Regular Elements of Semigroups of Continuous Functions and Differentiable Functions

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Abstract: In 1974, Magill and Subbiah gave a characterization of the regular elements of $C(X)$, the semigroup of all continuous selfmaps of a topological space $X$. In this paper, their result is applied to determine the regular elements of $C(I)$ where $I$ is an interval in $\mathbb{R}$, as follows: An element $f \in C(I)$ is regular if and only if $\text{ran } f$ is a closed interval in $I$ and there is a closed interval $J$ in $I$ such that $f_{|J}$ is a strictly monotone function from $J$ onto $\text{ran } f$. In addition, their proof is helpful to characterize the regular elements of $D(I)$ where $|I| > 1$ and $D(I)$ is the semigroup of all differentiable selfmaps of $I$. We show that for a nonconstant function $f \in D(I)$, $f$ is regular if and only if $f$ is a strictly monotone function from $I$ onto itself and $f'(x) \neq 0$ for all $x \in I$.

Keywords: Regular elements, semigroups of continuous functions, semigroups of differentiable functions.

2000 Mathematics Subject Classification: 20M17, 20M20.

1 Introduction

For a set $A$, let $|A|$ and $1_A$ denote the cardinality of $A$ and the identity map on $A$, respectively. If $f$ is a function and $A$ is a subset of the domain of $f$, we let $f_{|A}$ denote the restriction of $f$ to $A$.

An element $x$ of a semigroup $S$ is called an idempotent of $S$ if $x^2 = x$. A regular element of $S$ is an element $x \in S$ such that $x = yxy$ for some $y \in S$. Following [3], let $E(S)$ and $\text{Reg } (S)$ denote respectively the set of all idempotents and the set of all regular elements of $S$. If $\text{Reg } (S) = S$, then $S$ is called a regular semigroup. Note that $E(S) \subseteq \text{Reg } (S)$ and if $x = yxy$, then $xy, yx \in E(S)$.

For a set $X$, let $T(X)$ denote the full transformation semigroup on $X$, that is, $T(X)$ is the semigroup, under composition, of all selfmaps of $X$. It is known that $T(X)$ is a regular semigroup ([3], page 4) and it is clearly seen that for $f \in T(X), f \in E(T(X))$ if and only if $f(x) = x$ for all $x \in \text{ran } f$ where $\text{ran } f$ is the range (image) of $f$. For $f, g \in T(X)$, if $f = fgf$, then $fg, gf \in E(T(X))$. Also, $\text{ran } f = \text{ran } (fg)$ since $\text{ran } f = \text{ran } (fgf) \subseteq \text{ran } (fg)$ $\subseteq \text{ran } f$. Moreover,
\[(fg)(f(x)) = f(x)\) and \((gf)(gf)(x) = (gf)(x)\) for all \(x \in X\) which imply that \((fg)_{\text{ran}} f = 1_{\text{ran}} f\) and \((gf)_{\text{ran}} (gf) = 1_{\text{ran}} (gf)\), respectively.

For a topological space \(X\), let \(C(X)\) be the subsemigroup of \(T(X)\) consisting of all continuous functions \(f : X \to X\). A subset \(A\) of \(X\) is called a retract of \(X\) if \(A = \text{ran} f\) for some \(f \in E(C(X))\).

In 1974, Magill and Subbiah [6] characterized the regular elements of \(C(X)\) as follows:

**Theorem 1.1** ([6]) Let \(X\) be a topological space and \(f \in C(X)\). Then \(f \in \text{Reg}(C(X))\) if and only if

1. \(f\) is a retract of \(X\) and \(\text{ran} f\).
2. There is a retract \(A\) of \(X\) such that \(f|_A\) is a homeomorphism from \(A\) onto \(\text{ran} f\).

Recall that a homeomorphism from a topological space \(X\) onto a topological space \(Y\) is a bijection \(f : X \to Y\) such that \(f\) and \(f^{-1}\) are continuous.

Next, let \(I\) be an interval in \(\mathbb{R}\), the set of real numbers. By a nontrivial interval in \(\mathbb{R}\) we mean an interval \(I\) in \(\mathbb{R}\) with \(|I| > 1\). Consider \(I\) as a metric space with the usual metric on \(\mathbb{R}\). Then

\[
C(I) = \left\{ f : I \to I \mid f \text{ is continuous on } I \right\}
\]

and we have

\[
D(I) = \left\{ f : I \to I \mid f \text{ is differentiable on } I \right\}
\]

with \(|I| > 1\) is a subsemigroup of \(C(I)\). By an interval in \(I\) we mean a nonempty subset \(J\) of \(I\) having the property that for \(x \in I\), \(a \leq x \leq b\) for some \(a, b \in J\) implies \(x \in J\). Hence all intervals of \(I\) are precisely all intervals in \(\mathbb{R}\) of the form \(K \cap I\) where \(K\) is an interval of \(\mathbb{R}\) with \(K \cap I \neq \emptyset\). Also, by a closed interval in \(I\) we mean an interval in \(I\) which is a closed set in \(I\). It follows as consequences of the main results in [2] that neither \(C(I)\) nor \(D(I)\) is a regular semigroup for every nontrivial interval \(I\) in \(\mathbb{R}\).

