Common Coupled Coincidence and Coupled Fixed Point of \( C \)-Contractive Mappings in Generalized Metric Spaces

Mujahid Abbas\(^1\), Wasfi Shatanawi\(^2\) and Talat Nazir\(^1\)

\(^1\)Department of Mathematics, Lahore University of Management Sciences
Lahore 54792, Pakistan
e-mail: mujahid@lums.edu.pk (M. Abbas)
talat@lums.edu.pk (T. Nazir)

\(^2\)Department of Mathematics, The Hashemite University
Zarqa 13115, Jordan
e-mail: wshatanawi@yahoo.com (W. Shatanawi)

Abstract: In this paper, study of necessary conditions for existence of common coupled coincidence and coupled fixed point results for \( C \)-contractive type mappings in the context of generalized metric space equipped with a partial order is initiated. These results generalize comparable results from the current literature. We also provide illustrative example in support of our new results.

Keywords: generalized metric spaces; coupled fixed point; ordered metric spaces.

2010 Mathematics Subject Classification: 47H10.

1 Introduction

The study of fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity [1–3]. Mustafa and Sims \(^3\) generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al. [5–7] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades \(^8\) initiated

\(^1\)Corresponding author.

Copyright © 2015 by the Mathematical Association of Thailand. All rights reserved.
the study of common fixed point theory in generalized metric spaces (see also [9, 10]). While Gajić and Crvenković [11, 12] initiated the study of fixed point results for mappings with contractive iterate at a point in $G$-metric spaces. Recently, many mathematicians have considered fixed point and common fixed point problem in generalized metric spaces (see, e.g., [13–17]). The existence of fixed points in partially ordered metric spaces has been investigated in 2004 by Ran and Reurings [18], and then further results in this direction were proved (see [19–20]). Results on weak contractive mappings on such spaces, together with applications to differential equations, were obtained by Harjani and Sadarangani in [21].

Bhashkar and Lakshmikantham in [22] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. They also discussed applications of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. Afterwards, Lakshmikantham and Ćirić [2] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ in partially ordered complete metric spaces. Then, later [23] and [24] obtained interesting results in this direction. Abbas et al. [25] have proved coupled coincidence and coupled common fixed point results in cone metric spaces for $w-$ compatible mappings.

Very recently, Cho et al [26] obtained some coupled fixed point results in generalized metric spaces (see also, [17, 27–32] and references therein). Recently, Harjani et al. [33] obtained some fixed point theorems for weakly $C-$ contractive mappings in ordered metric spaces.

The aim of this paper is to prove some common coupled coincidence and coupled fixed points results for $C-$ contractive mappings defined on a partial ordered set equipped with a generalized metric. Our results extend and unify various comparable results.

Consistent with Mustafa and Sims [4], the following definitions and results will be needed in the sequel.

**Definition 1.1.** Let $X$ be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

(a) $G(x, y, z) = 0$ if $x = y = z$;

(b) $0 < G(x, y, z)$ for all $x, y \in X$, with $x \neq y$;

(c) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;

(d) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots$ (symmetry in all three variables);

and

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then $G$ is called a $G-$ metric on $X$ and $(X, G)$ is called a $G-$ metric space.

**Definition 1.2.** A sequence $\{x_n\}$ in a $G-$ metric space $X$ is:
(i) a \(G\)-Cauchy sequence if, for any \(\varepsilon > 0\), there is an \(n_0 \in \mathbb{N}\) such that for all \(n, m, l \geq n_0\), \(G(x_n, x_m, x_l) < \varepsilon\).

(ii) a \(G\)-convergent sequence if, for any \(\varepsilon > 0\), there is an \(x \in X\) and an \(n_0 \in \mathbb{N}\), such that for all \(n, m \geq n_0\), \(G(x, x_n, x_m) < \varepsilon\).

A \(G\)-metric space on \(X\) is said to be \(G\)-complete if every \(G\)-Cauchy sequence in \(X\) is \(G\)-convergent in \(X\). It is known that \(\{x_n\}\) \(G\)-converges to \(x \in X\) if and only if \(G(x_m, x_n, x) \to 0\) as \(n, m \to \infty\) \([4]\).

