On \((LCS)_{2n+1}\)-Manifolds Satisfying Certain Conditions on the Concircular Curvature Tensor

Sunil Kumar Yadav\(^\dagger\), Praduman Kumar Dwivedi\(^\ddagger\) and Dayalal Suthar\(^\dagger\)

\(^\dagger\)Department of Mathematics
Alwar Institute of Engineering & Technology
North Ext., MIA, Alwar, Rajasthan, India
e-mail: prof_sky16@yahoo.com, dd_suthar@yahoo.co.in

\(^\ddagger\)Department of Mathematics,
Institute of Engineering & Technology
North Ext., MIA, Alwar, Rajasthan, India
e-mail: drpkdwivedi@yahoo.co.in

Abstract: We classify Lorentzian concircular structure manifolds, which satisfy the condition \(\hat{C}(\xi, X) \cdot \hat{C} = 0, \hat{C}(\xi, X) \cdot R = 0, \hat{C}(\xi, X) \cdot S = 0\) and \(\hat{C}(\xi, X) \cdot C = 0\).

Keywords: \((LCS)_{2n+1}\)-manifold; Concircular curvature tensor; Weyl Conformal curvature tensor; Einstein manifolds.

2010 Mathematics Subject Classification: 53C25.

1 Introduction

An \((2n + 1)\)-dimensional Lorentzian manifold \(M\) is smooth connected para contact Hausdorff manifold with Lorentzian metric \(g\), that is, \(M\) admits a smooth symmetric tensor field \(g\) of type \((0,2)\) such that for each point \(p \in M\), the tensor \(g_p : T_p M \times T_p M \to R\) is a non degenerate inner
product of signature \((-, +, \ldots, +\) where \(T_p M\) denotes the tangent space of \(M\) at \(p\) and \(R\) is the real number space. In a Lorentzian manifold \((M, g)\) a vector field \(\rho\) defined by

\[
g(X, \rho) = A(X)
\]

for any vector field \(X \in \chi(M)\) is said to be concircular vector field \([1]\), if

\[
(\nabla_X A)(Y) = \alpha [g(X, Y) + \omega(X)A(Y)]
\]

where \(\alpha\) is a non zero scalar function, \(A\) is a 1-form and \(\omega\) is a closed 1-form.

Let \(M\) be a Lorentzian manifold admitting a unit time like concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have

\[
g(\xi, \xi) = -1 \quad (1.1)
\]

Since \(\xi\) is the unit concircular vector field, there exists a non-zero 1-form such that

\[
g(X, \xi) = \eta(X) \quad (1.2)
\]

and hence the equation

\[
(\nabla_X \eta)(Y) = \alpha [g(X, Y) + \eta(X)\eta(Y)] \quad (\alpha \neq 0) \quad (1.3)
\]

holds for all vector field \(X, Y\), where \(\nabla\) denotes the operator of covariant differentiation with respect to Lorentzian metric \(g\) and \(\alpha\) is a non zero scalar function satisfying

\[
(\nabla_X \alpha) = (X\alpha) = \rho \eta(X), \quad (1.4)
\]

where \(\rho\) being a scalar function. If we put

\[
\phi X = \frac{1}{\alpha} \nabla_X \xi \quad (1.5)
\]

Then from (1.3) and (1.5), we have

\[
\phi^2 X = X + \eta(X)\xi, \quad (1.6)
\]

from which it follows that \(\phi\) is a symmetric \((1, 1)\)-tensor. Thus the Lorentzian manifold \(M\) together with unit time like concircular vector field \(\xi\), it’s associate 1-form \(\eta\) and \((1, 1)\)-tensor field \(\phi\) is said to be Lorentzian concircular structure manifolds (briefly \((LCS)_{2n+1}\)-manifold) \([2]\). In particular if \(\alpha = 1\), then the manifold becomes LP-Sasakian structure of Matsumoto \([3]\).
2 Preliminaries

A differentiable manifold $M$ of dimension $(2n + 1)$ is called $(LCS)_{2n+1}$-manifold if it admits a $(1, 1)$-tensor $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ which satisfy the following

\[
\eta(\xi) = -1, \tag{2.1}
\]
\[
\phi^2 = I + \eta \otimes \xi, \tag{2.2}
\]
\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}
\]
\[
g(X, \xi) = \eta(X), \tag{2.4}
\]
\[
\phi\xi = 0, \eta(\phi X) = 0. \tag{2.5}
\]

for all $X, Y$ in $TM$. Also in a $(LCS)_{2n+1}$-manifold the following relations are satisfied [4].

