Embedding of Special Semigroup Amalgams

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Abstract: After showing that any special semigroup amalgam in the class of all left [right] regular bands is strongly embeddable in the class of all regular bands, we show that the class of all semigroups satisfying the identity $axy = axay$ [$yxa = yaxa$] has the special amalgamation property.

Keywords: epimorphism; special semigroup amalgam; left [right] regular band; regular band; zigzag equations.

2010 Mathematics Subject Classification: 20M17.

1 Introduction

In [1], Scheiblich has shown that the class of normal bands is closed. In [2], the authors have generalized this result and have shown that the class of all left [right] regular bands is closed. In this paper, we further extend this result and show, by using zigzag manipulations, that the class of all left [right] regular bands is closed within the class of all regular bands. However, it is not known whether the class of all regular bands is closed.

In [3, Theorem 2.2], the authors have shown that the class of all left [right] quasinormal bands has the special amalgamation property. Now, we generalize this result, by showing that the class of all semigroups satisfying the identity

¹This research was supported by University Grant Commission of India
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Let $S$ and spine is said to be closed if for all $S \in C$ such that $U$ is a subsemigroup of $S$, $Dom(U, S) = U$. Let $B$ and $C$ be classes of semigroups such that $B \subseteq C$. Then $B$ is said to be $C$-closed if every member of $B$ is $C$-closed. A class $C$ of semigroups is said to be closed if for all $U, S \in C$ with $U$ a subsemigroup of $S$, $Dom(U, S) = U$.

A morphism $\alpha : A \rightarrow B$ in the category $C$ of all semigroups is called an epimorphism (epi for short) if for all $C \in C$ and for all morphisms $\beta, \gamma : B \rightarrow C$, $\alpha \beta = \alpha \gamma$ implies $\beta = \gamma$. It may easily be seen that a morphism $\alpha : S \rightarrow T$ is epi if and only if the inclusion mapping $i : S \alpha \rightarrow T$ is epi, and an inclusion map $i : U \rightarrow S$ is epi if and only if $Dom(U, S) = S$. For more details, one may refer to [5-7].

A most useful characterization of semigroup dominions is provided by Isbell’s Zigzag Theorem.

**Result 2.1** ([4, Theorem 2.3] or [8, Theorem VII.2.13]). Let $U$ be a subsemigroup of a semigroup $S$ and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of $d$ as follows:

\[
d = a_0 y_1 = x_1 a_1 y_1 = x_1 a_2 y_2 = x_2 a_3 y_2 = \cdots = x_m a_{2m-1} y_m = x_m a_{2m},
\]

where $m \geq 1$, $a_i \in U$ ($i = 0, 1, \ldots, 2m$), $x_i, y_i \in S$ ($i = 1, 2, \ldots, m$); and

\[
a_0 = x_1 a_1, \quad a_{2m-1} y_m = a_{2m},
\]

\[
a_{2i-1} y_i = a_{2i} y_{i+1}, \quad x_i a_{2i} = x_{i+1} a_{2i+1} \quad (1 \leq i \leq m - 1).
\]

Such a series of factorization is called a zigzag in $S$ over $U$ with value $d$, length $m$ and spine $a_0, a_1, \ldots, a_{2m}$.

We refer to the equations in Result 2.1, in whatever follows, as the zigzag equations.

A (semigroup)amalgam $A = \{ [S_i : i \in I] ; U ; \{ \phi_i : i \in I \} \}$ consists of a semigroup $U$ (called the core of the amalgam), a family $\{ S_i : i \in I \}$ of semigroups disjoint from each other and from $U$, and a family $\phi_i : U \rightarrow S_i (i \in I)$ of monomorphisms. We shall simplify the notation to $\mathcal{U} = [S_i ; U ; \phi_i]$ or to $\mathcal{U} = [S_i ; U]$ when the context allows.

We shall say that the amalgam $A$ is embedded in a semigroup $T$ if there exist a monomorphism $\lambda : U \rightarrow T$ and, for each $i \in I$, a monomorphism $\lambda_i : S_i \rightarrow T$ such that $axy = axay[xya = yaxa]$ has special amalgamation property. Notice that this class of semigroups contains the class of all left [right] quasinormal bands.

