Certain Kummer’s Matrix Function of Two Complex Variables under Certain Differential and Integral Operators

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Abstract : The main aim of this paper is to define and study of a new matrix function, say, the Kummer’s matrix function of two complex variables. The domain of regularity, integral form, matrix recurrence relation and several new results on this function are established. We study the operation of differential operators of Kummer’s matrix function that solutions of certain matrix differential equations have been obtained. Finally the study of the integral operators with of Kummer’s matrix function is investigated.

Keywords : hypergeometric matrix function; Kummer’s matrix function; matrix recurrence relation; matrix differential equation; differential operator; integral operator.

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1 Introduction and Preliminaries

Special functions, mathematical physics, and orthogonal polynomials are closely related to [1, 2]. Special matrix functions appear in connection with statistics [3–5], theoretical physics, group representation theory, Lie groups theory [6]. In [7–12], the hypergeometric matrix function has been introduced as a matrix power series and an integral representation. Moreover, Jódar and Cortés introduced and studied the hypergeometric matrix function and the hypergeometric matrix differential equation in [13] and the explicit closed form general solution of it has been given in [14]. Recently, extension to the matrix framework of the classical families of Legendre, pseudo Legendre, Humbert, Bessel, and Gegenbauer matrix polynomials, Tricomi and Hermite-Tricomi matrix functions have been proposed and studied in a number of papers [15, 16, 17, 18, 19, 20, 21, 22]. The reason of interest for this family of hypergeometric function is due to their intrinsic mathematical importance and to the fact that these functions have applications in physics [23–27].

The primary goal of this paper is to consider a new system of matrix function, namely the Kummer’s matrix function. The structure of the paper is as follows: In Section 2 a definition of Kummer’s matrix function is given and the radius of regularity and an integral form are given. Some matrix recurrence relations are established in Section 3. The effect of differential operator on these function is investigated and the matrix differential equation satisfied by them is presented in Section 4. Finally, the study of the integral operators with of Kummer’s matrix function is investigated in Section 5.

Throughout this paper $D_0$ will denote the complex plane and its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of $A$ [28]. If $A$ is a matrix in $\mathbb{C}^{N \times N}$, its two-norm denoted by $\|A\|$ is defined by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

where for a vector $y$ in $\mathbb{C}^N$, $\|y\|_2 = (y^T y)^{1/2}$ is Euclidean norm of $y$.

Let us denote $\alpha(A)$ the real number [29]

$$\alpha(A) = \max\{\text{Re}(z) : z \in \sigma(A)\}. \quad (1.1)$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable $z$, which are defined in an open set $\Omega$ of the complex plane and $A$ is a matrix in $\mathbb{C}^{N \times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [30], it follows that

$$f(A)g(A) = g(A)f(A). \quad (1.2)$$

Hence, if $B$ in $\mathbb{C}^{N \times N}$ is a matrix for which $\sigma(B) \subset \Omega$ also and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \quad (1.3)$$
The reciprocal gamma function denoted by $\Gamma^{-1}(z) = 1/\Gamma(z)$ is an entire function of the complex variable $z$. Then for any matrix $A$ in $\mathbb{C}^{N \times N}$, the image of $\Gamma^{-1}(z)$ acting on $A$ denoted by $\Gamma^{-1}(A)$ is a well-defined matrix. Furthermore, if $A + nI$ is invertible for all integer $n \geq 0$ \hspace{1cm} (1.4)

then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets the formula

$$(A)_n = A(A + I) \cdots (A + (n - 1)I) = \Gamma(A + nI)\Gamma^{-1}(A); \quad n \geq 1; \quad (A)_0 = I. \hspace{1cm} (1.5)$$

Jódar and Cortés have proved in [29], that

$$\Gamma(A) = \lim_{n \to \infty} (n - 1)![(A)_n]^{-1}n^A. \hspace{1cm} (1.6)$$

Using relation (17) of [14] and [32] for any square complex matrix $A$, it follows that in the form

$$\|e^{tA}\| \leq e^{\alpha(A)} \sum_{i=0}^{r-1} \left(\frac{\|A\|^{\frac{1}{r+1}}i}{i!}\right)^i; \quad t \geq 0$$

and

$$\|n^A\| \leq n^{\alpha(A)} \sum_{i=0}^{r-1} \left(\frac{\|A\|^{\frac{1}{r}}\ln n}{i!}\right)^i; \quad n \geq 1. \hspace{1cm} (1.7)$$

In the following, we introduce to define and study of a new matrix function which represents of the Kummer’s matrix function as given by the relation and the radius of regularity and an integral form are given.