\[
\eta(R(X, Y)Z) = (\alpha^2 - \rho) \left[ g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \right], \tag{2.6}
\]
\[
R(X, Y)\xi = (\alpha^2 - \rho) \left[ \eta(Y)X - \eta(X)Y \right], \tag{2.7}
\]
\[
R(\xi, X)Y = (\alpha^2 - \rho) \left[ g(X, Y)\xi - \eta(Y)X \right], \tag{2.8}
\]
\[
R(\xi, X)\xi = (\alpha^2 - \rho) \left[ \eta(X)\xi + X \right], \tag{2.9}
\]
\[
(\nabla_X \phi)(Y) = \alpha \left[ g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X \right], \tag{2.10}
\]
\[
S(X, \xi) = 2n(\alpha^2 - \rho) \left[ \eta(X) \right], \tag{2.11}
\]
\[
S(\phi X, \phi Y) = S(X, Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y). \tag{2.12}
\]

**Definition 2.1.** A Lorentzian concircular structure manifold is said to be $\eta$-Einstein [5] if the Ricci operator $Q$ satisfies

\[ Q = aI + b\eta \otimes \xi, \]

where $a$ and $b$ are smooth functions on the manifolds, In particular if $b = 0$, then $M$ is an Einstein manifolds.

Let $(M, g)$ be an $n$-dimensional Riemannian manifold, then the Concircular curvature tensor $C$ and the Weyl Conformal curvature tensor $\tilde{C}$ are defined by [6]:

\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)} \left[ g(Y, Z)X - g(X, Z)Y \right], \tag{2.13}
\]
\[
C(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)} \left[ S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \right. \\
- \left. g(X, Z)QY \right] + \frac{r}{n(n-1)} \left[ g(Y, Z)X - g(X, Z)Y \right], \tag{2.14}
\]

for all $X, Y, Z \in TM$, respectively, where $r$ is the scalar curvature of $M$. 

Main Results

In this section, we obtain a necessary and sufficient condition for $(LCS)_{2n+1}$-manifolds satisfying the derivation conditions $C(\xi, X) \cdot C = 0, C(\xi, X) \cdot R = 0, C(\xi, X) \cdot S = 0$ and $C(\xi, X) \cdot C = 0$.

**Theorem 3.1.** An $(2n + 1)$-dimensional Lorentzian concircular structure manifold $M$ satisfies $C(\xi, X) \cdot C = 0$

if and only if either the scalar curvature $r$ of $M$ is $r = 2n(2n + 1)$ or $M$ is locally isometric to the Hyperbolic sphere $H^{2n+1}(\rho - \alpha^2)$.

**Proof.** In a Lorentzian concircular structure manifold $M$, we have

\[
C(\xi, Y)Y = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right] \{g(X, Y)\xi - \eta(X)Y\}; \tag{3.1}
\]

\[
C(X, Y)\xi = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right] \{\eta(Y)X - \eta(X)Y\}. \tag{3.2}
\]

The condition $C(\xi, X) \cdot C = 0$ implies that

\[
C(\xi, U)C(X, Y)\xi - C(C(\xi, U)X, Y)\xi - C(X, C(\xi, U)Y)\xi = 0.
\]

In view of (3.2), we get

\[
0 = \left[(\alpha^2 - \rho) - \frac{r}{n(n-1)}\right] \times [g(U, C(X, Y)\xi)\xi - C(X, Y)\xi\eta(U) - g(U, X)C(\xi, Y)\xi + \eta(X)C(U, Y)\xi - g(U, Y)C(X, \xi)\xi + \eta(Y)C(X, U)\xi - C(X, Y)U].
\]

Using (3.1), we have

\[
0 = \left[(\alpha^2 - \rho) - \frac{r}{n(n+1)}\right] \times [C(X, Y)U - \left((\alpha^2 - \rho) - \frac{r}{n(n+1)}\right)(g(U, Y)X - g(U, X)Y)].
\]

Therefore either the scalar curvature $r = 2n(2n + 1)(\alpha^2 - \rho)$ or

\[
[C(X, Y)U - \left((\alpha^2 - \rho) - \frac{r}{n(n+1)}\right)(g(U, Y)X - g(U, X)Y)] = 0.
\]
In view of (2.13), we get

\[ R(X, Y)U = (\alpha^2 - \rho)[g(Y, U)X - g(X, U)Y] \]

This equation implies that \( M \) is of constant curvature \((\rho - \alpha^2)\). Consequently it is locally isometric to the Hyperbolic space \( H^{2n+1}(\rho - \alpha^2) \).

Conversely, if it has the scalar curvature \( r = 2n(2n + 1)(\alpha^2 - \rho) \) then from (3.2) it follows that \( \hat{C}(\xi, X) = 0 \). Similarly, in the second case, since constant \( r = 2n(2n + 1)(\alpha^2 - \rho) \), therefore again we get \( \hat{C}(\xi, X) = 0 \).

Using the fact \( \hat{C}(\xi, X) \cdot R = 0 \), \( \hat{C}(\xi, X) \) is acting as a derivation, we have the following a corollary.

**Corollary 3.2.** An \((2n + 1)\)-dimensional Lorentzian concircular structure manifold \( M \) satisfies

\[ \hat{C}(\xi, X) \cdot R = 0 \]

if and only if either \( M \) is locally isometric to the Hyperbolic sphere \( H^{2n+1}(\rho - \alpha^2) \) or \( M \) has the scalar curvature \( r = 2n(2n + 1) \).