### 2 Preliminaries

Let $U$, $S$ be semigroups with $U \subseteq S$. Following Isbell [4], we say that $U$ dominates an element $d$ of $S$ if for every semigroup $T$ and for all homomorphisms $\beta, \gamma : S \rightarrow T$, $u \beta = u \gamma$ for all $u \in U$ implies $d \beta = d \gamma$. The set of all elements of $S$ dominated by $U$ is called the dominion of $U$ in $S$, and we denote it by $Dom(U, S)$. It may be easily seen that $Dom(U, S)$ is a subsemigroup of $S$ containing $U$. A semigroup $U$ is said to be $C$-closed if for all $S \in C$ such that $U$ is a subsemigroup of $S$, $Dom(U, S) = U$. Let $B$ and $C$ be classes of semigroups such that $B \subseteq C$. Then $B$ is said to be $C$-closed if every member of $B$ is $C$-closed. A class $C$ of semigroups is said to be closed if for all $U, S \in C$ with $U$ a subsemigroup of $S$, $Dom(U, S) = U$.

A morphism $\alpha : A \rightarrow B$ in the category $C$ of all semigroups is called an epimorphism (epi for short) if for all $C \in C$ and for all morphisms $\beta, \gamma : B \rightarrow C$, $\alpha \beta = \alpha \gamma$ implies $\beta = \gamma$. It may easily be seen that a morphism $\alpha : S \rightarrow T$ is epi if and only if the inclusion mapping $i : S \alpha \rightarrow T$ is epi, and an inclusion map $i : U \rightarrow S$ is epi if and only if $Dom(U, S) = S$. For more details, one may refer to [5-7].

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\]

where $m \geq 1$, $a_i \in U$ ($i = 0, 1, \ldots, 2m$), $x_i, y_i \in S$ ($i = 1, 2, \ldots, m$); and

\[
a_0 = x_1 a_1,
\]

\[
a_{2m-1} y_m = a_{2m},
\]

\[
a_{2i-1} y_i = a_{2i} y_{i+1}, \quad x_i a_{2i} = x_{i+1} a_{2i+1} \quad (1 \leq i \leq m - 1).
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We shall say that the amalgam $A$ is embedded in a semigroup $T$ if there exist a monomorphism $\lambda : U \rightarrow T$ and, for each $i \in I$, a monomorphism $\lambda_i : S_i \rightarrow T$ such that
(a) $\phi_i \lambda_i = \lambda$ for each $i \in I$;

(b) $S_i \lambda_i \cap S_j \lambda_j = U \lambda$ for all $i, j \in I$ such that $i \neq j$.

A semigroup amalgam $U = \{\{S, S\}'\}; U; \{i, \alpha \mid U\}$ consisting of a semigroup $S$, a subsemigroup $U$ of $S$, an isomorphic copy $S'$ of $S$, where $\alpha : S \to S'$ be an isomorphism and $i$ is the inclusion mapping of $U$ into $S$, is called a special semigroup amalgam. A class $C$ of semigroups is said to have the special amalgamation property if every special semigroup amalgam in $C$ is embeddable in $C$.

**Result 2.2** ([8, Theorem VII.2.3]). Let $U$ be a subsemigroup of a semigroup $S$. Let $S'$ be a semigroup disjoint from $S$ and let $\alpha : S \to S'$ be an isomorphism. Let $P = S \ast_U S'$, be the free product of the amalgam

$$U = \{\{S, S\}'\}; U; \{i, \alpha \mid U\},$$

where $i$ is the inclusion mapping of $U$ into $S$, and let $\mu, \mu'$ be the natural monomorphisms from $S, S'$ respectively into $P$. Then

$$(S \mu \cap S' \mu') \mu^{-1} = \text{Dom}(U, S).$$

From the above result, it follows that a special semigroup amalgam $\{\{S, S\}'\}; U; \{i, \alpha \mid U\}$ is embeddable in a semigroup if and only if $\text{Dom}(U, S) = U$. Therefore, the above amalgam with core $U$ is embeddable in a semigroup if and only if $U$ is closed in $S$.

Recall that a band $B$ (a semigroup in which every element is an idempotent) is called left [right] regular if it satisfies the identity $axa = ax[axa = xa]$, left[right] quasi-normal if it satisfies the identity $axy = ax[yxa = yxa]$ and regular if it satisfies the identity $axya = axaya$ respectively (see [9]).