Proposition 1.3 \([4]\). Let \(X\) be a \(G\)-metric space. Then the following are equivalent:

1. \(\{x_n\}\) is \(G\)-convergent to \(x\).
2. \(G(x_n, x_n, x) \to 0\) as \(n \to \infty\).
3. \(G(x_n, x, x) \to 0\) as \(n \to \infty\).
4. \(G(x_n, x_m, x) \to 0\) as \(n, m \to \infty\).

Proposition 1.4. A \(G\)-metric on \(X\) is said to be symmetric if \(G(x, y, y) = G(y, x, x)\) for all \(x, y \in X\).

Proposition 1.5. Every \(G\)-metric on \(X\) will define a metric \(d_G\) on \(X\) by

\[
d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall \, x, y \in X. \tag{1.1}
\]

For a symmetric \(G\)-metric

\[
d_G(x, y) = 2G(x, y, y), \quad \forall \, x, y \in X. \tag{1.2}
\]

However, if \(G\) is non-symmetric, then the following inequality holds:

\[
\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \forall \, x, y \in X. \tag{1.3}
\]

Recall that if \((X, \leq)\) is a partially ordered set and \(f : X \to X\) is such that for \(x, y \in X\), \(x \leq y\) implies \(f(x) \leq f(y)\), then a mapping \(f\) is said to be nondecreasing. Similarly, a nonincreasing mapping is defined.

Definition 1.6 \([22]\). An element \((x, y) \in X \times X\) is called a coupled fixed point of mapping \(F : X \times X \to X\) if \(x = F(x, y)\) and \(y = F(y, x)\).

Definition 1.7 \([13]\). An element \((x, y) \in X \times X\) is called:

\begin{itemize}
  \item[(c1)] a coupled coincidence point of mappings \(F : X \times X \to X\) and \(g : X \to X\) if \(g(x) = F(x, y)\) and \(g(y) = F(y, x)\), and \((gx, gy)\) is called coupled point of coincidence.
  \item[(c2)] a common coupled fixed point of mappings \(F : X \times X \to X\) and \(g : X \to X\) if \(x = g(x) = F(x, y)\) and \(y = g(y) = F(y, x)\).
\end{itemize}
Definition 1.8 ([2]). Let \((X, \preceq)\) be a partially ordered set. A map \(F : X \times X \to X\) is said to have a \(g\)-mixed monotone property where \(g : X \to X\) if for \(x_1, x_2, y_1, y_2 \in X\)

\[ gx_1 \preceq gx_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X \]

and

\[ gy_1 \preceq gy_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X. \]

If we take \(g = I_X\) (an identity mapping on \(X\)), then \(F\) is said to have the mixed monotone property (2.2).

2 Main Results

We obtain common coupled coincidence and coupled fixed points results for \(C\)-contractive mappings defined on a partially ordered set equipped with generalized metric space. We also extend some recent results of Choudhury and Maity [34] for two maps in generalized metric space.

We start with following result.

Theorem 2.1. Let \((X, \preceq)\) be a partially ordered set such that there exists a complete \(G\)-metric on \(X\). Let \(F : X \times X \to X\) and \(g : X \to X\) be continuous mappings such that \(F\) has the mixed \(g\)-monotone property, \(g\) commutes with \(F\) and \(F(X \times X) \subseteq g(X)\). Suppose that there exist a continuous function \(\phi : [0, \infty) \times [0, \infty) \to [0, \infty)\) with \(\phi(t, s) = 0\) if and only if \(t = s = 0\) such that

\[ G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(gx, gu, gw), G(gy, gv, gz)\} - \phi(G(gx, gu, gw), G(gy, gv, gz)) \tag{2.1} \]

for all \(x, y, u, v, w, z \in X\) with \(gw \preceq gu \preceq gx\) and \(gy \preceq gv \preceq gz\). If there exist \(x_0, y_0 \in X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq gy_0\), then \(F\) and \(g\) have a coupled coincidence point.

