An ideal of $S$ is prime if and only if it is both weakly prime and semiprime. Furthermore, if $S$ is commutative, then the prime and weakly prime ideals coincide.

**Proof.** Suppose that $I$ is a prime ideal of $S$. It is trivial that $I$ is both weakly prime and semiprime.

Conversely, suppose that $T$ is an ideal of $S$ which is weakly prime and semiprime. Let $ab \in T$, we need to show that $a^* \in T$ or $b^* \in T$. First note that by Proposition 2.1 $(bSa)[bSa] \subseteq (S(aSb))^2 \subseteq (S(aSb))^2 = T$. Then $T$ is semiprime implies that $(bSa)^*: T$. Therefore $(Sa^*[SbS] \subseteq (Sa^*[SbS] \subseteq (Sa^*[SbS])^2 = T$. This shows that $T$ is prime.
\((S(\mathcal{B}^*)^a)^*S\) = \((S(bS^a)^*S) \subseteq (ST\mathcal{S}) \subseteq T\). Observe that \((Sa^*S)\), \((Sb^*S)\) are ideals, and \(T\) is weakly prime. Thus \((Sa^*S)^* \subseteq T\) or \((Sb^*S)^* \subseteq T\). Hence \((SaS) \subseteq T\) or \((SbS) \subseteq T\) by Proposition 2.2. To prove that \(T\) is prime, we just need to show that if \((SaS) \subseteq T\) then \(a^* \in T\). The other part is proved similarly.

If \((SaS) \subseteq T\) then \((I(a))^3 = (a \cup Sa \cup aS \cup SaS)^3 \subseteq ((a \cup Sa \cup aS \cup SaS)^3) \subseteq (SaS) \subseteq T\). Hence \((I(a))(I(a)) = (I(a))(I(a)) \subseteq ((I(a)^3) \subseteq (T) = T\) by Proposition 2.2. Note that \(T\) is weakly prime and \(I(a)\), \((I(a)I(a)\) are ideals. It follows that \((I(a))^* \subseteq T\) or \((I(a))I(a)^* \subseteq T\). Suppose \((I(a))^* \subseteq T\). Then \(a^* \in (I(a))^* \subseteq T\) and we complete the proof. Suppose \((I(a)I(a))^* \subseteq T\). Then \(a^* \in (I(a))I(a)^* \subseteq (I(a)I(a))^* \subseteq T\) because \(aa \in (I(a)I(a))\), whence \(a = (a^*)^* \in T\) because \(T\) is semiprime. Now \(T\) is an ideal implies that \(aa \in T\), hence \(a^* \in T\) by \(T\) is semiprime.

To prove the second statement, let \(T\) be an ideal of \(S\). If \(T\) is prime then obviously \(T\) is weakly prime. Conversely, suppose \(T\) is weakly prime. Let \(a \in T\). Since \(S\) is commutative, we have \(I(a)(b) = (a \cup Sa \cup aS \cup SaS)((b \cup Sb \cup bS \cup SbS) \subseteq (a \cup Sa \cup aS \cup SaS)(b \cup Sb \cup bS \cup SbS) \subseteq (a \cup Sa \cup aS \cup SaS)\). Observe that \((a \cup Sa \cup aS \cup SaS) = (a \cup Sa \cup aS \cup SaS)\). Thus \((ab \cup Sb) \subseteq T\) = \(T\). Hence \((I(a))(b) \subseteq T\), and conclude that \((I(a))^* \subseteq T\) or \((I(b))^* \subseteq T\) by \(T\) is weakly prime. Therefore \(a^* \in T\) or \(b^* \in T\); that is, \(T\) is prime.

\[\Box\]

**Proposition 2.7.** Let \(S\) be an ordered \(*\)-semigroup with order preserving involution \(*\). The following statements are equivalent:

1. \(A^*A^* = A\) for any ideal \(A\) of \(S\);
2. \(A^* \cap B^* = (AB)\) for any ideals \(A, B\) of \(S\);
3. \(I(a) \cap I(b) = ((I(a))^*(I(b))^*)\) for any \(a, b \in S\);
4. \(I(a) = (I(a)^*)(I(a))^*\) for any \(a \in S\);
5. \(a \in (S(a^*S))^*\) for any \(a \in S\).