In 1967, Magill [5] proved that every automorphism \(\varphi\) of \(D(\mathbb{R})\) is inner, that is, there is a unit (invertible element) \(g \in D(\mathbb{R})\) such that \(\varphi(f) = gfg^{-1}\) for all \(f \in D(\mathbb{R})\). The second author and Changphas [4] investigated the regularity of the subsemigroup \(OT(I)\) of \(T(I)\) consisting of all order-preserving functions \(f : I \to I\). It was proved that \(OT(I)\) is a regular semigroup if and only if \(I\) is closed and bounded.

Our first purpose is to determine the regular elements of \(C(I)\) by Theorem 1.1. Also, the following basic results are recalled to be referred.

**Proposition 1.2** ([1], page 177) Let \(I\) be an interval in \(\mathbb{R}\) and \(f : I \to \mathbb{R}\). If \(f\) is strictly increasing [decreasing] and continuous on \(I\), then \(f^{-1}\) is strictly increasing [decreasing] and continuous on \(\text{ran} f\).

**Proposition 1.3** ([1], page 179) Let \(I\) be an interval in \(\mathbb{R}\) and \(f : I \to \mathbb{R}\). If \(f\) is one-to-one and continuous on \(I\), then \(f\) is strictly monotone on \(I\).
Our second purpose is to give necessary and sufficient conditions for elements of $D(I)$ to be regular. The proof of Theorem 1.1 given in [6] is useful for this result. Beside Proposition 1.2 and 1.3, the following basic results are also needed.

**Proposition 1.4** ([1], page 198) Let $I$ be a nontrivial interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ strictly monotone on $I$. If $f$ is differentiable on $I$ and $f'(x) \neq 0$ for all $x \in I$, then $f^{-1}$ is differentiable on $\text{ran } f$ and

$$
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
$$

for all $x \in \text{ran } f$.

**Proposition 1.5** ([1], page 205) Let $I$ be a nontrivial interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ differentiable on $I$.

(i) $f$ is increasing [decreasing] on $I$ if and only if $f'(x) \geq 0$ [$f'(x) \leq 0$] for all $x \in I$.

(ii) If $f'(x) > 0$ [$f'(x) < 0$] for all $x \in I$, then $f$ is strictly increasing [strictly decreasing] on $I$.

**Proposition 1.6** ([1], page 209-210) Let $I$ be a nontrivial interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$, $c \in I$ and assume that $f'(c)$ exists.

(i) If $f'(c) > 0$, then there is a $\delta > 0$ such that $f(x) > f(c)$ for all $x \in I \cap (c, c + \delta)$ and $f(x) < f(c)$ for all $x \in I \cap (c - \delta, c)$.

(ii) If $f'(c) < 0$, then there is a $\delta > 0$ such that $f(x) < f(c)$ for all $x \in I \cap (c, c + \delta)$ and $f(x) > f(c)$ for all $x \in I \cap (c - \delta, c)$.

In the remainder, let $I$ be an interval in $\mathbb{R}$.

## 2 The Semigroup $C(I)$

First, we provide the following two lemmas which will be used to obtain the main results.

**Lemma 2.1** If $f \in \text{Reg}(C(I))$, then $\text{ran } f$ is a closed interval in $I$.

**Proof.** Let $g \in C(I)$ be such that $f = fgf$. Then $\text{ran } (fg) = \text{ran } f$ and $(fg)(x) = x$ for all $x \in \text{ran } f$. Since $f$ is continuous on $I$ and $\text{ran } f \subseteq I$, $\text{ran } f$ is an interval in $I$. Let $x \in \text{ran } f$ where $\text{ran } f$ is the closure of $\text{ran } f$ in $I$. Then there is a sequence $(x_n)$ in $\text{ran } f$ such that $\lim_{n \to \infty} x_n = x$. By the continuity of $fg$ at $x$ in $I$, $\lim_{n \to \infty} (fg)(x_n) = (fg)(x)$. But $(fg)(x_n) = x_n$ for every $n$, so $x = (fg)(x) \in \text{ran } (fg) = \text{ran } f$. \qed

**Lemma 2.2** For $A \subseteq I$, $A$ is a retract of $I$ if and only if $A$ is a closed interval in $I$. 
Proof. Since $E(C(I)) \subseteq \text{Reg}(C(I))$, by Lemma 2.1, every retract of $I$ is a closed interval in $I$.