## 2 On Kummer’s Matrix Function

The kummer’s matrix function $\Phi_1(A; B; C; z, w)$ of two complex variables is written in the form

$$\Phi_1(A; B; C; z, w) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(B)m[(C)_{m+n}]^{-1}}{m!n!} z^m w^n = \sum_{m,n=0}^{\infty} U_{m,n}(z, w)$$

\hspace{1cm} (2.1)

where $U_{m,n}(z, w) = (A)_{m+n}(B)m[(C)_{m+n}]^{-1} z^m w^n$ and $U_{m,n} = (A)_{m+n}(B)m[(C)_{m+n}]^{-1}$.

For simplicity, we can write the $\Phi_1(A \pm I; B; C; z, w)$ in the form $\Phi_1(A \pm)$, $\Phi_1(A, B \pm I; C; z, w)$ in the form $\Phi_1(B \pm)$, . . . , and $\Phi_1(A, B; C \pm I; z, w)$ in the form $\Phi_1(C \pm)$.
The radius of regularity $R$ of the Kummer’s matrix function for this purpose, we recall relation (1.3.10) of [33, 34]. Hence

$$\frac{1}{R} = \lim_{m+n\to\infty} \sup_{\|U_{m,n}\|} \frac{\|U_{m,n}\|}{\sigma_{m,n}} = \lim_{m+n\to\infty} \sup_{\|U_{m,n}\|} \left( \frac{\|(A)_{m+n}(B)_{m}[C]_{m+n}\|}{m!n!\sigma_{m,n}} \right)^{\frac{1}{m+n}}$$

$$= \lim_{m+n\to\infty} \sup \left\| \frac{(m+n)^{-A}(A)_{m+n}(m+n-1)!((m+n)^{A}m^{-B}(B)_{m})}{(m+n-1)!} \right\|^{\frac{1}{m+n}} \times \left\| (m-1)!m^{B}\frac{m+n-1)^{-C}}{(m+n-1)!}(m+n)^{1\cdot B}I_{m+n}m+n \right\|^{\frac{1}{m+n}}$$

$$= \lim_{m+n\to\infty} \sup \left\| (\Gamma(A)\Gamma(A))^{-1}[\Gamma(B)]^{-1}(m+n)^{1\cdot C}m+n\right\|^{\frac{1}{m+n}} = 0$$

where

$$\sigma_{m,n} = \begin{cases} \frac{(m+n)^{2}}{(m+n)^{2}}, & m, n \neq 0; \\ 1, & m, n = 0. \end{cases}$$

and

$$1 \leq \sigma_{m,n} \leq (\sqrt{2})^{m+n}.$$
3 Matrix Recurrence Relation for Kummer’s Matrix Function

Some matrix recurrence relation is carried out on the Kummer’s matrix function. In this connection the following matrix contiguous functions relations follow, directly by increasing or decreasing one in original relation

\[
\Phi_1(A+) = \sum_{m,n=0}^{\infty} \frac{(A+I)_{m+n}(B)^m(C)^n}{m!n!} z^m w^n\]

\[
= A^{-1} \sum_{m,n=0}^{\infty} (A + (m + n)I)U_{m,n}(z, w) .
\]

Similarly, we have

\[
\Phi_1(A-) = \sum_{m,n=0}^{\infty} (A - I)((A + (m + n - 1)I)^{-1}U_{m,n}(z, w),
\]

\[
\Phi_1(C+) = \sum_{m,n=0}^{\infty} C(C + (m + n)I)^{-1}U_{m,n}(z, w),
\]

\[
\Phi_1(C-) = \sum_{m,n=0}^{\infty} (C - I)^{-1}(C + (m + n - 1)I)U_{m,n}(z, w).
\]