**Proof.** 1) \(\Rightarrow\) 2). Since \(A^*\) and \(B^*\) are ideals, by hypothesis and Proposition 2.1 we have \((AB) \subseteq (AS) \subseteq (A) = (A^*A^*) = (A^*A^*) \subseteq (A^*) = A^*\). Similarly \((AB) \subseteq (SB) \subseteq (B) = (B^*B^*) = (B^*B^*) \subseteq (B^*) = B^*\). Thus \((AB) \subseteq A^* \cap B^*\). Furthermore \(A^* \cap B^*\) is an ideal implies \(A^* \cap B^* = ((A^* \cap B^*)^*)(A^* \cap B^*)^* = ((A^* \cap B^*) \cap (A^* \cap B^*)) \subseteq (AB)\). Therefore we have \((AB) \subseteq A^* \cap B^*\) and \(A^* \cap B^* \subseteq (AB)\). So \(A^* \cap B^* = (AB)\).

2) \(\Rightarrow\) 3). Proposition 2.2 implies that \((I(a))^*\) and \((I(b))^*\) are ideals. Then the statement is clear by this fact.

3) \(\Rightarrow\) 4). Since \((I(a)) = ((I(a))^*(I(a))^*)\) by hypothesis, we just need to show that \((I(a))^* = I(a^*).\) Clearly \(a^* \in (I(a))^*\). Hence \((I(a))^* \subseteq (I(a))^*\) because \((I(a))^*\) is an ideal. Now let \(x \in (I(a))^*\). We have \(x^* \in I(a) = (a \cup aS \cup Sa \cup SaS)\). This means that \(x^* \leq a\) or \(x^* \leq au\) or \(x^* \leq ua\) or \(x^* \leq uv\) for some \(u, v \in S\). Thus \(x \leq a^*\) or \(x \leq u^*a^* \in Sa^*\) or \(x \leq a^*u^* \in aS^*\) or \(x^* \leq v^*a^*u^* \in Sa^*S^*\) for some \(u^*, v^* \in S\), whence \(x \in (a^*)\) or \(x \in (Sa^*)\) or \(x \in (a^*S)\) or \(x \in (Sa^*)\). Therefore \(x \in (a^*) \cup (Sa^*) \cup (a^*S) \cup (a^*S^*) \subseteq (a^* \cup Sa^* \cup a^*S \cup Sa^*S^*) = I(a^*),\) i.e. \((I(a))^* \subseteq I(a^*).\) Consequently \((I(a))^* = I(a^*)\).
4)\(\implies\) 5). It suffices to prove two notions. (i) \(I(a) = ((I(a^*))^0I(a))\). (ii) \(((I(a^*))^0I(a)) \subseteq (Sa*Sa*S)\). Then we can conclude that \(a \in I(a) \subseteq (Sa*Sa*S)\) and complete the proof.

(ii) By hypothesis and Proposition 2.1, we have \(I(a) = (I(a^*)I(a^*)) = ((I(a)I(a))(I(a))I(a)) = (I(a)I(a)I(a))\). Furthermore \(I(a) I(a)I(a)I(a) = ((I(a^*))^0I(a))\). Consequently \(I(a) \subseteq ((I(a^*))^0I(a)) \subseteq (S(a^*)^0I(a))\) and hence \(I(a) \subseteq ((I(a^*))^0I(a)) \subseteq I(a)\). Thus \(I(a) = ((I(a^*))^0I(a))\).

(i) Since \((I(a))^3 \subseteq (SaS)\) (has shown in Theorem 2.6), we have \((I(a))^5 = (I(a))^3\). If \((I(a))^3 \subseteq (SaS)\), then \(I(a) \subseteq ((I(a)^3)^0I(a)) \subseteq (SaS)^0I(a)\). Clearly \(I(a) \subseteq (SaS)^0I(a)\). Finally \((I(a))^3 \subseteq (SaS)\) and \((I(a)^3)^0I(a) \subseteq (SaS)^0I(a)\).