Proof. Let \(x_0, y_0 \in X\) be such that \(gx_0 \preceq F(x_0, y_0)\) and \(F(y_0, x_0) \preceq gy_0\). Since \(F(X \times X) \subseteq g(X)\), we can choose \(x_1, y_1 \in X\) such that \(gx_1 = F(x_0, y_0)\) and \(gy_1 = F(y_0, x_0)\). Similarly we can choose \(x_2, y_2 \in X\) such that \(gx_2 = F(x_1, y_1)\) and \(gy_2 = F(y_1, x_1)\). Since \(F\) has the mixed \(g\)-monotone property, we have \(gx_0 \preceq gx_1 \preceq gx_2\) and \(gy_1 \preceq gy_2 \preceq gy_0\). Continuing this process, we can construct two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[ gx_n = F(x_{n-1}, y_{n-1}) \preceq gx_{n+1} = F(x_n, y_n) \]

and

\[ gy_{n+1} = F(y_n, x_n) \preceq gy_n = F(y_{n-1}, x_{n-1}). \]

If for some integer \(k\), we have \((gx_{k+1}, gy_{k+1}) = (gx_k, gy_k)\), then \(F(x_k, y_k) = gx_k\) and \(F(y_k, x_k) = gy_k\), therefore \((x_k, y_k)\) is a coincidence point of \(F\) and \(g\). So, we
assume that \((g_{x_{n+1}}, g_{y_{n+1}}) \neq (g_x, g_y)\) for all \(n \in \mathbb{N}\), that is, either \(g_{x_{n+1}} \neq g_x\) or \(g_{y_{n+1}} \neq g_y\). For \(n \in \mathbb{N}\), we have

\[
G(g_{x_{n+1}}, g_{x_n}, g_{x_n}) = G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\
\leq \max\{G(g_x, g_x, g_{x_n-1}), G(g_y, g_y, g_{y_n-1})\} \\
- \phi(G(g_x, g_x, g_{x_n-1}), G(g_y, g_y, g_{y_n-1})) \\
\leq \max\{G(g_x, g_x, g_{x_n-1}), G(g_y, g_y, g_{y_n-1})\}.
\]

(2.2)

On other hand,

\[
G(g_{y_{n+1}}, g_{x_n}, g_{y_{n+1}}) = G(F(y_{n-1}, x_{n-1}), F(y_n, x_n), F(y_n, x_n)) \\
\leq \max\{G(g_{y_{n-1}}, g_{y_n}, g_{y_n}), G(g_{x_{n-1}}, g_{x_n}, g_{x_n})\} \\
- \phi(G(g_{y_{n-1}}, g_{y_n}, g_{y_n}), G(g_{x_{n-1}}, g_{x_n}, g_{x_n})) \\
\leq \max\{G(g_{y_{n-1}}, g_{x_n}, g_{x_n}), G(g_{y_{n-1}}, g_{y_n}, g_{y_n})\}.
\]

(2.3)

By (2.2) and (2.3), we have

\[
\max\{G(g_{x_{n+1}}, g_{x_n}, g_{x_n}), G(g_{y_{n+1}}, g_{y_n}, g_{y_n})\} \\
\leq \max\{G(g_x, g_x, g_{x_n-1}), G(g_y, g_y, g_{y_n-1})\} \\
- \min\{\phi(G(g_x, g_x, g_{x_n-1}), G(g_y, g_y, g_{y_n-1})), \phi(G(g_{y_{n-1}}, g_{y_n}, g_{y_n}), G(g_{x_{n-1}}, g_{x_n}, g_{x_n}))\} \\
\leq \max\{G(g_x, g_x, g_{x_n-1}), G(g_y, g_y, g_{y_n-1})\}.
\]

(2.4)

Thus \(\{\max\{G(g_{x_{n-1}}, g_{x_n}, g_{x_n}), G(g_{y_{n-1}}, g_{y_n}, g_{y_n})\}\}\) is a nonnegative decreasing sequence. Hence there exists \(r \geq 0\) such that

\[
\lim_{n \to \infty} \max\{G(g_{x_{n-1}}, g_{x_n}, g_{x_n}), G(g_{y_{n-1}}, g_{y_n}, g_{y_n})\} = r.
\]

On taking limit as \(n \to \infty\) in (2.4), we get

\[
r \leq r - \min\{\lim_{n \to \infty} \phi(G(g_{x_{n-1}}, g_{x_n}, g_{x_n-1}), G(g_{y_{n-1}}, g_{y_n}, g_{y_n-1})), \lim_{n \to \infty} \phi(G(g_{x_{n-1}}, g_{x_n}, g_{x_n}), G(g_{y_{n-1}}, g_{y_n}, g_{y_n})))\} \\
\leq r.
\]