We shall be using standard notations and refer the reader to Clifford and Preston [10] and Howie [8] for any unexplained symbols and terminology. Further, in whatever follows, bracketed statements or notions are dual to the other statements or notions.

## 3 Main Results

**Lemma 3.1.** Let $U$ be a left regular band and $S$ be any regular band such that $U$ be a subband of $S$. If for $d \in \text{Dom}(U, S) \setminus U$ and (2.1) be a zigzag in $S$ over $U$ of minimal length $m$, then

$$\left(\prod_{i=0}^{m-1} a_{2i}\right) y_m = \left(\prod_{i=0}^{m-1} a_{2i}\right) a_{2m-1}(a_{2m-4}a_{2m-6} \cdots a_2 a_0) \left(\prod_{i=0}^{m-1} a_{2i}\right) y_m.$$
Proof. Now

\[
\left( \prod_{i=0}^{m-1} a_{2i} \right) y_m
\]

\[= a_0 a_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m \]

\[= x_1 a_1 a_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(by zigzag equations)} \]

\[= (x_1 a_1 a_2)^2 a_4 \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(as } S \text{ is a band)} \]

\[= x_1 a_1 (a_2 x_1 a_1 a_2) a_4 \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(as } S \text{ is a regular band)} \]

\[= (x_1 a_1 a_2) (x_1 a_2 x_1 a_1 a_2) a_4 \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(by zigzag equations)} \]

\[= (x_1 a_1 a_2) (x_2 a_3 x_1 a_1 a_2 a_4) \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(as } S \text{ is a band)} \]

\[= (x_1 a_1 a_2) (x_2 a_3 x_1 a_1 a_2 a_4) (x_3 a_5 x_2 a_3 x_1 a_1 a_2 a_4 a_6) \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(as } S \text{ is a regular band)} \]

\[= (x_1 a_1 a_2) (x_2 a_3 x_1 a_1 a_2 a_4) (x_3 a_5 x_2 a_3 x_1 a_1 a_2 a_4 a_6)^2 \quad \text{(by zigzag equations)} \]

\[= \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(as } S \text{ is a band)} \]

\[= (x_1 a_1 a_2) (x_2 a_3 x_1 a_1 a_2 a_4) \cdots (x_{m-1} a_{2m-3} x_{m-3} a_{2m-5} \cdots x_2 a_3 x_1 a_1 a_2 a_4) \cdots a_{2m-2} y_m \]

\[= (x_1 a_1 a_2) (x_2 a_3 x_1 a_1 a_2 a_4) \cdots (x_{m-1} a_{2m-3} \cdots x_2 a_3 x_1 a_1 a_2 a_4 \cdots a_{2m-2})^2 y_m \quad \text{(as } S \text{ is a band)} \]

\[= (x_1 a_1 a_2) \cdots \left( x_{m-1} a_{2m-3} \cdots x_2 a_3 \left( \prod_{i=0}^{m-1} a_{2i} \right) \right) \left( x_{m-1} a_{2m-3} \cdots x_2 a_3 \left( \prod_{i=0}^{m-1} a_{2i} \right) \right) y_m \]

\[= (x_1 a_1 a_2) \cdots x_{m-1} a_{2m-3} \cdots x_2 a_3 \left( \prod_{i=0}^{m-1} a_{2i} \right) a_{2m-2} x_{m-1} \]

\[= (x_1 a_1 a_2) \cdots x_{m-1} a_{2m-3} \cdots x_2 a_3 \left( \prod_{i=0}^{m-1} a_{2i} \right) a_{2m-2} x_{m-1} y_m \]

