On the Diamond Operator Related Nonlinear Beam Equation

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Abstract: In this paper, we study the nonlinear equation of the form

\[ \frac{\partial^2}{\partial t^2} u(x, t) + c^2(-\Diamond)^k u(x, t) = f(x, t, u(x, t)), \]

with the initial conditions

\[ u(x, 0) = f(x), \quad \frac{\partial}{\partial t} u(x, 0) = g(x) \]

where \( u(x, t) \in \mathbb{R}^n \times (0, \infty) \), \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space and \( \Diamond^k \) is the Diamond operator iterated \( k \) times and is defined by (1.1). By \( \varepsilon \) approximation we also obtain the asymptotic solution for such equations. Moreover, if we put \( n = 1, p = 0, q = 1 \) and \( k = 1 \) we obtain the asymptotic solution of the nonlinear beam equation.

Keywords: diamond operator; beam equation; Fourier transform; tempered distribution.

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1 Introduction

In 1996, A. Kananthai [1] first introduced the operator $\diamondsuit^k$ and is named Diamond operator and is defined by

$$
\diamondsuit^k = \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k.
$$

(1.1)

The operator $\diamondsuit^k$ can be written as the product of the operators in the form

$$
\diamondsuit^k = \bigtriangleup^k \bigcirc^k = \bigcirc^k \bigtriangleup^k,
$$

(1.2)

where $\bigtriangleup^k$ is the Laplacian operator iterated $k$ times and is defined by

$$
\bigtriangleup^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k,
$$

(1.3)

and $\bigcirc^k$ is the ultra-hyperbolic operator iterated $k$ times and is defined by

$$
\bigcirc^k = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k.
$$

(1.4)

It is well known that for the 1-dimensional wave equation

$$
\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t),
$$

(1.5)

we obtain $u(x,t) = f(x+ct) + g(x-ct)$ as a solution of the equation where $f$ and $g$ are continuous.

Also for the $n$-dimensional wave equation

$$
\frac{\partial^2}{\partial t^2} u(x,t) + c^2 (-\bigtriangleup) u(x,t) = 0,
$$

(1.6)

with the initial condition $u(x,0) = f(x)$ and $\frac{\partial}{\partial t} u(x,0) = g(x)$ where $\bigtriangleup$ is defined by (1.3) with $k = 1$, $f$ and $g$ are given continuous functions. By solving the Cauchy problem for such an equation, the Fourier transform has been applied and the solution is given by

$$
\hat{u}(\xi, t) = \hat{f}(\xi) \cos (2\pi |\xi|) t + \hat{g}(\xi) \frac{\sin (2\pi |\xi|) t}{2\pi |\xi|},
$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ [see 2, p177]. By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$
u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x)
$$

(1.7)
where \( \hat{\Phi}_t \) is an inverse Fourier transform of \( \hat{\Phi}_t(\xi) = \frac{\sin(2\pi|\xi|)}{2\pi|\xi|} t \) and \( \hat{\Psi}_t \) is an inverse Fourier transform of \( \hat{\Psi}_t(\xi) = \cos(2\pi|\xi|) t \).

In this paper, we study the equation

\[
\frac{\partial^2}{\partial t^2} u(x,t) + c^2 (-\hat{\diamond}^k) u(x,t) = f(x,t,u(x,t)),
\]

(1.8)

with

\[
u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x,0) = g(x)
\]

where the operator \( \hat{\diamond}^k \) is defined by (1.1), \( c \) is a positive constant, \( k \) is a non-negative integer, \( f \) and \( g \) are continuous functions and absolutely integrable. The equation (1.8) is motivated by replacing the \( \triangle \) by \( \hat{\diamond} \) in (1.6) and extend it to the nonlinear form. We consider (1.8) with the following conditions on \( u \) and \( f \) as follows:

1. \( u(x,t) \in C^{(4k)}(\mathbb{R}^n) \) for any \( t > 0 \) where \( C^{(4k)}(\mathbb{R}^n) \) is the space of continuous function with \( 4k \)-derivatives.
2. \( f \) satisfies the Lipchitz condition,

\[
|f(x,t,u) - f(x,t,w)| \leq A|u - w|
\]

where \( A \) is constant with \( 0 < A < 1 \).
3. \( \int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))| \, dx \, dt < \infty \) for \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( 0 < t < \infty \) and \( u(x,t) \) is continuous function on \( \mathbb{R}^n \times (0, \infty) \).