5)\(\implies\) 1). If \(x \in (A^*A^*)\), then \(x \subseteq yz\) for some \(y, z \in A^*\). By hypothesis \(y \in (Sy^*Sy)\), then \(y \subseteq u_1y^*u_2y^*u_3\) for some \(u_i \in S, i = 1, 2, 3\). Similarly, \(z \subseteq v_1z^*v_2z^*v_3\) for some \(v_i \in S, i = 1, 2, 3\). Consequently, \(yz \subseteq u_1y^*u_2y^*u_3v_1z^*v_2z^*v_3 \in Sy^*S \subseteq SAS \subseteq A\). Therefore \(x \in (A)\) because \(x \subseteq yz\), whence \((A^*A^*) \subseteq (A) = A\). If \(x \in A\), then we have \(x \subseteq w_1x^*w_2x^*w_3\) for some \(w_i \in S, i = 1, 2, 3\) because \(x \in (Sx^*S^x)\).

Conversely, let \(A, B\) and \(T\) be ideals of \(S\) with \(AB \subseteq T\). Since \(A^* = (AA)\), we have \(A^* \subseteq (AA)\). On the other hand let \(x \in (AA)\). Then \(x \subseteq a_1a_2 \in AA\) for some \(a_1, a_2 \in A\). Since \(A^* \subseteq (AA)\), we have \(a_1^* \subseteq u_1v_1 \in AA\) and \(a_2^* \subseteq u_2v_2 \in AA\) for some \(u_1, u_2, v_1, v_2 \in A\). Thus \(a_1 \subseteq (u_1v_1)\) and \(a_2 \subseteq (u_2v_2)\). This implies that \(x \subseteq a_1a_2 \subseteq (u_1v_1)^*(u_2v_2)^* \in (AA)^*(AA) = A^*A^*A^*A^* \subseteq A^*\) because \(A^*\) is an ideal. Consequently \(x \in (A^*) = A^*\). Therefore \((AA) \subseteq A^*\).

Conversely, let \(A, B\) and \(T\) be ideals of \(S\) with \(AB \subseteq T\).

\begin{definition} An ordered -semigroup \(S\) is called intra-regular if \(a \in (Sa*a*S)\) for any \(a \in S\).
\end{definition}
Proposition 2.10. Let $S$ be an ordered $*$-semigroup. Then $S$ is intra-regular if and only if the ideals of $S$ are semiprime.

Proof. Suppose that $I$ is an ideal of $S$ with $aa \in I$ for some $a \in S$. Since $S$ is intra-regular, we have $a^* \in (SaaS) \subseteq (SIS) \subseteq (I) = I$ and hence $I$ is semiprime.

Conversely, suppose that $a$ is an element of $S$. Clearly $(Sa^*a^*)^*a$ is an ideal. So $(Sa^*a^*)^*a$ is semiprime by hypothesis. This implies $aa = (a^*a^*)^*a \in (Sa^*a^*)^*$ because $(a^*a^*)^*(a^*a^*) \in (Sa^*a^*)^*a \subseteq (Sa^*a^*)^*$. Therefore $a^* \in (Sa^*a^*)^*$ whence it follows that $(a^*a^*)^*a \in (Sa^*a^*)^*$. Hence $a \in (Sa^*a^*)^*$ and we conclude that $S$ is intra-regular.

$\square$

Proposition 2.11. Let $S$ be an ordered $*$-semigroup. If $S$ is intra-regular, then $(SxyS) = (Sx^*y^*S)$ for any $x, y \in S$.

Proof. Let $x, y \in S$. Since $S$ is intra-regular, we have $xy \in (S(x)y^*)(xy)^*S = (Sy^*x^*y^*x^*S) \subseteq (Sx^*y^*S)$. Thus $xy \leq u_1x^*y^*u_2$ for some $u_1, u_2 \in S$. Hence $u_3u_4x^*u_2 \leq u_3u_4x^*u_2^*u_4 \in (Sa^*a^*)^*S \subseteq (Sx^*y^*S)$ for any $u_3, u_4 \in S$. This implies $SxyS \subseteq (Sx^*y^*S)$, so $(SxyS) \subseteq (Sx^*y^*S)$ by Proposition 2.1. By symmetry we have $(Sx^*y^*S) \subseteq (SxyS)$. Therefore $(SxyS) = (Sx^*y^*S)$.