Hence

\[
\lim_{n \to \infty} \phi(G(g_{x_n}, g_{x_n}, g_{x_n-1}), G(g_{y_n}, g_{y_n}, g_{y_n-1})) = 0.
\]

By using the properties of \(\phi\), we have

\[
\lim_{n \to \infty} G(g_{x_{n-1}}, g_{x_n}, g_{x_n-1}) = 0
\]

and

\[
\lim_{n \to \infty} G(g_{y_{n-1}}, g_{y_n}, g_{y_n}) = 0.
\]
Therefore, \( r = 0 \) and hence
\[
\lim_{n \to \infty} \max \{G(gx_{n-1}, gx_n, gx_n), G(gy_{n-1}, gy_n, gy_n)\} = 0. \tag{2.5}
\]

Now we shall show that \( \{gx_n\} \) and \( \{gy_n\} \) are \( G \)-Cauchy sequences.

Assume on Contrary that \( \{gx_n\} \) or \( \{gy_n\} \) is not a \( G \)-Cauchy sequence, that is
\[
\lim_{n, m \to \infty} G(gx_m, gx_n, gx_n) \neq 0
\]
or
\[
\lim_{n, m \to \infty} G(gy_m, gy_n, gy_n) \neq 0.
\]

This means that there exists \( \varepsilon > 0 \) for which we can find subsequences of integers \( m_k \) and \( n_k \) with \( n_k > m_k > k \) such that
\[
\max \{G(gx_{n_k}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{m_k}, gy_{m_k-1}, gy_{m_k-1})\} \geq \varepsilon. \tag{2.6}
\]

Further, corresponding to \( m_k \) we can choose \( n_k \) in such a way that it is the smallest integer with \( n_k > m_k \) which satisfy (2.6). Then
\[
\max \{G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}), G(gy_{n_k}, gy_{n_k-1}, gy_{n_k-1})\} < \varepsilon. \tag{2.7}
\]

By using the proper (e) of generalized metric and (2.7), we have

\[
G(gx_{n_k}, gx_{n_k}, gx_{n_k}) \\
\leq G(gx_{m_k}, gx_{n_k-1}, gx_{n_k-1}) + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) \\
\leq G(gx_{m_k}, gx_{m_k-1}, gx_{m_k-1}) + G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}) \\
+ G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) \\
\leq 2G(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}) \\
+ G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}) \\
< 2G(gx_{m_k}, gx_{m_k}, gx_{m_k-1}) + \varepsilon + G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}), \tag{2.8}
\]

and

\[
G(gy_{m_k}, gy_{n_k}, gy_{n_k}) \\
\leq G(gy_{m_k}, gy_{n_k-1}, gy_{n_k-1}) + G(gy_{n_k-1}, gy_{n_k}, gy_{n_k}) \\
\leq G(gy_{m_k}, gy_{m_k-1}, gy_{m_k-1}) + G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}) \\
+ G(gy_{n_k-1}, gy_{n_k}, gy_{n_k}) \\
\leq 2G(gy_{m_k}, gy_{m_k}, gy_{m_k-1}) + G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}) \\
+ G(gy_{n_k-1}, gy_{n_k}, gy_{n_k}) \\
< 2G(gy_{m_k}, gy_{m_k}, gy_{m_k-1}) + \varepsilon + G(gy_{n_k-1}, gy_{n_k}, gy_{n_k}). \tag{2.9}
\]
By (2.6)-(2.9), we have
\[ \varepsilon \leq \max\{G(gx_{m_k}, gx_{n_k}, gx_{m_k}), G(gy_{m_k}, gy_{n_k}, gy_{m_k})\} \]
\[ \leq 2\max\{G(gx_{m_k}, gx_{m_k}, gx_{m_k}), G(gy_{m_k}, gy_{m_k}, gy_{m_k})\} \]
\[ + \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{m_k-1})\} \]
\[ + \max\{G(gx_{n_k-1}, gx_{n_k}, gx_{n_k}), G(gy_{n_k-1}, gy_{n_k}, gy_{n_k})\} \]
\[ \leq 2\max\{G(gx_{m_k}, gx_{m_k}, gx_{m_k}), G(gy_{m_k}, gy_{m_k}, gy_{m_k})\} + \varepsilon \]
\[ + \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{m_k}), G(gy_{n_k-1}, gy_{n_k}, gy_{n_k})\} \]