By \( \epsilon \)-approximation and under such conditions of \( f \) and \( u \), we obtain asymptotic solution of (1.8) in the convolution form

\[
u(x,t) = O(\epsilon^{-\frac{1}{2}}) * f(x,t,u(x,t))
\]

(1.9)

Moreover, if we put \( k = 1 \), \( n = 1 \), \( p = 0 \) and \( q = 1 \) in (1.8) then (1.8) reduces to the nonlinear beam equation

\[
\frac{\partial^2}{\partial t^2} u(x,t) + c^2 \frac{\partial^4}{\partial x^4} u(x,t) = f(x,t,u(x,t)),
\]

(1.10)

and also we obtain

\[
u(x,t) = O(\epsilon^{-\frac{1}{2}}) * f(x,t,u(x,t))
\]

(1.11)

is the asymptotic solution of (1.10). We also study the boundness of \( E(x,t) \) where \( E(x,t) \) is defined by (2.11) in the Sobolev space. That is in (1.8) by setting the conditions \( f(x) \in H_s(\mathbb{R}^n) \) and \( g(x) \in H_{s-1}(\mathbb{R}^n) \) then \( E(x,t) \in H_s(\mathbb{R}^n \times (0, \infty)) \) where \( H_s(\mathbb{R}^n) \) is the Sobolev space of order \( s \) and is defined by

\[
H_k = H_k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2(\mathbb{R}^n) \}
\]
where $k$ is a nonnegative integer and norm

$$
\|f\|_k^2 = \int_{\mathbb{R}^n} |f(\xi)|^2 (1 + |\xi|^2)^k d\xi < \infty
$$

$L^2(\mathbb{R}^n)$ is space of the square integrable in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i$ is a nonnegative integer and

$$
\partial^{\alpha} f = \frac{\partial^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} f(x).
$$

Before going to that point, the following definitions and some concepts are needed.

## 2 Preliminaries

We shall need the following definitions

**Definition 2.1.** Let $f \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$
\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx
$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the inner product in $\mathbb{R}^n$ and $dx = dx_1 dx_2 \cdots dx_n$.

Also, the inverse of Fourier transform is defined by

$$
f(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \hat{f}(x) dx.
$$

If $f$ is a distribution with compact support by Eq(2.1) can be written as

$$
\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i(\xi, x)} \rangle
$$

**Definition 2.2.** Let $t > 0$ and $p$ is a real number

- $f(t) = O(t^p)$ as $t \to 0 \iff t^{-p} |f(t)|$ is bounded as $t \to 0$
- $f(t) = o(t^p)$ as $t \to 0 \iff t^{-p} |f(t)| \to 0$ as $t \to 0$

**Definition 2.3.** Let $H_k = H_k(\mathbb{R}^n)$ be the space of the Sobolev space of order $k$ on $\mathbb{R}^n$ and is defined by

$$
H_k = H_k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : \partial^{\alpha} f \in L^2(\mathbb{R}^n) \}
$$

where $k$ is a nonnegative integer and norm

$$
\|f\|_k^2 = \int_{\mathbb{R}^n} |f(\xi)|^2 (1 + |\xi|^2)^k d\xi < \infty
$$
Let $L^2(\mathbb{R}^n)$ be the space of square integrable functions in $\mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i$ is a nonnegative integer and 

$$
\partial^\alpha f = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} f(x)
$$

**Lemma 2.4.** Given the function

$$
f(x) = \exp \left[ - \left( \sum_{i=1}^{p} x_i^2 \right)^2 + \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^2 \right]
$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $p + q = n$, $\sum_{i=1}^{p} x_i^2 < \sum_{j=p+1}^{p+q} x_j^2$.