$\square$

Proposition 2.12. Let $S$ be an ordered $*$-semigroup with order preserving involution $*$. If the ideals of $S$ are semiprime, then

1. $I(x) = (SxS)$ for any $x \in S$, and
2. $I(xy) = I(x) \cap I(y)$ for any $x, y \in S$.

Proof. 1) Let $x$ be an element in $S$. Note that $(SxS)$ is an ideal whence is semiprime. Applying this fact and $x^2x^2 = x^4 \in (SxS)$ yields $x^*x^* = (x^2)^* \in (SxS)$. Similarly $x \in (SxS)$ so that $I(x) \subseteq (SxS)$. Furthermore $(SxS) \subseteq (x \cup xS \cup Sx \cup SzS) = I(x)$. Hence $I(x) = (SxS)$.

2) Since $xy \in I(x)S \subseteq I(x)$, we have $I(xy) \subseteq I(x)$. Also $I(xy) \subseteq I(y)$ because $xy \in SI(y) \subseteq I(y)$. Thus $I(xy) \subseteq I(x) \cap I(y)$.

We now show that $I(x) \cap I(y) \subseteq I(xy)$. If $z \in I(x) \cap I(y)$, then $z \in (SxS) \cap (SyS)$ by 1), whence $z \leq u_1xu_2$ and $z \leq v_1v_2$ for some $u_1, u_2, v_1, v_2 \in S$. Note that $(yv_2u_1x)^2 = yv_2u_1xv_2u_1x \in (SyS) = I(x)$ and that $I(xy)$ is semiprime. Thus $(yv_2u_1x)^* \in I(xy)$. Therefore $z^*z^* \leq (u_1xu_2)^*(v_1v_2)^* = u_2^*(yv_2u_1x)^*v_1^* \in I(xy)$. Hence $z^*z^* \in (I(xy)) = I(xy)$. It follows that $z \in I(xy)$, then $I(x) \cap I(y) \subseteq I(xy)$.

$\square$

Theorem 2.13. Let $S$ be an ordered $*$-semigroup with order preserving involution $*$. The ideals of $S$ are prime if and only if $S$ is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$.

Proof. If the ideals are prime, then they are weakly prime, and hence Theorem 2.8 implies that any two ideals are comparable. Let $a \in S$. Note that $(Sa^*a^*)^*$ is an ideal by Proposition 2.1, whence is prime. Therefore $a^4 \in (Sa^*a^*)^*$ because $(a^*a^*)^{a^4} \in (Sa^*a^*)^*$. A similar argument shows $(a^*)^2 \in (Sa^*a^*)^*$ and $a \in (Sa^*a^*)^*$; that is, $S$ is intra-regular.
Conversely suppose that $S$ is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$. Let $T$ be an ideal of $S$ and $ab \in T$. We claim that $a^* \in T$ or $b^* \in T$. By virtue of Proposition 2.10, $I(a)$ is semiprime. Thus $aa \in I(a)$ implies $a^* \in I(a)$. $b^* \in I(b)$ is proved similarly. Furthermore by hypothesis we have either $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a^* \in I(a) = I(a) \cap I(b) = I(ab) \subseteq T$ by Proposition 2.12. If $I(b) \subseteq I(a)$, then $b^* \in I(b) = I(a) \cap I(b) = I(ab) \subseteq T$.

\[ \square \]

3 Characterization of Intra-Regular Ordered $\ast$-Semigroups

In Section 2 we considered ideals. In this section we shall introduce the notion of filters which will be used to establish some congruence. Once some properties are well made it is not difficult to establish the characterization. For convenience we define $aIb$ if and only if $I(a) = I(b)$.

**Definition 3.1.** Let $S$ be an ordered $\ast$-semigroup. A subsemigroup $F$ of $S$ is called a filter if

1. for any $a, b \in S$, $ab \in F$ implies $a^* \in F$ and $b^* \in F$,
2. for any $a \in F$, $c \in S$, $c \geq a$ implies $c \in F$.

Let $N(x)$ be the least filter of $S$ containing $x$ and $N$ be defined by $N := \{(x, y) \in S \times S \mid N(x) = N(y)\}$. A congruence on ordered $\ast$-semigroup $S$ is an equivalence relation $\sigma$ on $S$ which preserves both $\cdot$ and $\ast$. In other words, if $(a, b) \in \sigma$, then $(a^*, b^*) \in \sigma$ [1].