By the same way, yields

\[
\Phi_1(A+, B+) = A^{-1}B^{-1} \sum_{m,n=0}^{\infty} (A + (m + n)I)(B + mI)U_{m,n}(z, w),
\]

\[
\Phi_1(A-, B-) = (A - I)(B - I)
\]

\[
\times \sum_{m,n=0}^{\infty} [(A + (m + n - 1)I)^{-1}(B + (m - 1)I)^{-1}U_{m,n}(z, w),
\]

\[
\Phi_1(B+; C+) = \sum_{m,n=0}^{\infty} CB^{-1}(B + mI)(C + (m + n)I)^{-1}U_{m,n}(z, w),
\]

\[
\Phi_1(B-; C-) = \sum_{m,n=0}^{\infty} (B - I)(B + (m - 1)I)^{-1}(C - I)^{-1}(C + (m + n - 1)I)U_{m,n}(z, w).
\]

4 Differential Operator for Kummer’s Matrix Function

Consider the differential operator \( D \), as given in [33, 35], takes the form

\[
D = \begin{cases} 
  d_1 + d_2, & m, n \geq 1, \\
  1, & \text{otherwise}.
\end{cases}
\]
where $d_1 = \frac{\partial}{\partial z}$ and $d_2 = \frac{\partial}{\partial w}$.

For the Kummer's matrix function the following matrix contiguous function relations can be defined

$$
(D I + A) \Phi_1 = \sum_{m,n=0}^{\infty} \frac{(A + (m+n)I)(A)_{m+n}(B)_{m}[(C)_{m+n}]^{-1}}{m!n!} z^m w^n
$$

$$
= \sum_{m,n=0} (A + (m+n)I)U_{m,n}(z, w) = A \Phi_1(A+).
$$

(4.2)

By the same way, we get

$$
(d_1 I + B) \Phi_1 = B \Phi_1(B+),
$$

$$
(D I + C - I) \Phi_1 = (C-I) \Phi_1(C-).
$$

(4.3)

From (4.2) and (4.3), it follows at once that

$$
(A - C + I) \Phi_1 = A \Phi_1(A+) - (C-I) \Phi_1(C-),
$$

$$
(A - B) \Phi_1 = A \Phi_1(A+) - B \Phi_1(B+) - d_2 \Phi_1(B),
$$

$$
(B - C + I) \Phi_1 = B \Phi_1(B+) + d_2 \Phi_1 - (C-I) \Phi_1(C-).
$$

(4.4)

Acting the operating with $D$ on the Kummer’s matrix function of two complex variables yields

$$
D \Phi_1 = \sum_{m,n=1}^{\infty} \frac{(m+n)(A)_{m+n}(B)_{m}[(C)_{m+n}]^{-1}}{m!n!} z^m w^n
$$

$$
= zB \Phi_1(A, B+; C; z, w) + (zB + w)(A-C)[C]^{-1} \Phi_1(A, B; C+; z, w)
$$

$$
+ z(A-C)[C]^{-1}d_1 \Phi_1(A, B; C+; z, w) + w \Phi_1(A, B; C; z, w).
$$

(4.5)

In addition, we can be written another form of equation (4.5),

$$
D \Phi_1 = zABC^{-1} \Phi_1(A+, B+; C+; z, w) + wAC^{-1} \Phi_1(A+, B; C+; z, w).
$$

(4.6)

Writing $\Phi_1(A, B; C; z, w)$ we see that

$$
D(D I + C - I) \Phi_1 = \sum_{m,n=1}^{\infty} \frac{(m+n)(C + (m+n-1)I)(A)_{m+n}(B)_{m}[(C)_{m+n}]^{-1}}{m!n!} z^m w^n
$$

$$
= \left[ z(D I + A)(d_1 I + B) + w(D I + A) \right] \Phi_1(A, B; C; z, w)
$$

i.e.,

$$
\left[ D(D I + C - I) - z(D I + A)(d_1 I + B) - w(D I + A) \right] \Phi_1(A, B; C; z, w) = 0
$$

(4.7)
and $\Phi_1(A, B; C; z, w)$ is a solution of the above matrix differential equation.