We consider four cases

**Case 1:** $p$ odd and $q$ even ($n$ odd).

**Case 2:** $p$ even and $q$ odd ($n$ odd).

**Case 3:** $p$ and $q$ are both even ($n$ even), $n \neq 4k$, $k = 1, 2, 3, \ldots$.

For case (1)-(3), we obtain

$$
| \int_{\mathbb{R}^n} f(x) dx | \leq \frac{\Omega_p \Omega_q}{8} \frac{\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{q}{4}\right)\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)}
$$

**Case 4:** $p$ and $q$ are both even ($n$ even), we obtain

$$
| \int_{\mathbb{R}^n} f(x) dx | \leq \frac{\Omega_p \Omega_q}{8} \frac{\Gamma\left(\frac{p}{4}\right)\Gamma\left(\frac{q}{4}\right)\Gamma\left(\frac{4}{4}\right)}{(-1)^m \Gamma\left(\frac{4}{4} + m\right)}
$$

where $\Gamma$ denotes the Gamma function. That is $\int_{\mathbb{R}^n} f(x) dx$ is bounded.

**Proof.**

$$
\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} \exp \left[ - \left( \sum_{i=1}^{p} x_i^2 \right)^2 + \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^2 \right] dx
$$

Let us transform to bipolar coordinates defined by

$x_1 = r \omega_1$, $x_2 = r \omega_2$, $\ldots$, $x_p = r \omega_p$

$dx_1 = rd\omega_1$, $dx_2 = rd\omega_2$, $\ldots$, $dx_p = rd\omega_p$

and

$x_{p+1} = s \omega_{p+1}$, $x_{p+2} = s \omega_{p+2}$, $\ldots$, $x_{p+q} = s \omega_{p+q}$
where $\omega_1^2 + \omega_2^2 + \ldots + \omega_p^2 = 1$ and $\omega_{p+1}^2 + \omega_{p+2}^2 + \ldots + \omega_{p+q}^2 = 1$.

Thus
\[
\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} \exp \left[ -\sqrt{s^4 - r^4} \right] r^{p-1} s^{q-1} dr ds \Omega_p d\Omega_q,
\]
where $dx = r^{p-1} s^{q-1} dr ds \Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively,

\[
|\int_{\mathbb{R}^n} f(x) dx| \leq \int_{\mathbb{R}^n} \exp \left[ -\sqrt{s^4 - r^4} \right] r^{p-1} s^{q-1} dr ds \Omega_p d\Omega_q.
\]

By computing directly, we obtain
\[
|\int_{\mathbb{R}^n} f(x) dx| \leq \frac{\Omega_p \Omega_q}{2} \int_0^{2\pi} \int_0^{\pi} e^{-\sqrt{s^4 - r^4} \sin^2 \theta} r^{p-2} s^{q-1} \cos \theta \sin \theta d\theta ds,
\]
where $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus
\[
|\int_{\mathbb{R}^n} f(x) dx| \leq \Omega_p \Omega_q \int_0^{\pi/2} \int_0^s e^{-s^2 \cos^2 \theta} r^{p+q-1} (\sin \theta)^{p/2} \cos \theta d\theta ds. \tag{2.4}
\]

Put $r^2 = s^2 \sin \theta$, $2r dr = s^2 \cos \theta d\theta$, and $0 \leq \theta \leq \frac{\pi}{2}$,

\[
|\int_{\mathbb{R}^n} f(x) dx| \leq \frac{\Omega_p \Omega_q}{2} \int_0^{\pi/2} \int_0^s e^{-s^2 \cos^2 \theta} y^{p+q-1} (\sin \theta)^{p/2} \cos \theta d\theta \cos \theta dy ds.
\]

Put $y = s^2 \cos \theta$, $ds = \frac{dy}{2s \cos \theta}$ into (2.4), we obtain
\[
|\int_{\mathbb{R}^n} f(x) dx| \leq \frac{\Omega_p \Omega_q}{4} \int_0^{\pi/2} \int_0^{\infty} e^{-y} \left( \frac{y}{\cos \theta} \right)^{p+q-1} (\sin \theta)^{p/2} \cos \theta d\theta \cos \theta dy ds
\]
\[
= \frac{\Omega_p \Omega_q}{4} \int_0^{\pi/2} \int_0^{\infty} e^{-y} \left( \frac{y}{\cos \theta} \right)^{p+q-1} (\sin \theta)^{p/2} \cos \theta dy d\theta
\]
\[
= \frac{\Omega_p \Omega_q}{4} \Gamma \left( \frac{n}{2} \right) \int_0^{\pi/2} \left( \frac{y}{\cos \theta} \right)^{p+q-1} (\sin \theta)^{p/2} d\theta
\]
\[
= \frac{\Omega_p \Omega_q}{8} \Gamma \left( \frac{n}{2} \right) \beta \left( \frac{p}{4}, \frac{4-n}{4} \right)
\]
\[
|\int_{\mathbb{R}^n} f(x) dx| \leq \frac{\Omega_p \Omega_q}{8} \frac{\Gamma(\frac{p}{4}) \Gamma(\frac{2-n}{4})}{\Gamma(\frac{4-n}{4})}. \tag{2.5}
\]
We consider the boundness of (2.5) in four cases:

**Case 1:** \( p \) odd and \( q \) even (\( n \) odd). If \( q = 4 \) then

\[
\frac{\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-n}{4} - \frac{q}{4}\right)} = \frac{\Gamma(1 - \frac{n}{4})}{\Gamma(0)} = 0.
\]

(2.6)

Thus (2.6) is bounded.

**Case 2:** \( p \) even and \( q \) odd (\( n \) odd).

In this case

\[
\frac{\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-n}{4} - \frac{q}{4}\right)} \neq \infty
\]

(2.7)

Thus (2.5) is bounded.

**Case 3:** \( p \) and \( q \) are both odd (\( n \) even and \( n \neq 4k \)).

For \( n \neq 4k, k = 1, 2, 3, \ldots \) Therefore

\[
\frac{\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-n}{4} - \frac{q}{4}\right)} \neq \infty
\]

(2.8)

Thus (2.5) is bounded.

**Case 4:** \( p \) and \( q \) are both even (\( n \) even). In this case using the formula

\[
\frac{\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-n}{4} - \frac{q}{4}\right)} = \Gamma(1 - \left(\frac{4-n}{4} - \frac{q}{4}\right))
\]

(2.9)

Putting (2.9) into (2.4), we obtain

\[
\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\Omega_p \Omega_q}{8} \frac{\Gamma\left(\frac{4}{4}\right) \Gamma\left(\frac{4}{4}\right)}{(-1)^m \Gamma\left(\frac{4}{4} + m\right)}
\]

where \( \Gamma \) denotes the Gamma function. By (2.6)-(2.9) we conclude \( \int_{\mathbb{R}^n} f(x) dx \) is bounded.

**Lemma 2.5.** (The Fourier transform of \( \varphi^k \delta \))

\[
\mathcal{F}\varphi^k = \frac{(-1)^{2k}}{(2\pi)^{n/2}} \left[ (\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2)^2 \right]^k
\]
where \( \mathcal{F} \) is the Fourier transform defined by \( (2.1) \) and if the norm of \( \xi \) is given by \( \| \xi \| = (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^{1/2} \) then

\[
\mathcal{F}\hat{\phi} \leq \frac{M}{(2\pi)^{n/2}} \| \xi \|^4k.
\]

Since \( M \) is constant thus \( \mathcal{F}\hat{\phi} \) is bounded and continuous on the space \( S' \) of the tempered distribution. Moreover, by Eq. \( (2.2) \)

\[
\hat{\phi} = \mathcal{F}^{-1} \left\{ \frac{1}{(2\pi)^{n/2}} \left( (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2)^2 \right) \right\}^k.
\]

Proof. By Eq. \( (2.3) \)

\[
\mathcal{F}\hat{\phi} = \frac{1}{(2\pi)^{n/2}} \langle \hat{\phi}, e^{-i\xi x} \rangle
\]

\[
= \frac{1}{(2\pi)^{n/2}} \langle \delta, \hat{\phi} e^{-i\xi x} \rangle
\]

\[
= \frac{1}{(2\pi)^{n/2}} \langle \delta, -i
\]

\[
= \frac{1}{(2\pi)^{n/2}} \left\{ \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 \right\} \hat{\phi} e^{-i\xi x} \}
\]

\[
= \frac{1}{(2\pi)^{n/2}} \left\{ \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 \right\} \hat{\phi} e^{-i\xi x} \}
\]

\[
= \frac{1}{(2\pi)^{n/2}} \left\{ \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 \right\} \hat{\phi} e^{-i\xi x} \}
\]