Letting \( k \to \infty \) in above inequalities and using (2.5), we obtain
\[ \lim_{k \to \infty} \max\{G(gx_{m_k}, gx_{n_k}, gx_{m_k}), G(gy_{m_k}, gy_{n_k}, gy_{m_k})\} = \lim_{k \to \infty} \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{m_k-1})\} = \varepsilon. \] (2.10)

Since \( gx_{n_k-1} \geq gx_{n_k-1} \geq gx_{m_k-1} \) and \( gy_{n_k-1} \leq gy_{n_k-1} \leq gy_{m_k-1} \), by (2.1) we have
\[ G(gx_{n_k}, gx_{n_k}, gx_{m_k}) = G(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k-1}, y_{n_k-1})) \]
\[ \leq \max\{G(gx_{n_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{n_k-1}, gy_{n_k-1}, gy_{m_k-1})\} - \phi(G(gx_{n_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{n_k-1}, gy_{n_k-1}, gy_{m_k-1})) \] (2.11)

and
\[ G(gy_{m_k}, gy_{n_k}, gy_{n_k}) = G(F(y_{m_k-1}, x_{m_k-1}), F(y_{n_k-1}, x_{n_k-1})) \]
\[ \leq \max\{G(gy_{m_k-1}, gy_{m_k-1}, gy_{n_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1})\} - \phi(G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1})). \] (2.12)

By (2.11) and (2.12), we get
\[ \max\{G(gx_{m_k}, gx_{n_k}, gx_{m_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k})\} \]
\[ \leq \max\{G(gx_{m_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{m_k-1})\} \]
\[ - \min\{\phi(G(gx_{m_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{m_k-1}))\}, \]
\[ \phi(G(gy_{m_k-1}, gy_{n_k-1}, gy_{m_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{m_k-1}))\} \]
\[ \leq \max\{G(gx_{m_k}, gx_{n_k}, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k})\} \].

On taking limit as \( k \to \infty \) in the above inequalities and using (2.10), we have
\[ \varepsilon \leq \varepsilon - \min\{\lim_{k \to \infty} \phi(G(gx_{m_k-1}, gx_{n_k-1}, gx_{m_k-1}), G(gy_{m_k-1}, gy_{n_k-1}, gy_{m_k-1})), \]
\[ \lim_{k \to \infty} \phi(G(gy_{m_k-1}, gy_{n_k-1}, gy_{n_k-1}), G(gx_{m_k-1}, gx_{n_k-1}, gx_{n_k-1}))\} \]
\[ \leq \varepsilon. \]
By (2.10), we obtain that
\[ \lim_{k \to \infty} \phi(G(gx_{m_k}, g_{x_{m_k}}-1, gx_{n_k}), G(gy_{m_k}, gy_{n_k}, gy_{n_k}-1)) = 0 \]
or
\[ \lim_{k \to \infty} \phi(G(gy_{m_k}-1, gy_{n_k}, gy_{n_k}-1), G(gx_{m_k}, gx_{n_k}, gx_{n_k}-1)) = 0. \]
It now follows that
\[ \lim_{k \to \infty} G(gx_{m_k}, gx_{n_k}, gx_{n_k}-1) = 0 \]
By (2.10), we obtain that \( \varepsilon = 0 \), a contradiction. Therefore \( \{gx_n\} \) and \( \{gy_n\} \) are both \( G \)-Cauchy sequences in \( X \). Since \( (X, G) \) is \( G \)-complete, there are \( x, y \in X \) such that \( \{gx_n\} \) and \( \{gy_n\} \) are \( G \)-convergent to \( x \) and \( y \) respectively, that is,
\[ \lim_{n \to \infty} G(gx_n, gx_n, x) = \lim_{n \to \infty} G(gx_n, x, x) = 0 \] (2.13)
and
\[ \lim_{n \to \infty} G(gy_n, gy_n, y) = \lim_{n \to \infty} G(gy_n, y, y) = 0. \] (2.14)
Using (2.13), (2.14) and the continuity of \( g \), we have
\[ \lim_{n \to \infty} G(g(gx_n), g(gx_n), gx) = \lim_{n \to \infty} G(g(gx_n), gx, gx) = 0 \] (2.15)
and
\[ \lim_{n \to \infty} G(g(gy_n), g(gy_n), gy) = \lim_{n \to \infty} G(g(gy_n), gy, gy) = 0. \] (2.16)
Therefore \( \{g(gx_n)\} \) is \( G \)-convergent to \( gx \) and \( \{g(gy_n)\} \) is \( G \)-convergent to \( gy \). Since \( F \) and \( g \) commute, we get
\[ g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n) \] (2.17)
and
\[ g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n). \] (2.18)
As \( F \) is continuous, so taking limit as \( n \to \infty \) in (2.17) and (2.18) implies that
\( gx = F(x, y) \) and \( gy = F(y, x) \). That is, \((gx, gy)\) is a coupled coincidence point of \( F \) and \( g \).