**Definition 3.2.** A congruence $\sigma$ on ordered $\ast$-semigroup $S$ is called semilattice congruence if $(a^*a^*, a) \in \sigma$ and $(ab, ba) \in \sigma$ for any $a, b \in S$. A semilattice congruence $\sigma$ on $S$ is called complete if $a \leq b$ implies $(a, ab) \in \sigma$.

**Proposition 3.3.** Let $S$ be an ordered $\ast$-semigroup. Then the relation $N$ is a complete semilattice congruence on $S$.

**Proof.** Trivially $N$ is an equivalence relation on $S$. Let $(a, b) \in N$. In order to show that $N$ is a congruence, it suffices to prove that $(ac, bc) \in N$ for any $c \in S$ since $(ca, cb) \in N$ can be proved similarly. If $N(ac) = N(ab)$, $N(bc) = N(ba)$ and $N(ab) = N(ba)$ for any $c \in S$, then $N(ac) = N(bc)$, whence $(ac, bc) \in N$.

We first show that $N(ac) = N(ab)$. Obviously $ac \in N(ac)$. Thus $a^* \in N(ac)$ and hence $a^*a^* \in N(ac)$. It follows that $a \in N(ac)$, whence $N(a) \subseteq N(ac)$. Therefore $b \in N(b) = N(a) \subseteq N(ac)$ because $(a, b) \in N$. Consequently $ab \in N(ac)$ and $N(ab) \subseteq N(ac)$. $N(ac) \subseteq N(ab)$ is proved similarly.

$N(bc) = N(ba)$ is completed by similar arguments.

Next we show that $N(ab) = N(ba)$. Since $ab \in N(ab)$, we have $a^*, b^* \in N(ab)$ by Definition 3.1. Then $a^*a^*, b^*b^* \in N(ab)$ because $N(ab)$ is a subsemigroup.
Again \( a, b \in N(ab) \) follows directly from Definition 3.1. Hence \( ba \in N(ab) \) and \( N(ba) \subseteq N(ab) \). Similarly \( N(ab) \subseteq N(ba) \).

Now we turn to prove that \( \mathcal{N} \) is a semilattice congruence. In addition to the fact that \( N(ab) = N(ba) \) we shall need to show that \( (a, a^* a^*) \in \mathcal{N} \). Clearly \( aa \in N(a) \), and hence \( a^* \in N(a) \). Therefore \( N(a^* a^*) \subseteq N(a) \) because \( a^* \in N(a) \). On the other hand \( a^* a^* \in N(a^* a^*) \), whence \( a \in N(a^* a^*) \). This implies \( N(a) \subseteq N(a^* a^*) \). Consequently \( N(a) = N(a^* a^*) \); that is, \( (a, a^* a^*) \in \mathcal{N} \).

To complete the proof we claim that \( a \leq b \) implies \( (a, b) \in \mathcal{N} \). Observe that \( ab \in N(ab) \), whence \( a^* \in N(ab) \) and \( a^* a^* \in N(ab) \). Therefore \( a \in N(ab) \), whence \( N(a) \subseteq N(ab) \). Furthermore since \( a \leq b \) and \( a \in N(a) \), this implies that \( b \in N(a) \) by Definition 3.1. Thus \( ab \in N(a) \), whence \( N(ab) \subseteq N(a) \). We conclude that \( N(a) = N(ab) \), and \( (a, b) \in \mathcal{N} \).

\[ \square \]

**Proposition 3.4.** Let \( S \) be an ordered \( \ast \)-semigroup with order preserving involution \( \ast \). Then \( S \) is intra-regular if and only if \( N(x) = \{ y \in S \mid x \in \langle Sy^* S \rangle \} \).