\[= (x_1 a_1 a_2) \cdots \left( x_{m-1} a_{2m-2} x_{m-1} \right) y_m \]
\[
\begin{align*}
\text{(where } z_1 &= a_{2m-3} \cdots a_{2} \prod_{i=0}^{m-2} a_{2i}) \text{ and } z_2 = a_{2m-3} \cdots a_{2} \prod_{i=0}^{m-1} a_{2i} \\
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_1 x_{m-1} a_2 \cdots a_{m-2} x_{m-2}) z_2 y_m \text{ (as } S \text{ is a regular band)}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_1 x_{m-1} a_2) z_2 y_m \text{ (by zigzag equations)}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_1 a_2 \cdots a_{m-2} x_{m-2} a_{m-1}) z_2 y_m \text{ (as } U \text{ is a band)}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_1 a_2 \cdots a_{m-2} x_{m-2} a_{m-1} x_{m-1}) z_2 y_m \text{ (by zigzag equations)}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_2 \cdots a_{m-1} x_{m-1}) (a_{2m-3} \cdots a_{2} \prod_{i=0}^{m-1} a_{2i}) y_m \\
&\quad \text{(since } z_2 = a_{2m-3} \cdots a_{2} \prod_{i=0}^{m-1} a_{2i})
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_{2m-3} \cdots x_{m-3} a_{2m-4} \cdots x_{m-2}) z_3 y_m \\
&\quad \text{(as } z_1 = a_{2m-3} \cdots a_{2} \prod_{i=0}^{m-2} a_{2i}) \text{ and where}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_{2m-3} \cdots x_{m-3} a_{2m-4} \cdots x_{m-2}) z_3 y_m \\
&\quad \text{(as } z_1 = a_{2m-3} \cdots x_2 a_3 \prod_{i=0}^{m-3} a_{2i}) \text{ and as}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_{2m-3} \cdots x_{m-3} a_{2m-4} \cdots x_{m-2}) z_3 y_m \\
&\quad \text{(by zigzag equations)}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_{m-1} a_{2m-3} \cdots x_{m-3} a_{2m-4} \cdots x_{m-2}) z_3 y_m \\
&\quad \text{(since } x_{m-1}, z_1 a_{2m-1}, a_{2m-6} \in S \text{ and } S \text{ is a regular band)}
\end{align*}
\]

\[
\begin{align*}
\vdots
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots (x_2 a_3 a_{2m-3} \cdots a_{2} \prod_{i=0}^{m-1} a_{2i}) a_{2m-1}
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) \cdots z_6 a_{2m-1} z_5 y_m \quad \text{(where } z_5 = a_{2m-4} a_{2m-6} \cdots a_2 a_0 \prod_{i=0}^{m-3} a_{2i})
\end{align*}
\]

\[
\begin{align*}
&\quad \text{and } z_6 = x_{m-1} a_{2m-3} \cdots a_2 a_3 \prod_{i=0}^{m-1} a_{2i})
\end{align*}
\]

\[
\begin{align*}
&= (x_1 a_1 a_2) (x_2 a_3 x_1 a_1 a_2 a_4) \cdots z_6 a_{2m-1} z_5 y_m \\
&= (x_1 a_1 a_2 x_1 a_2 x_1) a_{2m-1} z_5 y_m \quad \text{(by zigzag equations)}
\end{align*}
\]
Let $S$ (2.1) be a zigzag in regular bands. Then

$$S = (a \cdots a)$$

(as $S$ is a regular band)

$$= (a_1a_2)(a_1a_2)(a_4 \cdots z_6a_{2m-1}z_5y_m)$$

(by zigzag equations)

$$= a_0a_2a_4 \cdots z_6a_{2m-1}z_5y_m$$

(as $a_2 \in S$)

$$= a_0a_2a_4 \cdots z_6a_{2m-1}z_5y_m$$

(by zigzag equations)

$$= a_0a_2a_4 \cdots (x_{m-1}a_{2m-3} \cdots x_2a_3(\prod_{i=0}^{m-1}a_{2i})) a_{2m-1}z_5y_m$$

(as $z_6 = x_{m-1}a_{2m-3} \cdots x_2a_3(\prod_{i=0}^{m-1}a_{2i})$)

$$\vdots$$

$$= (a_0a_2a_4 \cdots a_{2m-4}a_{2m-2})a_{2m-1}z_5y_m$$

$$= (a_0a_2a_4 \cdots a_{2m-6}a_{2m-4}a_{2m-2})a_{2m-1}(a_{2m-4}a_{2m-6} \cdots a_{2}a_0(\prod_{i=0}^{m-1}a_{2i}))y_m$$

(as $z_5 = a_{2m-4}a_{2m-6} \cdots a_{2}a_0(\prod_{i=0}^{m-1}a_{2i})$)

$$= (\prod_{i=0}^{m-1}a_{2i})a_{2m-1}(a_{2m-4}a_{2m-6} \cdots a_{2}a_0(\prod_{i=0}^{m-1}a_{2i}))y_m,$$

as required.