Now, relation (3.2) yields,

$$\Phi_1(A-) = \sum_{m,n=0}^{\infty} (A - I)[A + (m + n - 1)I]^{-1}U_{m,n}(z, w)$$

$$= \Phi_1 - (A - I)^{-1}D \Phi_1(A-)$$

i.e. the $\Phi_1(A, B; C; z, w)$ is a solution to the matrix partial differential equation

$$D \Phi_1(A-) - (A - I) \Phi_1 + (A - I) \Phi_1(A-) = 0. \quad (4.8)$$

Next let us operator with $D$ on the series defining $\Phi_1(A-)$, we thus obtain

$$D \Phi_1(A-) = \sum_{m,n=0}^{\infty} \frac{(A - I)(m + n)[(A + (m + n - 1)I)]^{-1}(A)_{m+n}(B)_{m}[C]_{m+n}^{-1}}{m!n!}z^m w^n$$

$$= z(A - I)\Phi_1(A, B; C; z, w) + z(B - C)(A - I)[C]^{-1}\Phi_1(A, B; C+_z, w)$$

$$- z(B - C)(A - I)[C]^{-1}d_2 \Phi_1(A, B; C+_z, w) + w(A - I)[C]^{-1}\Phi_1(A, B; C+_z, w) \quad (4.9)$$

From (4.8) and (4.9) we get

$$(1 - z) \Phi_1 = \Phi_1(A-, B; C; z, w) + z(B - C)[C]^{-1}\Phi_1(A, B; C+_z, w)$$

$$- z(B - C)[C]^{-1}d_2 \Phi_1(A, B; C+_z, w) + w[C]^{-1}\Phi_1(A, B; C+_z, w). \quad (4.10)$$

We can easily see to $\Phi_1(C+)$ with the aid of (3.2) that

$$\Phi_1(C+) = \Phi_1 - C^{-1}D \Phi_1(C+)$$

i.e., the $\Phi_1(A, B; C; z, w)$ is a solution to the matrix partial differential equation

$$D \Phi_1(C+) = C \Phi_1 - C \Phi_1(C+). \quad (4.11)$$

Operate with $d_1$ on the function $\Phi_1(A, B; C; z, w)$, we obtain

$$d_1 \Phi_1 = \sum_{m=1, n=0}^{\infty} \frac{m(A)_{m+n}(B)_{m}[(C)_{m+n}]^{-1}}{m!n!}z^m w^n$$

$$= zAB[C]^{-1}\Phi_1(A + I, B + I; C + I; z, w), \quad (4.12)$$

$$d_1^{(2)} \Phi_1 = \sum_{m=2, n=0}^{\infty} \frac{m(m - 1)(A)_{m+n}(B)_{m}[(C)_{m+n}]^{-1}}{m!n!}z^m w^n$$

$$= z^2(A)_{2}[(B)_{2}]^{-1}\Phi_1(A + 2I, B + 2I; C + 2I; z, w) \quad (4.13)$$
and

$$d_1^2 \Phi_1 = d_1 \left[ AB[C]^{-1} \sum_{m,n=0}^{\infty} \frac{(A + I)_{m+n}(B + I)_m(C + I)_{m+n}^{-1}}{m!n!} z^{m+1} w^n \right]$$

$$= AB[C]^{-1} \sum_{m=1,n=0}^{\infty} \frac{m(A + I)_{m+n}(B + I)_m(C + I)_{m+n}^{-1}}{m!n!} z^{m+1} w^n$$

$$+ AB[C]^{-1} \sum_{m=1,n=0}^{\infty} \frac{(A + I)_{m+n}(B + I)_m(C + I)_{m+n}^{-1}}{m!n!} z^{m+1} w^n$$

$$= d_1^{(2)} \Phi_1 + d_1 \Phi_1$$

i.e.,

$$d_1^{(2)} \Phi_1 - d_1 (d_1 - 1) \Phi_1 = 0. \quad (4.14)$$

Similarly, for $d_2$, we have

$$d_2 \Phi_1 = wA[(C)]^{-1} \Phi_1 (A + I, B + I, C + I; z, w), \quad (4.15)$$

$$d_2^{(2)} \Phi_1 = w^2(A)_2[(C)]_2^{-1} \Phi_1 (A + 2I, B + 2I, C + 2I; z, w), \quad (4.16)$$

and

$$d_1 d_2 \Phi_1 = zw(A)_2B[(C)]_2^{-1} \Phi_1 (A + 2I, B + I, C + 2I; z, w). \quad (4.18)$$

Hence the Kummer’s matrix function is a solution for the matrix partial differential equations in the forms (4.14) and (4.17).