**Proof.** Suppose \( S \) is intra-regular. Let \( T_x = \{ y \in S \mid x \in \langle Sy^* S \rangle \} \) for any \( x \in S \). We shall show that \( T_x \) is a filter, then claim that \( T_x \subseteq F \) for any filter \( F \) containing \( x \). To show that \( T_x \) is a filter, we first prove that \( T_x \) is a subsemigroup. Let \( a, b \in T_x \). Then \( x \in \langle Sx^* x \rangle \) since \( S \) is intra-regular. Thus \( x \leq v_1 x^* x \) for some \( v_1, v_2 \in S \), and \( x \in \langle Sx \rangle \). By definition \( x \in T_x \), whence \( T_x \neq \emptyset \). Let \( a, b \in T_x \). Then \( x \in \langle Sa \rangle \) and \( x \in \langle Sb \rangle \), hence \( x \leq u_1 a^* u_2 \) and \( x \leq u_3 b^* u_4 \) for some \( u_1 \in S, i = 1, \ldots, 4 \). This implies that \( x^* \leq u_1^* a u_1^* \) and \( x^* \leq u_4^* b u_4^* \) because \( \ast \) is an order preserving involution. Note that \( x \in \langle Sx^* x \rangle \) and therefore \( x \leq u_5 x^* x \) for some \( u_5 \). Thus \( x \in \langle S \rangle \). Consequently \( x \leq u_5^* a u_5^* \) and \( x \leq u_4^* b u_4^* \) because \( \ast \) is an order preserving involution. Finally we obtain that \( x \leq u_5^* a u_5^* \) and \( x \leq u_4^* b u_4^* \) because \( \ast \) is an order preserving involution. Therefore \( x \leq u_5^* a u_5^* \) and \( x \leq u_4^* b u_4^* \) because \( \ast \) is an order preserving involution. Now we claim that \( T_x \) is the least filter containing \( x \), i.e. \( T_x = N(x) \). Let \( F \) be a filter of \( S \) containing \( x \) and \( t \) an element of \( T_x \). By definition \( x \in \langle St \rangle \), then \( x \leq u_1 t^* u_2 \) for some \( u_1, u_2 \in S \). Since \( S \) is intra-regular, this implies \( t \leq u_3 t^* u_4 \) for some \( u_3, u_4 \in S \). Thus \( t^* \leq u_3^* t^* u_4^* \) and \( x \leq u_1 t^* u_2 \leq u_1 (u_3^* t^* u_4^*) u_2 = u_1 u_3^* u^* u_2^* u_2 \). Therefore \( u_1 u_3^* u^* u_2^* u_2 = (u_1 u_3^* t^*) (u_4^* u_2^* u_2) \in F \) because \( F \) is a filter containing \( x \). Definition 3.1 implies \( (u_1 u_3^* t^*) \ast t^* u_4^* u_2^* = (t^*) (t^* u_4^* u_2^*) \in F \). Again \( t = (t^*) \ast \in F \) by the same reason. We conclude that \( T_x \subseteq F \), whence \( T_x \) is the filter generated by \( x \).
Conversely, suppose $N(x) = \{y \in S \mid x \in (S^*yS)\}$. Let $x \in S$. Observe that $N(x)$ is a subsemigroup and $x \in N(x)$. Thus $x^2 \in \{y \in S \mid x \in (S^*yS)\}$. This implies $x \in (S(x^2)^*S) = (Sx^*x^*S)$, i.e. $S$ is intra-regular. \hfill \Box

**Theorem 3.5.** Let $S$ be an ordered $*$-semigroup with order preserving involution $\ast$. Then $S$ is intra-regular if and only if $N = I$.

**Proof.** Suppose that $S$ is intra-regular. To show that $I \subseteq N$ we let $(a, b) \in I$ and $x \in N(a)$. Since $I(a) = I(b)$, we have $a \leq u_1x^*u_2$ for some $u_1, u_2 \in S$ by Proposition 3.4. Furthermore $b \in (a \cup aS \cup Sa \cup SaS)$ because $b \in I(b)$. Thus $b \leq a$ or $b \leq au_3$ or $b \leq u_3au_4$ or $b \leq u_3au_4$ for some $u_3, u_4 \in S$. This implies that $b \leq u_1x^*u_2 \in Sx^*S$ or $b \leq u_1x^*u_2$ or $b \leq u_3u_1x^*u_2 \in Sx^*S$ or $b \leq u_3u_1x^*u_2 \in Sx^*S$. Hence $b \in (Sx^*S)$, whence $x \in \{y \in S \mid b \in (Sx^*S)\} = N(b)$ by Proposition 3.4. We conclude that $N(a) \subseteq N(b)$. Similarly $N(b) \subseteq N(a)$. This means that $N(a) = N(b)$, hence $(a, b) \in N$.