**Theorem 3.2.** Let $V$ be the class of all left regular bands and $C$ be the class of all regular bands. Then $V$ is $C$-closed.

**Proof.** Let $U$ and $S$ be a left regular band and a regular band respectively with $U$ a subband of $S$. Take any $d \in \text{Dom}(U, S) \setminus U$. Then, by Result 2.1, we may let (2.1) be a zigzag in $S$ over $U$ with value $d$ of minimal length $m$. Now

$$d = a_0y_1$$

(by zigzag equations)

$$= x_1a_1y_1$$

(by zigzag equations)

$$= x_1a_1a_1y_1$$

(by zigzag equations)

$$= x_1a_1a_2y_2$$

(by zigzag equations)

$$= (x_1a_1a_2)^2y_2$$

(as $S$ is a band)

$$= (x_1a_1a_2x_1)(a_1a_2y_2)$$

(as $S$ is a band)

$$= (x_1a_1a_2x_1)a_1a_2y_2$$

(by zigzag equations)

$$= x_1a_1x_2a_3x_1a_1a_2y_2$$

(by zigzag equations)

$$= (x_1a_1x_2a_3x_1)a_1a_2y_2$$

(by zigzag equations)

$$= x_1a_1x_2a_3a_0a_2y_2$$

(by zigzag equations)

$$= a_0a_2a_3y_2$$

(by zigzag equations)

$$= x_1a_1a_2a_3y_3$$

(by zigzag equations)
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\[ = (x_1a_1a_2)^2 a_4 y_3 \quad \text{(as } S \text{ is a band)} \]
\[ = (x_1a_1x_1a_2a_1a_2a_4y_3 \quad \text{(as } S \text{ is a regular band)} \]
\[ = x_1a_1(x_2a_3x_1a_2a_4y_3) \quad \text{(by zigzag equations)} \]
\[ = x_1a_1(x_2a_3x_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(as } S \text{ is a band)} \]
\[ = x_1a_1(x_2a_3x_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(by zigzag equations)} \]
\[ = x_1a_1(x_2a_3x_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(as } S \text{ is a regular band)} \]
\[ = x_1a_1(x_2a_3x_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(by zigzag equations)} \]
\[ = x_1a_1(x_2a_3x_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(as } S \text{ is a regular band)} \]
\[ = (x_1a_1a_2a_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(by zigzag equations)} \]
\[ = (x_1a_1a_2a_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(as } S \text{ is a regular band)} \]
\[ = (x_1a_1a_2a_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(by zigzag equations)} \]
\[ = (x_1a_1a_2a_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(as } U \text{ is a left regular band)} \]
\[ = (x_1a_1a_2a_1a_2a_4a_5x_2)(a_3x_1a_1a_2a_4y_3) \quad \text{(by zigzag equations)} \]
\[ = (a_0a_2a_4 \cdots a_{2m-3} a_{2m-3} y_{m-1}) \]
\[ = (a_0a_2a_4 \cdots a_{2m-3} a_{2m-2} y_{m}) \quad \text{(by zigzag equations)} \]
\[ = \left( \prod_{i=0}^{m-1} a_{2i} \right) y_m \]
\[ = \left( \prod_{i=0}^{m-1} a_{2i} \right) (a_{2m-1} a_{2m-4} a_{2m-5} \cdots a_0) \left( \prod_{i=0}^{m-1} a_{2i} \right) y_m \quad \text{(by Lemma 2.1)} \]
\[ = \left( \prod_{i=0}^{m-1} a_{2i} \right) w_1 \left( \prod_{i=0}^{m-1} a_{2i} \right) y_m \quad \text{where } w_1 = a_{2m-1} a_{2m-4} a_{2m-5} \cdots a_0 \]
\[ = \left( \prod_{i=0}^{m-1} a_{2i} \right) w_1 y_m \quad \text{(as } \prod_{i=0}^{m-1} a_{2i}, w_1 \in U \text{ and } U \text{ is a left regular band)} \]
\[ = \left( \prod_{i=0}^{m-1} a_{2i} \right) (a_{2m-4} a_{2m-2} a_{2m-1} a_{2m-1}) a_{2m-6} \cdots a_0 y_m \quad \text{(as } w_1 = a_{2m-1} a_{2m-4} a_{2m-5} \cdots a_0) \]
Let $a_{2i}$ of all regular bands. Then bands is closed within the class of all left quasinormal bands. We now

\[ \text{Theorem 3.3.} \]

\[ \text{Dom} \]