The $\beta(D)$ differential operator has been defined by Sayyed [33] in the form

$$\beta(D) = 1 + \sum_{k=1}^{N} D^k; \quad D^k = D D^{k-1}. \quad (4.19)$$

From (4.3), we obtain

$$D \Phi_1 = (C - I) \left[ \Phi_1 (A, B; C - I; z, w) - \Phi_1 (A, B; C; z, w) \right] \quad (4.20)$$

and

$$D^2 \Phi_1 = (C - I)(C - 2I) \Phi_1 (A, B; C - 2I; z, w)$$

$$- [(C - I)(C - 2I) + (C - I)^2] \Phi_1 (A, B; C - I; z, w) \quad (4.21)$$

$$+ (C - I)^2 \Phi_1 (A, B; C; z, w).$$
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Thus by mathematical induction, we have the following general form

\[ \beta(D)\Phi_1 \]

\[ = (1 + \sum_{k=1}^{N} D^k)\Phi_1(A, B; C; z, w) \]

\[ = \Phi_1(A, B; C; z, w) + \sum_{k=1}^{N} \left( \prod_{j=1}^{k} (C - jI) \Phi_1(A, B; C - jI; z, w) \right. \]

\[ - \left[ \prod_{j=1}^{k-1} (C - jI) + \prod_{j=1}^{N-1} (C - jI) \sum_{k=1}^{N} (C - jI) \right] \Phi_1(A, B; C - (j - 1)I; z, w) \]

\[ + \left[ \prod_{j=1}^{k-2} (C - jI) \sum_{j=1}^{k-1} (C - jI) + \prod_{j=1}^{k-2} (C - jI) \left( \sum_{j=1}^{k-2} (C - jI)^2 \right) \right. \]

\[ + \sum_{j=1}^{k-4} (C - jI)(C - (j + 1)I) + \sum_{j=1}^{k-4} (C - jI)(C - (j + 1)I) \cdots \] (4.22)

\[ \left. \times \Phi_1(A, B; C - (j - 2)I; z, w) + \cdots + (-1)^k(C - I)^k \Phi_1(A, B; C; z, w) \right] \]

where \( N \) is a finite positive integer.

Special cases: if we put \( A = C \) in (4.6) we get

\[ D \Phi_1(-, B; -, z, w) = zB \Phi_1(-, B+; -, z, w) + w \Phi_1(-, B; -, z, w) \]

\[ = zB \Phi_1(-, B; -, z, w) + zd_1 \Phi_1(-, B; -, z, w) \] (4.23)

\[ + w \Phi_1(-, B; -, z, w) \]

i.e.,

\[ \left[ DI - zId_1 - zB - wI \right] \Phi_1(-, B; -, z, w) = 0. \] (4.24)

This means that the power series \( \Phi_1(-, B; -, z, w) \) is a solution of the matrix partial differential equation (4.24).

Also, we consider the function \( \Phi_1(A, -, C; z, w) \), then

\[ D \Phi_1(A, -, C; z, w) = (z + w) \left[ \Phi_1(A, -, C; z, w) + (A - C)C^{-1} \Phi_1(A, -, C+; z, w) \right] \]

i.e.,

\[ \left[ D - (z + w) \right] \Phi_1(A, -, C; z, w) - (z + w)(A - C)C^{-1} \Phi_1(A, -, C+; z, w) = 0 \] (4.25)
thus the matrix differential equation (4.25) has a solution of the form $\Phi_1(A, -; C; z, w)$.

Now, we take the operator $D^2$ in consideration, as follows

$$D^2 \Phi_1(-, B; -; z, w) = z^2(d_1 I + B)(d_1 I + B + I) \Phi_1(-, B; -; z, w) + w^2 \Phi_1(-, B; -; z, w)$$

i.e.,

$$\left[D^2 - z^2(d_1 I + B)(d_1 I + B + I) - w^2\right] \Phi_1(-, B; -; z, w) = 0 \quad (4.26)$$

and $\Phi_1(-, B; -; z, w)$ is a solution for the matrix differential equation.