Thus,

\[
|\mathcal{F}\hat{\phi}| = \frac{1}{(2\pi)^{n/2}} \left( (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2)^2 \right)^k
\]

\[
\leq \frac{M}{(2\pi)^{n/2}} \| \xi \|^4k.
\]

where \( M \) is constant and \( \| \xi \| = (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^{1/2}, \ \xi_i(i = 1, 2, \ldots, n) \in \mathbb{R}. \)

Hence we obtain \( \mathcal{F}\hat{\phi} \) which is bounded and continuous on the space \( S' \) of the tempered distribution. Since \( \mathcal{F} \) is \( -1 \) transformation from the space \( S' \) of the tempered distribution to the real space \( \mathbb{R} \), then by \( (2.2) \)

\[
\hat{\phi} = \mathcal{F}^{-1} \left\{ \frac{1}{(2\pi)^{n/2}} \left( (\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)^2 - (\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2)^2 \right) \right\}^k.
\]

That completes the proof. \( \Box \)
Lemma 2.6. Given the operator

\[ L = \frac{\partial^2}{\partial t^2} + c^2 (-\Diamond)^k , \]  

(2.10)

where \( \Diamond^k \) is the diamond operator and is defined by (1.1). Then we obtain

\[ E(x, t) = O(e^{-\frac{n}{2}t}) \]  

(2.11)

an elementary asymptotic solution for the operator defined by (2.10).

Proof. Let

\[ LE(x,t) = \delta(x,t), \]

where \( E(x, t) \) is the elementary solution of the operator \( L \) and \( \delta \) is the Dirac-delta distribution. Thus

\[ \frac{\partial^2}{\partial t^2} E(x,t) + c^2 (-\Diamond)^k E(x,t) = \delta(x)\delta(t). \]  

(2.12)

Take applying the Fourier transform defined by (2.1) to both sides of (2.12), we obtain

\[ \frac{\partial^2}{\partial t^2} \hat{E}(\xi,t) + c^2 \left( \sum_{i=1}^{p} \xi_i^2 \right)^2 \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \hat{E}(\xi,t) = \frac{1}{(2\pi)^n} \delta(t) \]

The solution of the above equation is

\[ \hat{E}(\xi,t) = H(t) \omega(\xi,t), \]  

(2.13)

where \( H(t) \) is the Heaviside function and \( \omega(\xi,t) \) is a solution of homogeneous equation.

Now, we are solving the solution of homogeneous equation. Given the homogeneous equation

\[ \frac{\partial^2}{\partial t^2} \hat{E}(\xi,t) + c^2 \left( \sum_{i=1}^{p} \xi_i^2 \right)^2 \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \hat{E}(\xi,t) = 0 \]  

(2.14)

Let \( \omega(\xi,t) \) be the solution of (2.14), we have

\[ \frac{\partial^2}{\partial t^2} \omega(\xi,t) + c^2 \left( \sum_{i=1}^{p} \xi_i^2 \right)^2 \left( \sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \omega(\xi,t) = 0 \]  

(2.15)

with the initial condition

\[ \omega(x,0) = f(x), \quad \frac{\partial}{\partial t} \omega(x,0) = g(x). \]  

(2.16)
Now, we put $r^2 = \xi_{1}^{2} + \xi_{2}^{2} + \ldots + \xi_{p}^{2}$ and $s^2 = \xi_{p+1}^{2} + \xi_{p+2}^{2} + \ldots + \xi_{p+q}^{2}$, we obtain

$$
\frac{\partial^2}{\partial t^2} \hat{\omega}(\xi, t) + c^2 \left(s^4 - r^4\right)^k \hat{\omega}(\xi, t) = 0
$$

$$
\hat{\omega}(\xi, t) = A(\xi) \cos \left(\sqrt{s^4 - r^4}\right)^k c t + B(\xi) \sin \left(\sqrt{s^4 - r^4}\right)^k c t.
$$