To show that $N \subseteq I$ we let $(a, b) \in N$ and $x \in I(a)$. Note that $N(a) = N(b)$ and $I(a) = (a \cup aS \cup Sa \cup SaS)$. Then $x \leq a$ or $x \leq au_1$ or $x \leq u_1a$ or $x \leq u_1au_2$ for some $u_1, u_2 \in S$. Since $b \in N(b) = N(a) = \{y \in S \mid a \in (Sy^*S)\}$, we get $a \leq u_3b^*u_4$ for some $u_3, u_4 \in S$. This implies that $x \leq u_3b^*u_4$ or $x \leq u_3b^*u_4u_1$ or $x \leq u_3u_1b^*u_4$ or $x \leq u_3u_1b^*u_4u_2$. Also $b^* \leq u_5b^*u_6$ for some $u_5, u_6 \in S$ because $S$ is intra-regular. Therefore $x \leq u_3u_5b^*u_6u_4 \in SbS$ or $x \leq u_3u_5b^*u_6u_4u_1 \in SbS$ or $x \leq u_1u_3(u_5b^*u_6u_4) \in SbS$ or $x \leq u_1u_3(u_5b^*u_6u_4)u_2 \in SbS$. Thus $x \in (SbS) \subseteq I(b)$, hence $I(a) \subseteq I(b)$. Similarly $I(b) \subseteq I(a)$. We conclude that $I(a) = I(b)$ and $(a, b) \in I$.

To prove the converse let $a \in S$. Observe that $(a, a^*a^*) \in N$ by Definition 3.2 and Proposition 3.3. Thus $N = I$ implies that $(a, a^*a^*) \in I$. Therefore $a \in I(a) = I(a^*a^*)$, hence $a \in (a^*a^* \cup a^*a^*S \cup Sa^*S \cup Sa^*a^*)$. We now consider the four possibilities: (i) $a \leq a^*a^*$; (ii) $a \leq a^*a^*u_1$; (iii) $a \leq u_1a^*a^*$; (iv) $a \leq u_1a^*a^*u_2$ for some $u_1, u_2 \in S$. In case (i) clearly $a \leq a^*a^*$; (ii) $a \leq a^*a^*u_1 \leq a^*u_1^*a^*u_1 \leq a^*u_1^*(a^*a^*)u_1 = a^*a^*a^*u_1 \leq a^*u_1^*(a^*a^*)u_1 = a^*a^*a^*u_1 \leq a^*a^*a^*u_1 \leq a^*a^*a^*u_1 \leq a^*a^*a^*$. Hence $a \leq u_1a^*a^*u_2 \in SbS$. In case (iv) it is easy to see that $a \leq u_1a^*a^* \leq u_1a^*(u_1a^*a^*)u_1 \leq a^*a^*a^*u_1 \leq a^*a^*a^*$. Therefore $a \in (Sa^*a^*)$; that is, $S$ is intra-regular. \hfill \Box

**Example 3.6.** Let $S = \{a, b, c\}$ be an ordered semigroup. The multiplication “$\cdot$” and the corresponding Hasse diagram are given below. Define the involution $\ast$ by $a^* = a$ and $b^* = c$ (hence $c^* = b$). It is easy to check that $S$ is an ordered $*$-semigroup with order preserving involution $\ast$.

\[
\leq: \{(a, a), (b, a), (b, b), (c, a), (c, c)\}
\]
By Definition 2.9, \( S \) is intra-regular because \((S a^* a^* S) = (S b^* b^* S) = (S c^* c^* S) = S\). Also, by Definition 3.1, \( N(a) = N(b) = N(c) = S \), thus \( \mathcal{N} := \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\} \). Furthermore \( I(a) = (a \cup Sa \cup aS \cup SaS) \) implies \( I(a) = S \).

Similarly, \( I(b) = I(c) = S \). Therefore \( \mathcal{I} := \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\} \), whence \( \mathcal{N} = \mathcal{I} \).

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References


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