5 Integral Operator for Kummer’s Matrix Function

Define an integral operator $\hat{D}$ acting on the above hypergeometric function such that [33, 36]

$$\hat{D} = \begin{cases} 1; \\ \frac{1}{z} \int_0^z dz + \frac{1}{w} \int_0^w dw; & m, n = 0, \\ \text{otherwise.} \end{cases} \quad (5.1)$$

where the integration is carried out with respect to each variable individually on the assumption that the other is constant.

Let $\hat{D}$ acts on the Kummer’s matrix function, then we have

$$\hat{D} \Phi_1(A, B; C; z, w) = \sum_{m,n} \left( \frac{1}{m+1} + \frac{1}{n+1} \right) \frac{(A)_{m+n}(B)^m(C)^{m+n} - 1}{m!n!} z^m w^n$$

$$= \frac{1}{z} [(B - I)^{-1} \Phi_1(A, B--; C; z, w) + \frac{1}{z} (C - A) [(A - I)^{-1}$$

$$\times [(B - I)^{-1} \Phi_1(A-, B--; C; z, w)$$

$$+ \frac{1}{w} \Phi_1(A, B; C; z, w) + \frac{1}{w} (C - A) [(A - I)]^{-1}$$

$$\times \Phi_1(A-, B; C; z, w). \quad (5.2)$$

The relation (5.2) can be written in the form

$$\hat{D} \Phi_1(A, B; C; z, w) = \frac{1}{z} (C - I) [(A - I)]^{-1} [(B - I)]^{-1} \Phi_1(A-, B--; C--; z, w)$$

$$+ \frac{1}{w} (C - I) [(A - I)]^{-1} \Phi_1(A-, B; C--; z, w)$$

and

$$\hat{D} \Phi_1(A, B; C; z, w) = \frac{1}{zw} (C - I)_2 [(A - I)_2]^{-1} [(B - I)_2]^{-1} D$$

$$\times \Phi_1(A - 2I, B - I; C - 2I; z, w). \quad (5.3)$$
We can written \( \hat{D} = \hat{d}_1 + \hat{d}_2 \) where \( \hat{d}_1 = \frac{1}{z} \int_0^z dz \) and \( \hat{d}_2 = \frac{1}{w} \int_0^w dw \), then the operator \( \hat{D}^2 \) is such that

\[
\hat{D}^2 = \hat{D}\hat{D} = (\hat{d}_1)^2 + 2\hat{d}_1\hat{d}_1 + (\hat{d}_1)^2 = \frac{1}{z^2} \int_0^z \int_0^z dz dz + \frac{2}{z w} \int_0^z \int_0^z dz dw \\
+ \frac{1}{w^2} \int_0^w \int_0^w dwdw.
\]

We see that,

\[
\hat{D}^2 \Phi_1 = \sum_{m,n} \frac{(A)_{m+n-2}}{m!n!} (B)_{m-2} [(C)_{m+n-2}]^{-1} z^m w^n \\
+ 2 \sum_{m,n} \frac{(A)_{m+n-2}}{m!n!} (B)_{m-1} [(C)_{m+n-2}]^{-1} z^m w^n \\
+ \sum_{m,n} \frac{(A)_{m+n-2}}{m!n!} (B)_{m} [(C)_{m+n-2}]^{-1} z^m w^n
\]

\[
= \frac{1}{z^2} \left[ [(B - 2I)z]^{-1} \Phi_1(A, B - ; C; z, w) \\
+ (C - A)(A - C + I)[(A - I)]^{-1} [(B - 2I)z]^{-1} \Phi_1(A, A - I, B - ; C; z, w) \\
+ (C - A)(A - C + I)[(A - I)]^{-1} [(B - 2I)z]^{-1} \Phi_1(A - I, A - I, B - ; C; z, w) \\
+ \frac{2}{zw} \left[ [(B - I)z]^{-1} \Phi_1(A, B - ; C; z, w) \\
+ (C - A)(A - C + I)[(A - I)]^{-1} [(B - I)z]^{-1} \Phi_1(A, A - I, B - ; C; z, w) \\
+ (C - A)(A - C + I)[(A - I)]^{-1} [(B - I)z]^{-1} \Phi_1(A - I, A - I, B - ; C; z, w) \\
+ \frac{1}{w^2} \left[ \Phi_1(A, B; C; z, w) + (C - A)(A - C + I)[(A - I)]^{-1} \Phi_1(A - I, B; C; z, w) \\
+ (C - A)(A - C + I)[(A - I)]^{-1} \Phi_1(A - I, A - I, B; C; z, w) \right] \right].
\]