By (2.16), $\hat{\omega}(\xi, 0) = A(\xi) = \hat{f}(\xi)$

$$
\frac{\partial \hat{\omega}(\xi, t)}{\partial t} = -c \left(\sqrt{s^4 - r^4}\right)^k A(\xi) \sin \left(\sqrt{s^4 - r^4}\right)^k c t + c \left(\sqrt{s^4 - r^4}\right)^k B(\xi) \cos \left(\sqrt{s^4 - r^4}\right)^k c t.
$$

$$
\frac{\partial \hat{\omega}(\xi, 0)}{\partial t} = 0 + c \left(\sqrt{s^4 - r^4}\right)^k B(\xi) = \hat{g}(\xi)
$$

$$
B(\xi) = \frac{\hat{g}(\xi)}{c \left(\sqrt{s^4 - r^4}\right)^k}
$$

$$
\hat{\omega}(\xi, t) = \hat{f}(\xi) \cos \left(\sqrt{s^4 - r^4}\right)^k c t + \frac{\hat{g}(\xi)}{c \left(\sqrt{s^4 - r^4}\right)^k} \sin \left(\sqrt{s^4 - r^4}\right)^k c t. \quad (2.17)
$$

By applying the inverse Fourier transform (2.17), we obtain the solution $\omega(x, t)$ in the convolution form Then (2.15) has a solution in the convolution form

$$
\omega(x, t) = f(x) \ast \psi_t(x) + g(x) \ast \phi_t(x).
$$

Now we need to show the existence of $\Phi_t(x)$ and $\Psi_t(x)$.

Let us consider the Fourier transform

$$
\hat{\Phi}_t(x) = \frac{\sin \left(\sqrt{s^4 - r^4}\right)^k c t}{c \left(\sqrt{s^4 - r^4}\right)^k} \text{ and } \hat{\Psi}_t(x) = \cos \left(\sqrt{s^4 - r^4}\right)^k c t.
$$

They are all tempered distributions but they are not $L_1(\mathbb{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform $\Phi_t(x)$ and $\Psi_t(x)$ directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of $\epsilon-$approximation.

Let us define

$$
\tilde{\phi}_t(\xi) = e^{-\epsilon c \left(\sqrt{s^4 - r^4}\right)^k} \hat{\phi}_t(\xi) = e^{-\epsilon c \left(\sqrt{s^4 - r^4}\right)^k} \frac{\sin \left(\sqrt{s^4 - r^4}\right)^k c t}{c \left(\sqrt{s^4 - r^4}\right)^k} \text{ for } \epsilon > 0. \quad (2.18)
$$

We see that $\phi_t(x) \in L_1(\mathbb{R}^n)$ and $\tilde{\phi}_t(x) \to \hat{\phi}_t(x)$ uniformly as $\epsilon \to 0$. So that
\( \phi_t(x) \) will be the limit in the topology of tempered distribution of \( \phi_t(x) \). Now

\[
\Phi_\varepsilon t(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \mathcal{F}_\varepsilon(t(\xi)) d\xi
\]

\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-c(\sqrt{s^4 - r^4})^k} \frac{\sin \left( \sqrt{s^4 - r^4} \right)^k}{c \left( \sqrt{s^4 - r^4} \right)^k} d\xi
\]

\[
|\Phi_\varepsilon t(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-c(\sqrt{s^4 - r^4})^k} c \left( \sqrt{s^4 - r^4} \right)^k d\xi
\]

By changing to bipolar coordinates. Now, put

\[
\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p
\]

and \( \xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_p = sw_{p+q}, \ p + q = n \)

where \( w_1^2 + w_2^2 + \cdots + w_p^2 = 1 \) and \( w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1 \),

\[
|\Phi_\varepsilon t(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-c(\sqrt{s^4 - r^4})^k} c \left( \sqrt{s^4 - r^4} \right)^k \Omega_p \Omega_q \int_0^\infty \int_0^{\pi/2} e^{-c(\sqrt{s^4 - r^4})^k} \frac{r^{p-1}s^{q-1}drdsd\Omega_p}{c \left( \sqrt{s^4 - r^4} \right)^k},
\]

where \( d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q \), \( d\Omega_p \) and \( d\Omega_q \) are the elements of surface area of the unit sphere in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively, where \( \Omega_p = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} \), \( \Omega_q = \frac{(2\pi)^{q/2}}{\Gamma(q/2)} \).