If we take \( u = w \) and \( v = z \) in Theorem 2.1, then we obtain the following corollary.

**Corollary 2.2.** Let \( (X, \leq) \) be a partially ordered set such that there exists a complete \( G \)-metric space on \( X \). Let \( F : X \times X \to X \) and \( g : X \to X \) be continuous mappings such that \( F \) has the mixed \( g \)-monotone property, \( g \) commutes with \( F \) and \( F(X \times X) \subseteq g(X) \). Suppose that there exist a continuous function \( \phi : [0, \infty) \times [0, \infty) \to [0, \infty) \) with \( \phi(t, s) = 0 \) if and only if \( t = s = 0 \) such that
\[
G(F(x, y), F(u, v), F(u, v)) \leq \max\{G(gx, gu, gu), G(gy, gv, gv)\} + \phi(G(gx, gu, gu), G(gy, gv, gv))
\] (2.19)
for all \(x, y, u, v \in X\) with \(gw \leq gu\) and \(gy \leq gv\). If there exist \(x_0, y_0 \in X\) such that \(gx_0 \leq F(x_0, y_0)\) and \(F(y_0, x_0) \leq gy_0\), then \(F\) and \(g\) have a coupled coincidence point.

If we take \(g = I_X\) (the identity mapping) in Theorem 2.1, we obtain the following coupled fixed point result.

**Corollary 2.3.** Let \((X, \leq)\) be a partially ordered set such that there exists a complete \(G\)-metric space on \(X\). Let \(F : X \times X \to X\) be a continuous mapping satisfying the mixed monotone property. Suppose that there exist a continuous function \(\phi : [0, \infty) \times [0, \infty) \to [0, \infty)\) with \(\phi(t, s) = 0\) if and only if \(t = s = 0\) such that

\[
G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(x, u, w), G(y, v, z)\} - \phi(G(x, u, w), G(y, v, z))
\]

(2.20)

for all \(x, y, u, v, w, z \in X\) with \(w \leq u \leq x\) and \(y \leq v \leq z\). If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq F(x_0, y_0)\) and \(F(y_0, x_0) \leq y_0\), then \(F\) has a coupled fixed point.

**Corollary 2.4.** Let \((X, \leq)\) be a partially ordered set such that there exists a complete \(G\)-metric space on \(X\). Let \(F : X \times X \to X\) be a continuous mapping satisfying the mixed monotone property. Suppose that there exist a continuous function \(\phi : [0, \infty) \times [0, \infty) \to [0, \infty)\) with \(\phi(t, s) = 0\) if and only if \(t = s = 0\) such that

\[
G(F(x, y), F(u, v), F(w, z)) \leq \frac{1}{2}(G(x, u, w) + G(y, v, z)) - \phi(G(x, u, w), G(y, v, z))
\]

(2.21)

for all \(x, y, u, v, w, z \in X\) with \(w \leq u \leq x\) and \(y \leq v \leq z\). If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq F(x_0, y_0)\) and \(F(y_0, x_0) \leq y_0\), then \(F\) has a coupled fixed point.