In $D$, the authors have shown that the class of all left quasinormal bands is a left regular band $U$. Hence

\[ \text{Dually, we may prove the following:} \]

\[ \text{as } a_{2m-4}, a_{2m-2a_2m-1} \in U \text{ and } U \text{ is a left regular band} \]

\[ = \left( \prod_{i=0}^{m-3} a_{2i} \right) a_{2m-4a_{2m-2a_2m-1}} a_{2m-6} \cdots a_{2a_0 y_m} \]

\[ = \left( \prod_{i=0}^{m-4} a_{2i} \right) a_{2m-6a_{2m-4a_{2m-2a_2m-1}}} \cdots a_{2a_0 y_m} \]

\[ = \left( \prod_{i=0}^{m-4} a_{2i} \right) a_{2m-6a_{2m-4a_{2m-2a_2m-1}}} a_{2m-8} \cdots a_{2a_0 y_m} \]

\[ = \left( \prod_{i=0}^{m-1} a_{2i} \right) a_{2m-10a_{2m-8}} a_{2a_0 y_m} \]

\[ \vdots \]

\[ = \left( \prod_{i=0}^{m-1} a_{2i} \right) a_{2m-10a_2 a_0 y_m} \]

\[ = a_0 a_2 (a_4 \cdots a_{2m-4a_{2m-2a_2m-1}}) a_{2a_0 y_m} \]

\[ = a_0 a_2 (a_4 \cdots a_{2m-4a_{2m-2a_2m-1}}) a_{a_0 y_m} \]

\[ \text{as } a_2, (a_4 \cdots a_{2m-4a_{2m-2a_2m-1}}) \in U \text{ and } U \text{ is a left regular band} \]

\[ = a_0 (a_2 a_4 \cdots a_{2m-4a_{2m-2a_2m-1}}) a_{a_0 y_m} \]

\[ = a_0 (a_2 a_4 \cdots a_{2m-4a_{2m-2a_2m-1}}) y_m \]

\[ \text{as } a_0, (a_2 a_4 \cdots a_{2m-4a_{2m-2a_2m-1}}) \in U \text{ and } U \text{ is a left regular band} \]

\[ = a_0 a_2 a_4 \cdots a_{2m-4a_{2m-2a_2m-1}} y_{m} \]

\[ = a_0 a_2 a_4 \cdots a_{2m-4a_{2m-2a_2m-1}} (a_{2m-1} y_m) \]

\[ = a_0 a_2 a_4 \cdots a_{2m-4a_{2m-2a_2m-1}} (a_{2m}) \]

\[ \text{by zigzag equations} \]

\[ = \prod_{i=0}^{m} a_{2i} \in U \]

\[ \Rightarrow d \in U. \]

Hence $\text{Dom}(U, S) = U$. \( \square \)

Dually, we may prove the following:

**Theorem 3.3.** Let $V$ be the variety of all right regular bands and $C$ be the variety of all regular bands. Then $V$ is $C$-closed.

In [3], the authors have shown that the class of all left [right] quasinormal bands is closed within the class of all left [right] quasinormal bands. We now
generalize this result and show that the variety $V = [axy = axay]$ of semigroups is closed.