Conversely, assume that $A$ is a closed interval in $I$.

Case 1 : $A = (a, b)$ for some $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$. Then $A = I = \text{ran}(1_I)$ and $1_I \in E(C(I))$.

Case 2 : $A = [a, b)$ for some $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$. Then $I = [c, b)$ for some $c \in \mathbb{R}$ or $I = (c, b]$ for some $c \in \mathbb{R} \cup \{-\infty\}$. Then $f : I \to I$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in A, \\ a & \text{if } x < a, \end{cases}$$

belongs to $E(C(I))$ whose range is $A$.

Case 3 : $A = (a, b]$ for some $a \in \mathbb{R}$ and $b \in \mathbb{R}$. It can be shown similarly to Case 2 that $A = \text{ran} f$ for some $f \in E(C(I))$.

Case 4 : $A = [a, b]$ for some $a, b \in \mathbb{R}$. Then $f : I \to I$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in A, \\ a & \text{if } x < a, \\ b & \text{if } x > b, \end{cases}$$

is an element of $E(C(I))$ whose range is $A$. \hfill \Box

Now, we are ready to provide the first main result.

**Theorem 2.3** For $f \in C(I)$, $f \in \text{Reg}(C(I))$ if and only if

(i) $\text{ran} f$ is a closed interval in $I$ and

(ii) there is a closed interval $J$ in $I$ such that $f|_J$ is a strictly monotone function from $J$ onto $\text{ran} f$.

Proof. Assume that $f \in \text{Reg}(C(I))$. Then (i) holds by Lemma 2.1. By Theorem 1.1 and Lemma 2.2, there is a closed interval $J$ in $I$ such that $f|_J$ is a homeomorphism from $J$ onto $\text{ran} f$. Then $f|_J$ is one-to-one and continuous on $J$, so we have by Proposition 1.3 that $f|_J$ is a strictly monotone function from $J$ onto $\text{ran} f$.

Conversely, assume that (i) and (ii) hold. By Lemma 2.2, $\text{ran} f$ and $J$ are retracts of $I$. From Proposition 1.2, $(f|_J)^{-1} : \text{ran} f \to J$ is continuous. Hence $f|_J$ is a homeomorphism from $J$ onto $\text{ran} f$. We therefore deduce from Theorem 1.1 that $f \in \text{Reg}(C(I))$, as desired. \hfill \Box

As a consequence of Theorem 2.3, we have

**Corollary 2.4** If $|I| > 1$, then $C(I)$ is not a regular semigroup.
Assume that \( D \) is the semigroup from Lemma 2.1.

Proof. Let \( a, b, c, d \in I \) be such that \( a < b < c < d \). Define \( f : I \to I \) by

\[
f(x) = \begin{cases} 
  a & \text{if } x < a, \\
  x & \text{if } x \in [a, b], \\
  b & \text{if } x \in (b, c], \\
  \frac{d-b}{d-c} (x - c) + b & \text{if } x \in (c, d], \\
  d & \text{if } x > d.
\end{cases}
\]

Then \( f \in C(I) \), \( \text{ran } f = [a, d] \) and \( f \) is increasing on \( I \). It is clearly seen that there is no interval \( J \) in \( I \) such that \( f_{|J} \) is a strictly increasing function from \( J \) onto \( \text{ran } f \). Hence from Theorem 2.3, \( f \) is not regular in \( C(I) \). \( \square \)

Example 2.5 Consider the following functions in \( C(\mathbb{R}) \):

\[
f(x) = \sin x, \quad g(x) = x^2, \quad h(x) = e^x, \quad k(x) = x \sin x.
\]

Since \( \text{ran } f = [-1, 1] \) and \( f_{\lfloor \frac{\pi}{2}, \frac{3\pi}{2} \rfloor} \) is a strictly increasing function from \([\frac{\pi}{2}, \frac{3\pi}{2}] \) onto \([-1, 1] \), by Theorem 2.3, \( f \in \text{Reg } (C(\mathbb{R})) \). Also, \( g \in \text{Reg } (C(\mathbb{R})) \) since \( \text{ran } g = [0, \infty) \) and \( g_{\lfloor 0, \infty \rfloor} \) is a strictly increasing function from \([0, \infty) \) onto \([0, \infty) \). Since \( \text{ran } h = (0, \infty) \) which is not closed in \( \mathbb{R} \), by Theorem 2.3, \( h \notin \text{Reg } (C(\mathbb{R})) \).