**Proof.** Follows from Corollary 2.3 by noting that

\[
\frac{1}{2}(G(x, u, w) + G(y, v, z)) \leq \max\{G(x, u, w), G(y, v, z)\}.
\]

(2.22)

In our next result, we drop the continuity of \(F\).

**Theorem 2.5.** Let \((X, \leq)\) be a partially ordered set and \((X, G)\) such that there exists a complete \(G\)-metric space on \(X\). Suppose that there exist a continuous function \(\phi : [0, \infty) \times [0, \infty) \to [0, \infty)\) with \(\phi(t, s) = 0\) if and only if \(t = s = 0\) such that

\[
G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(gx, gu, gw), G(gy, gv, gz)\} - \phi(G(gx, gu, gw), G(gy, gv, gz))
\]

(2.23)

for all \(x, y, u, v, w, z \in X\) with \(gw \leq gu \leq gx\) and \(gy \leq gv \leq gz\). Assume that \(X\) satisfies:
we obtain $G$ for all $n$.

Proof. Following the proof of Theorem 2.1, we construct two $\{\mathcal{F} \}$. On taking limit as $\mathcal{G} \xrightarrow{n \to \infty} \mathcal{G}$ and $\mathcal{G}$, we have $\mathcal{G} \xrightarrow{n \to \infty} \mathcal{G}$. Thus $(x, y)$ is a coupled coincidence point.

Suppose also that $(g(X), G)$ is $G$-complete, $\mathcal{F}$ has the mixed $g$-monotone property and $F(X \times X) \subseteq g(X)$. If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then $\mathcal{F}$ and $g$ have a coupled coincidence point.

Corollary 2.6. Let $(X, \leq)$ be a partially ordered set such that there exists a complete $G$-metric space on $X$. Suppose that there exist a continuous function $\phi: [0, \infty) \times [0, \infty) \to [0, \infty)$ with $\phi(t, s) = 0$ if and only if $t = s = 0$ such that $G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(gx, gu, gw), G(gy, gv, gz)\}$

$$G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(gx, gu, gw), G(gy, gv, gz)\}$$

$$- \phi(G(gx, gu, gw), G(gy, gv, gz))$$

for all $x, y, u, v, w, z \in X$ with $gw \leq gu \leq gx$ and $gy \leq gv \leq gz$. Assume that $X$ satisfies:

1. if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$ for all $n$,

2. if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \leq y_n$ for all $n$.

Suppose also that $(g(X), G)$ is $G$-complete, $\mathcal{F}$ has the mixed $g$-monotone property and $F(X \times X) \subseteq g(X)$. If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq gy_0$, then $\mathcal{F}$ and $g$ have a coupled coincidence point.
If we take \( g = I_X \) (identity map) in Theorem 2.5, we obtain the following result.

**Corollary 2.7.** Let \((X, \leq)\) be a partially ordered set such that there exists a complete \(G\)-metric space on \(X\). Suppose that there exist a continuous function \( \phi : [0, \infty) \times [0, \infty) \to [0, \infty) \) with \( \phi(t, s) = 0 \) if and only if \( t = s = 0 \) such that

\[
G(F(x, y), F(u, v), F(w, z)) \leq \max\{G(x, u, w), G(y, v, z)\} - \phi(G(x, u, w), G(y, v, z))
\]  (2.25)

for all \( x, y, u, v, w, z \in X \) with \( w \leq u \leq x \) and \( y \leq v \leq z \). Assume that \( X \) satisfies:
1. if a non-decreasing sequence \( \{x_n\} \) is such that \( x_n \to x \), then \( x_n \leq x \) for all \( n \),
2. if a non-increasing sequence \( \{y_n\} \) is such that \( y_n \to y \), then \( y \leq y_n \) for all \( n \).

Suppose \( F \) has the mixed monotone property. If there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( F(y_0, x_0) \leq y_0 \), then \( F \) has a coupled fixed point.