**Theorem 3.4.** The variety $V = [axy = axay]$ of semigroups is closed.

**Proof.** Take any $U, S \in V$ with $U$ a subsemigroup of $S$ and let $d \in \text{Dom}(U, S) \setminus U$. Then, by Result 2.1, we may let (2.1) be a zigzag in $S$ over $U$ with value $d$ of minimal length $m$. Now

\[
d = a_0y_1
= x_1a_1y_1 \\
= x_1a_1x_1y_1 \\
= x_1a_1x_1a_1y_1 \\
= x_1a_1a_1y_1 \\
= x_1a_1a_2y_2 \\
= x_1a_1x_1a_2y_2 \\
= x_1a_1x_2a_3y_2 \\
= x_1a_1x_2a_3a_3y_2 \\
= x_1a_1x_2a_3a_3a_3y_2 \\
= x_1a_1x_2a_3a_3a_3y_2 \\
= x_1a_1x_2a_3a_3a_3y_2 \\
= x_1a_1x_2a_3a_3a_3y_2 \\
= x_1a_1x_2a_3a_3a_3y_2 \\
= x_1a_1x_2a_3a_3a_3y_2 \\
= a_0a_2a_3y_2
= \left(\prod_{i=0}^{m-2} a_{2i}\right) (a_{3}y_2)
= \left(\prod_{i=0}^{m-2} a_{2i}\right) (a_{2m-3}y_{m-1})
= a_0a_2a_3 \cdots a_{2m-4}(a_{2m-2}y_m)
= (x_1a_1a_2a_4 \cdots a_{2m-4}a_{2m-2}y_m)
= (x_1a_1x_2a_3a_4 \cdots a_{2m-4}a_{2m-2}y_m)
= (x_1a_1x_2a_3a_4 \cdots a_{2m-4}a_{2m-2}y_m)
= x_1a_1(x_2a_3a_4 \cdots a_{2m-4}a_{2m-2}y_m)
= \cdots
= x_1a_1(x_2a_3a_4 \cdots x_{m-2}a_{2m-4}a_{2m-2}y_m)
= x_1a_1x_2a_3a_4 \cdots (x_{m-1}a_{2m-3}a_{2m-2})y_m
= x_1a_1x_2a_3a_4 \cdots (x_{m-1}a_{2m-3}x_{m-1}a_{2m-2})y_m
= (x_{m-1}, a_{2m-3}, a_{2m-2} \in S)

...
\[ x_1a_1x_2a_3x_2a_4 \cdots (x_{m-1}a_{2m-3}x_m a_{2m-1})y_m \]  
(by zigzag equations)

\[ x_1a_1x_2a_3x_2a_4 \cdots x_{m-1}a_{2m-3}(x_m a_{2m-1}y_m) \]  
(since \( x_m, a_{2m-1}, y_m \in S \))

\[ x_1a_1x_2a_3x_2a_4 \cdots x_{m-1}a_{2m-3}x_m(a_{2m-1}x_m a_{2m-1}y_m) \]  
(since \( a_{2m-1}, x_m, y_m \in S \))

\[ x_1a_1x_2a_3x_2a_4 \cdots (x_{m-1}a_{2m-3}x_m a_{2m-2})a_{2m-1}y_m \]  
(by zigzag equations)

\[ x_1a_1x_2a_3x_2a_4 \cdots (x_{m-1}a_{2m-3}a_{2m-2})a_{2m-1}y_m \]  
(by zigzag equations)

\[ x_1a_1x_2a_3x_2a_4 \cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m-1}y_m \]  
(by zigzag equations)

\[ x_1a_1(x_2a_3x_2a_4) \cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m \]  
(since \( x_2, a_3, a_4 \in S \))

\[ x_1a_1x_2a_3a_4 \cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m \]  
(since \( x_2, a_3, a_4 \in S \))

\[ (x_1a_1a_2)a_4 \cdots a_{2m-4}a_{2m-2}a_{2m} \]  
(by zigzag equations)

\[ (x_1a_1a_2)a_4 \cdots a_{2m-4}a_{2m-2}a_{2m} \]  
(since \( x_1, a_1, a_2 \in S \))

\[ a_0a_1a_2 \cdots a_{2m-4}a_{2m-2}a_{2m} \]  
(by zigzag equations)

\[ \prod_{i=0}^{m} a_{2i} \in U. \]  
(3.1)

\[ \Rightarrow d \in U. \]

Hence \( \text{Dom}(U, S) = U. \) \( \square \)

Dually, we have the following:

**Theorem 3.5.** The variety \( \mathcal{V} = \{ yza = yaxa \} \) of semigroups is closed.

**Acknowledgement:** We would like to thank the referee for his valuable comments and suggestions that helped to improve the presentation of the paper.

**References**


(Accepted 9 October 2012)