**Theorem 3.3.** Let \((M, g)\) be an \((2n + 1)\)-dimensional Riemannian manifold. Then \( R \cdot \hat{C} = R \cdot R \)

**Proof.** Let \( X, Y, U, V, W \in TM \). Then

\[
(R(X, Y)\hat{C})(U, V, W) = R(X, Y)\hat{C}(U, V)W - \hat{C}(R(X, Y)U, V)W \\
- \hat{C}(U, R(X, Y)V)W - \hat{C}(U, V)R(X, Y)W.
\]

From (2.13) and symmetric properties of the curvature tensor \( R \), we have

\[
- R(U, R(X, Y)V)W - R(U, V)R(X, Y)W \\
= (R(X, Y) \cdot R)(U, V, W).
\]

which proves the Theorem 3.3. \( \square \)

**Theorem 3.4.** An \((2n + 1)\)-dimensional Lorentzian concircular structure manifold \( M \) satisfies

\[ \hat{C}(\xi, X) \cdot S = 0 \]

if and only if either \( M \) has the scalar curvature \( r = 2n(2n + 1)(\alpha^2 - \rho) \) or is an Einstein manifold with the scalar curvature \( r = 2n(2n + 1)(\alpha^2 - \rho) \).
Proof. The condition $\tilde{C}(\xi, X) \cdot S = 0$ implies that

$$S(\tilde{C}(\xi, X)Y, \xi) + S(Y, \tilde{C}(\xi, X)\xi) = 0.$$ 

In view of (3.2), it gives

$$\left[ (\alpha^2 - \rho) - \frac{r}{n(n-1)} \right] \times [g(X, Y)S(\xi, \xi) - S(X, \xi)\eta(Y) + S(Y, \xi)\eta(X) + S(X, \xi)] = 0.$$ 

By the use of (2.11), we have

$$\left[ (\alpha^2 - \rho) - \frac{r}{n(n-1)} \right] [S(X, Y) - 2n(\alpha^2 - \rho)g(X, Y)] = 0.$$ 

Therefore, either the scalar curvature of $M$ is $r = 2n(2n+1)(\alpha^2 - \rho)$ or $S(X, Y) = 2n(\alpha^2 - \rho)g(X, Y)$ which implies that $M$ is an Einstein manifold with the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$, which proves the Theorem 3.4.

**Theorem 3.5.** An $(2n + 1)$-dimensional Lorentzian concircular structure manifold $M$ satisfies

$$\tilde{C}(\xi, X) \cdot C = 0$$

if and only if either $M$ has the scalar curvature $r = 2n(2n+1)(\alpha^2 - \rho)$ or is an $\eta$-Einstein manifold.

Proof. The condition $\tilde{C}(\xi, X) \cdot C = 0$ implies that

$$\left[ \tilde{C}(\xi, U)C(X, Y)W - C(\tilde{C}(\xi, U)X, Y)W - C(X, \tilde{C}(\xi, U)Y)W \right] = 0$$

Thus in view of (3.2) gives

$$0 = \left[ (\alpha^2 - \rho) - \frac{r}{n(n-1)} \right] \times \left[ C(X, Y)W, U \right] \xi - \eta(C(X, Y)W)U$$

$$- g(U, X)C(\xi, Y)W + \eta(X)C(Y, U)W - g(U, Y)C(X, \xi)W$$

$$+ \eta(Y)C(X, U)W + \eta(W)C(X, Y)W - g(U, W)C(X, Y)\xi.$$ 

So either the scalar curvature of $M$ is $r = 2n(2n+1)(\alpha^2 - \rho)$ or the equation

$$0 = C(X, Y)W, U \xi - \eta(C(X, Y)W)U - g(U, X)C(\xi, Y)W$$

$$+ \eta(X)C(Y, U)W - g(U, Y)C(X, \xi)W + \eta(Y)C(X, U)W$$

$$+ \eta(W)C(X, Y)W - g(U, W)C(X, Y)\xi.$$
holds on \( M \). Taking the inner product of this equation with \( \xi \), we get
\[
0 = -C(X,Y,W,U) - g(U,X)\eta(C(\xi,Y)W)
+ \eta(X)\eta(C(Y,U)W) - g(U,Y)\eta(C(X,\xi)W)
+ \eta(Y)\eta(C(X,U)W) + \eta(W)\eta(C(X,Y)W) - g(U,W)\eta(C(X,Y)\xi).
\]

Using (2.6), (2.11) and (2.14) in (3.3), we get
\[
S(Y,W)
= \left[ (\alpha^2 - \rho) - \frac{r}{2n(2n+1)} \right] g(Y,W) + \left[ (\alpha^2 - \rho) + \frac{r}{2n(2n+1)} \right] \eta(Y)\eta(W),
\]
which proves the Theorem 3.5.

**Acknowledgement**: The authors are thankful to the referee for his comments in the improvement of this paper.

**References**


(Received 28 January 2011)
(Accepted 18 August 2011)