In addition, we obtain the following

\[
\hat{D}(\hat{D} - 1) \Phi_1 = \frac{1}{z^2} [(C - 2I)z]^{-1} [(A - 2I)]^{-1} [(B - 2I)z]^{-1} \Phi_1(A - 2I, B - 2I; C - 2I; z, w) \\
+ \frac{2}{zw} [(C - 2I)z]^{-1} [(A - 2I)]^{-1} [(B - I)z]^{-1} \Phi_1(A - 2I, B - I; C - 2I; z, w) \\
+ \frac{1}{w^2} [(C - 2I)z]^{-1} [(A - 2I)]^{-1} \Phi_1(A - 2I, B; C - 2I; z, w) \\
- \frac{1}{z} [(C - I)[(A - I)]^{-1} [(B - I)]^{-1} \Phi_1(A - I, B - I; C - I; z, w) \\
- \frac{1}{w} [(C - I)[(A - I)]^{-1} \Phi_1(A - I, B; C - I; z, w).
Also, we get relations concerning \( \hat{d}_1 \) and \( \hat{d}_2 \) individually,

\[
\hat{d}_1 \Phi_1 = \frac{1}{z}(C - I) [(A - I)]^{-1}[(B - I)]^{-1} \Phi_1(A-, B-; C-; z, w) \quad (5.4)
\]

and

\[
\hat{d}_1(\hat{d}_1 - 1) \Phi_1 = \frac{1}{z^2}(C - 2I)_2 [(A - 2I)_2]^{-1}[(B - 2I)_2]^{-1} \\
\times \Phi_1(A - 2I, B - 2I; C - 2I; z, w) - \frac{1}{z}(C - I) [(A - I)]^{-1}[(B - I)]^{-1} \Phi_1(A - I, B - I; C - I; z, w). \quad (5.5)
\]

Similarly, for \( \hat{d}_2 \), we have

\[
\hat{d}_2 \Phi_1 = \frac{1}{w}(C - I) [(A - I)]^{-1} \Phi_1(A - I, B; C - I; z, w), \quad (5.6)
\]

\[
\hat{d}_2(\hat{d}_2 - 1) \Phi_1 = \frac{1}{w^2}(C - 2I)_2 [(A - 2I)_2]^{-1} \Phi_1(A - 2I, B; C - 2I; z, w) \\
- \frac{1}{w}(C - I) [(A - I)]^{-1} \Phi_1(A - I, B; C - I; z, w) \quad (5.7)
\]

and

\[
\hat{d}_1 \hat{d}_2 \Phi_1 = \frac{1}{zw}(C - 2I)_2 [(A - 2I)_2]^{-1}[(B - I)]^{-1} \Phi_1(A - 2I, B - I; C - 2I; z, w). \quad (5.8)
\]

Now, consider the operator \( \hat{I} \) [36], where

\[
\hat{I} = \left\{ \begin{array}{ll}
1; & m, n = 0, \\
\int_0^1 dz + \int_0^w dw; & \text{otherwise}.
\end{array} \right. \quad (5.9)
\]

Acting by this operator on the Kummer’s matrix function, it follows that

\[
\hat{I} \Phi_1 = \sum_{m,n} \frac{(A)m+n(B)_m[(C)_{m+n}]^{-1}}{(m+1)n!} z^{m+1} w^n + \sum_{m,n} \frac{(A)m+n(B)_m[(C)_{m+n}]^{-1}}{m!(n+1)!} z^{m+1} w^{n+1} \\
= [(B - I)]^{-1} \Phi_1(A, B-, C; z, w) + (C - A) [(A - I)]^{-1} [(B - I)]^{-1} \\
\times \Phi_1(A-, B--; C; z, w) + \Phi_1(A, B; C; z, w) \quad (5.10)
\]

or

\[
\hat{I} \Phi_1 = (C - I) [(A - I)]^{-1} [(B - I)]^{-1} \Phi_1(A-, B--; C--; z, w) + (C - I) [(A - I)]^{-1} \Phi_1(A-, B; C--; z, w). \quad (5.11)
\]
Hence
\[ \hat{I}^2 \Phi_1 = (C - I)(A - I)^{-1}((B - I))^{-1}\Phi_1(A - 2I, B - 2I; C - 2I; z, w) \]
\[ + 2(C - I)(A - I)^{-1}((B - I))^{-1}\Phi_1(A - 2I, B - I; C - 2I; z, w) \]
\[ + (C - I)(A - I)^{-1}\Phi_1(A - 2I, B; C - 2I; z, w). \] (5.12)