By the definition of \( k \), for every \( n \in \mathbb{Z} \), \( k(2n\pi) = 0 \) and \( k(2n\pi + \frac{\pi}{2}) = 2n\pi + \frac{\pi}{2} \) where \( \mathbb{Z} \) is the set of integers. This implies that \( \text{ran } k = \mathbb{R} \) and for every \( a \in \mathbb{R} \), \( k_{\lfloor n, \infty \rfloor} \) and \( k_{\lfloor -\infty, a \rfloor} \) are not one-to-one. Since \( |k(x)| = |x \sin x| \leq |x| \) for every \( x \in \mathbb{R} \), it follows that for any \( a, b \in \mathbb{R} \) with \( a < b \), \( k([-a, b]) \subseteq [-c, c] \neq \mathbb{R} \) where \( c = \max\{|a|, |b|\} \). Consequently, there is no closed interval \( J \) in \( \mathbb{R} \) such that \( k_{|J} \) is a strictly monotone function from \( J \) onto \( \mathbb{R} \). Hence we have \( k \notin \text{Reg } (C(\mathbb{R})) \) by Theorem 2.3.

3 The Semigroup \( D(I) \)

Our purpose of this section is to prove the following theorem.

Theorem 3.1 Assume that \( I \) is a nontrivial interval in \( \mathbb{R} \). Then for a nonconstant function \( f \in D(I) \), \( f \in \text{Reg } (D(I)) \) if and only if \( f \) is a strictly monotone function from \( I \) onto itself and \( f'(x) \neq 0 \) for all \( x \in I \).

Proof. Let \( f \in D(I) \) be a nonconstant function and assume that \( f \in \text{Reg } (D(I)) \). Then \( |\text{ran } f| > 1 \). Let \( g \in D(I) \) be such that \( f = \text{fgf} \). Then \( \text{fgf} \in E(D(I)) \), \( \text{ran } f = \text{ran } (fg) \), \( (fg)_{\text{ran } f} = 1_{\text{ran } f} \) and \( (gf)_{\text{ran } (gf)} = 1_{\text{ran } (gf)} \). Let \( J = \text{ran } (gf) \).

From Lemma 2.1, \( J \) and \( \text{ran } f \) are closed intervals in \( I \). Hence

\[
\begin{align*}
(f_{|J})(g_{|\text{ran } J}) &= (f_{|\text{ran } J})(g_{|\text{ran } J}) = (fg)_{|\text{ran } J} = 1_{\text{ran } J}, \\
(g_{|\text{ran } J})(f_{|J}) &= (g_{|\text{ran } J})(f_{|\text{ran } (gf)}) = (gf)_{|\text{ran } (gf)} = 1_{|J}.
\end{align*}
\]

This implies that \( f_{|J} \) is a bijection from \( J \) onto \( \text{ran } f \) and \( g_{|\text{ran } J} = (f_{|J})^{-1} \). Since \( |\text{ran } f| > 1 \), both \( J \) and \( \text{ran } f \) are nontrivial closed intervals in \( I \). Then we deduce
that \((g_{\text{ran}f})'(x) = g'(x)\) for all \(x \in \text{ran} f\) and \((f_{\text{ran}g})'(x) = f'(x)\) for all \(x \in J\).

Therefore from (1), we have

\[
g'(f(x))f'(x) = 1 \text{ for all } x \in J \text{ and } f'(g(x))g'(x) = 1 \text{ for all } x \in \text{ran} f.\]

Hence

\[
f'(x) \neq 0 \text{ for all } x \in J \text{ and } g'(x) \neq 0 \text{ for all } x \in \text{ran} f. \tag{2}\]

From Proposition 1.3, \(f_{\text{ran}g}\) is strictly monotone on \(J\).

First, assume that \(f_{\text{ran}g}\) is strictly increasing on \(J\). But \(g_{\text{ran}f} = (f_{\text{ran}g})^{-1}\), so by Proposition 1.2, \(g_{\text{ran}f}\) is strictly increasing on \(\text{ran} f\). Proposition 1.5(i) and (2) imply that \(f'(x) > 0\) for all \(x \in J\) and \(g'(x) > 0\) for all \(x \in \text{ran} f\). It remains to show that \(J = I = \text{ran} f\). First, suppose that \(J \subsetneq I\). Then there is an element \(c \in I\) such that \(c > x\) for all \(x \in J\) or \(c < x\) for all \(x \in J\).