**Corollary 2.8.** Let \((X, \leq)\) be a partially ordered set such that there exists a complete \(G\)-metric space on \(X\). Suppose that there exist a continuous function \( \phi : [0, \infty) \times [0, \infty) \to [0, \infty) \) with \( \phi(t, s) = 0 \) if and only if \( t = s = 0 \) such that

\[
G(F(x, y), F(u, v), F(w, z)) \leq \frac{1}{2}(G(x, u, w) + G(y, v, z)) - \phi(G(x, u, w), G(y, v, z))
\]  (2.26)

for all \( x, y, u, v, w, z \in X \) with \( w \leq u \leq x \) and \( y \leq v \leq z \). Assume that \( X \) satisfies:
1. if a non-decreasing sequence \( \{x_n\} \) is such that \( x_n \to x \), then \( x_n \leq x \) for all \( n \),
2. if a non-increasing sequence \( \{y_n\} \) is such that \( y_n \to y \), then \( y \leq y_n \) for all \( n \).

Suppose \( F \) has the mixed monotone property. If there exist \( x_0, y_0 \in X \) such that \( x_0 \leq F(x_0, y_0) \) and \( F(y_0, x_0) \leq y_0 \), then \( F \) has a coupled fixed point.

**Proof.** Since

\[
\frac{1}{2}(G(x, u, w) + G(y, v, z)) \leq \max\{G(x, u, w), G(y, v, z)\}.
\]  (2.27)

So that the result follows from Corollary 2.7.

**Remark 2.9.**

1) \ref{34} Theorem 3.1] is a special case of Corollary 2.4 (by taking \( \phi(t, s) = (\frac{1}{2} - \frac{1}{k})(s + t) \).
2) [34, Theorem 3.2] is a special case of Corollary 2.8 (by taking \( \phi(t, s) = (\frac{1}{2} - \frac{1}{k})(s + t) \)).

**Example 2.10.** Let \( X = [0, 1] \) be partially ordered set with the natural ordering of real numbers and

\[
G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\} 
\]

be a complete \( G \)-metric on \( X \). Let \( F : X \times X \to X \) and \( g : X \to X \) be defined by

\[
F(x, y) = \begin{cases} 
\frac{x^2 - y^2}{4}, & \text{if } x \geq y, \\
0, & \text{if } x < y,
\end{cases} 
\]

\( g(x) = \frac{3}{4}x^2 \)

and \( \phi : [0, \infty) \times [0, \infty) \to [0, \infty) \) be given by

\[
\phi(s, t) = \frac{1}{10}(s + t), \text{ for } s, t \in [0, \infty).
\]

Notice that \( F(X \times X) \) is contained in the set \( g(X) \).

Now for \( g(x) \leq g(u) \) and \( g(y) \geq g(v) \),

\[
G(F(x, y), F(u, v), F(u, v)) 
\]

\[
= \frac{1}{4} \left| x^2 - y^2 - (u^2 - v^2) \right| 
\]

\[
= \frac{3}{10} \left| |x^2 - u^2| + |y^2 - v^2| \right| 
\]

\[
= \frac{3}{4} \left[ \frac{|x^2 - u^2| + |y^2 - v^2|}{2} \right] - \frac{1}{10} \left( \frac{3}{4} |x^2 - u^2| + \frac{3}{4} |y^2 - v^2| \right) 
\]

\[
\leq \max \left\{ \frac{3}{4} |x^2 - u^2|, \frac{3}{4} |y^2 - v^2| \right\} - \phi \left( \frac{3}{4} |x^2 - u^2|, \frac{3}{4} |y^2 - v^2| \right) 
\]

\[
= \max \left\{ G \left( \frac{3}{4}x^2, \frac{3}{4}u^2, \frac{3}{4}u^2 \right), G \left( \frac{3}{4}y^2, \frac{3}{4}v^2, \frac{3}{4}v^2 \right) \right\} 
\]

\[
- \phi \left( G \left( \frac{3}{4}x^2, \frac{3}{4}u^2, \frac{3}{4}u^2 \right), G \left( \frac{3}{4}y^2, \frac{3}{4}v^2, \frac{3}{4}v^2 \right) \right) 
\]

\[
= \max\{G(gx, gu, gu), G(gy, gv, gv)\} - \phi(G(gx, gu, gu), G(gy, gv, gv)) 
\]

Thus mappings \( F, g \) and \( \phi \) satisfy all the conditions of Corollary 2.2. Moreover \( (0, 0) \) is a coupled coincidence point of \( F \) and \( g \).

**Acknowledgement:** The authors thank the editor and the referees for their useful comments and suggestions.
References


(Received 8 May 2012)
(Accepted 30 October 2012)