Now, we consider the integral operator \( \hat{I}_z^\mu \), where \( \hat{I}_z = \int_0^z dz \) such that
\[ \hat{I}_z^\mu = \hat{I}_z \hat{I}_z^{-1} \]
in the form
\[ \hat{I}_z \Phi_1 = (C - I)(A - I)^{-1}((B - I))^{-1}\Phi_1(A - 1, B - 1; C - 1; z, w) \] (5.13)
and
\[ \hat{I}_z^\mu \Phi_1 = (C - I)(C - 2I)((A - I)^{-1}((B - 2I))^{-1} \cdot ... \cdot (B - I)\Phi_1(A - 1, B - 1; C - 1; z, w) \]
\[ \times (A - 2I) - 1[(B - 2I)]^{-1} \cdot ... \cdot (A - 2I)\Phi_1(A - 1, B - 1; C - 1; z, w) \] (5.14)

It follows by mathematical induction that
\[ \hat{I}_z^\mu \Phi_1 = (C - I)(C - 2I) \cdots (C - \mu I)][(A - I)^{-1}[(B - 2I)]^{-1} \cdot ... \cdot [(B - I)\Phi_1(A - 1, B - 1; C - 1; z, w) \]
\[ \times (A - 2I) - 1[(B - 2I)]^{-1} \cdot ... \cdot (A - 2I)\Phi_1(A - 1, B - 1; C - 1; z, w) \] (5.15)

Special cases: Consider the function \( \Phi_1(A, -; C; z, w) \), then
\[ \hat{D} \Phi_1(A, -; C; z, w) = \left( \frac{1}{z} + \frac{1}{w} \right) \Phi_1(A, -; C; z, w) \]
\[ + \left( \frac{1}{z} + \frac{1}{w} \right) (C - A)[(A - I)]^{-1} \Phi_1(A, -; C; z, w) \]
i.e., the \( \Phi_1(A, -; C; z, w) \) is a solution to the matrix integral equation
\[ \hat{D} - \left( \frac{1}{z} + \frac{1}{w} \right) \Phi_1(A, -; C; z, w) \]
\[ - \left( \frac{1}{z} + \frac{1}{w} \right) (C - A)[(A - I)]^{-1} \Phi_1(A, -; C; z, w) = 0. \] (5.16)

Putting \( A = C \) in equation (5.2), we get
\[ \hat{D} \Phi_1(-, B; -; z, w) = \frac{1}{z}[(B - I)]^{-1} \Phi_1(-, B; -; z, w) + \frac{1}{w} \Phi_1(-, B; -; z, w) \]
i.e. the \( \Phi_1(-, B; -; z, w) \) is a solution to the matrix integral equation
\[ \hat{D} \Phi_1(-, B; -; z, w) - \frac{1}{z}[(B - I)]^{-1} \Phi_1(-, B; -; z, w) - \frac{1}{w} \Phi_1(-, B; -; z, w) = 0 \]
In addition, we see that
\[
\hat{D}^2 \Phi_1(-, B; -; z, w) = \frac{1}{z^2} \left[ (B - 2I)z \right]^{-1} \Phi_1(-, B = -; -; z, w) \\
+ \frac{2}{zw} \left[ (B - I) \right]^{-1} \Phi_1(-, B; -; z, w) + \frac{1}{w^2} \Phi_1(-, B; -; z, w)
\]
has a solution to the matrix integral equation in this \( \Phi_1(-, B; -; z, w) \). The results of this paper are variant, significant and so it is interesting and capable to develop its study in the future. Further applications will be discussed in a forthcoming paper.

Open problem

One can use the same class of differential and integral operators for some other polynomials. Hence, new results and further applications can be obtained.

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