\textbf{Case 1 :} \(c > x\) for all \(x \in J\). Since \(J\) is a closed interval in \(I\), \(\max(J)\) exists, say \(b\). This implies that \(f(b) = \max(\text{ran} f)\) because \(f_{\text{ran}g}\) is a strictly increasing function from \(J\) onto \(\text{ran} f\). But \(c > b\) and \(f'(b) > 0\), thus by Proposition 1.6(i), there is an element \(x \in (b, c)\) such that \(f(x) > f(b)\). This is a contradiction since \(f(b) = \max(\text{ran} f)\).

\textbf{Case 2 :} \(c < x\) for all \(x \in J\). We obtain a contradiction dually to Case 1.

Therefore we have that \(J = I\). Thus \(f_{\text{ran}g} = f\) and \(g_{\text{ran}f} = f^{-1}\) which is a strictly increasing function from \(\text{ran} f\) onto \(I\). If we consider \(\text{ran} f\) and \(g_{\text{ran}f}\), replacing \(J\) and \(f_{\text{ran}g}\), respectively, then we can obtain analogously that \(\text{ran} f = I\).

If \(f_{\text{ran}g}\) is strictly decreasing on \(J\), it can be proved similarly by Proposition 1.2, Proposition 1.5(i), (2) and Proposition 1.6(ii) that \(J = I = \text{ran} f\).

The converse follows directly from Proposition 1.4.

\[\square\]

Theorem 3.1, Proposition 1.4 and Proposition 1.5 yield the following result.

**Corollary 3.2** If \(f\) is a nonconstant function in \(D(I)\), then the following statements are equivalent.

\[
(i) \ f \in \text{Reg} (D(I)).
(ii) \ f \text{ is a unit of } D(I).
(iii) \ \text{ran} f = I \text{ and either } f'(x) > 0 \text{ for all } x \in I \text{ or } f'(x) < 0 \text{ for all } x \in I.
\]

**Corollary 3.3** The semigroup \(D(I)\) is not regular for every nontrivial interval \(I\) in \(\mathbb{R}\).

**Proof.** Let \(I\) be a nontrivial interval in \(\mathbb{R}\) and let \(a, b \in \mathbb{R}\) satisfy the following conditions : \(a < b\), if \(I\) is bounded below, let \(a = \inf(I)\) and if \(I\) is bounded above, let \(b = \sup(I)\). Then the interval \((a, b)\) is always contained in \(I\). Define \(f : \mathbb{R} \to \mathbb{R}\) by
\[ f(x) = \begin{cases} \frac{1}{b-a}(x - \frac{a+b}{2})^2 + \frac{a+b}{2} & \text{if } I \text{ is bounded below,} \\ \frac{1}{a-b}(x - \frac{a+b}{2})^2 + \frac{a+b}{2} & \text{if } I \text{ is not bounded below.} \end{cases} \]

Then \( f \) is a parabola whose vertex is the point \( \left( \frac{a+b}{2}, \frac{a+b}{2} \right) \). If \( I \) is bounded below, then \( f(a) = f(b) = \frac{a+3b}{4} \in \left( \frac{a+b}{2}, b \right) \subseteq I \). Also, if \( I \) is not bounded below, then \( f(a) = f(b) = \frac{3a+b}{4} \in \left( a, \frac{a+b}{2} \right) \subseteq I \). Consequently, \( f_{|I} \in D(I) \).

Since \( f'(\frac{a+b}{2}) = 0 \), by Theorem 3.1, \( f_{|I} \) is not regular in \( D(I) \). □

**Example 3.4** All the functions in Example 2.5 belong to \( D(I) \) and it is clearly seen from Theorem 3.1 that none of them is regular in \( D(I) \). Define

\[
\begin{align*}
p(x) &= \frac{1}{x} \text{ for all } x \in (0, \infty), \\
q(x) &= x^3 \text{ and } r(x) = x^3 + x \text{ for all } x \in \mathbb{R}.
\end{align*}
\]

Then \( p'(x) = -\frac{1}{x^2} < 0 \) for all \( x \in (0, \infty) \), \( q'(0) = 0 \) and \( r'(x) = 3x^2 + 1 > 0 \) for all \( x \in \mathbb{R} \), \( \text{ran } p = (0, \infty) \) and \( \text{ran } r = \mathbb{R} \). Hence, by Corollary 3.2, \( p \in \text{Reg } (D((0, \infty))) \), \( q \notin \text{Reg } (D(\mathbb{R})) \) and \( r \in \text{Reg } (D(\mathbb{R})) \).

**References**


(Received 8 August 2005)
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