\[
|\Phi_\varepsilon t(x)| \leq \frac{\Omega_p \Omega_q}{(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} e^{-c(\sqrt{s^4 - r^4})^k} \frac{r^{p-1}s^{q-1}drds}{c \left( \sqrt{s^4 - r^4} \right)^k},
\]

putting \( r^2 = s^2 \sin \theta, \ 2rdr = s^2 \cos \theta d\theta \) and \( 0 \leq \theta \leq \frac{\pi}{2} \).

\[
|\Phi_\varepsilon t(x)| \leq \frac{\Omega_p \Omega_q}{2(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} e^{-c(\sqrt{s^4 - r^4} \sin^2 \theta)^k} \frac{\sin \theta \left( \sin \theta \right)^{2-p} s^{p+q-1} \cos \theta d\theta ds}{c \left( \sqrt{s^4 - r^4} \sin^2 \theta \right)^k},
\]

\[
= \frac{\Omega_p \Omega_q}{2c(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} \frac{e^{-c(\cos^2 \theta)^k} \left( \sin \theta \right)^{2-p} s^{p+q-1} \cos \theta d\theta ds}{c \left( \cos \theta \right)^k}
\]
Put $y = cc \left( s^2 \cos \theta \right)^k = cc x^{2k} \cos^k \theta$, $s^4 = \frac{y}{cc \cos^k \theta}$, $ds = \frac{dy}{2k y}$, thus

$$
|\Phi'_2(x)| \leq \frac{\Omega_p \Omega_q}{4e(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\infty} e^{-y} s^{n-1} \frac{\cos \theta}{ky} \cos \theta \frac{s}{ky} dy \ d\theta
$$

$$
= \frac{\Omega_p \Omega_q}{4(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\infty} e^{-y} y^n \left( \frac{y}{cc \cos^k \theta} \right)^{n/2k} (\sin \theta) \frac{x^n}{ky} \cos \theta \frac{s}{ky} dy \ d\theta
$$

$$
= \frac{\Omega_p \Omega_q}{4(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\infty} e^{-y} y^{n/2k} (\sin \theta) \cos \theta \frac{s}{ky} dy \ d\theta
$$

$$
= \frac{\Omega_p \Omega_q}{4(2\pi)^{n/2}} \Gamma \left( \frac{n}{2n} - 1 \right) \int_0^{\pi/2} (\sin \theta) \frac{x^n}{ky} \cos \theta \frac{s}{ky} \ d\theta
$$

Similarly, we define $\Psi'_2(x) = e^{-cc((\sqrt{s^4 - r^2})^k)} \cos \left( \sqrt{s^4 - r^2} \right)^k ct$ and

$$
\Psi'_2(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{i(\xi, x)} \Psi'_2(\xi) d\xi
$$

$$
= \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{i(\xi, x)} e^{-cc((\sqrt{s^4 - r^2})^k)} \cos \left( \sqrt{s^4 - r^2} \right)^k ct \ d\xi
$$

and

$$
|\Psi'_2(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{-cc((\sqrt{s^4 - r^2})^k)} d\xi
$$

$$
= \frac{1}{(2\pi)^{n/2}} \int_0^{\infty} \int_0^s e^{-cc((\sqrt{s^4 - r^2})^k)} r \ p - 1 s q - 1 dr ds,
$$

put $r^2 = s^2 \sin \theta$, $2rd\theta = s^2 \cos \theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$

$$
|\Psi'_2(x)| \leq \frac{\Omega_p \Omega_q}{4(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\pi/2} e^{-cc(s^4 \cos \theta)^k} (\sin \theta) \cos \theta \frac{s}{ky} \ d\theta ds
$$

$$
= \frac{\Omega_p \Omega_q}{4(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^{\pi/2} e^{-cc(s^4 \cos \theta)^k} s^{p - 1} \cos \theta \frac{s}{ky} \ d\theta ds,
$$
put \( y = c(s^4 \cos \theta)^k \), \( ds = \frac{dy}{4ky} \),

\[
|\Psi_{\epsilon}'(x)| \leq \frac{\Omega_y \Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}}{y} \left( \frac{y}{\epsilon c \cos^4 \theta} \right)^{n/2k} \sin \theta \cos \theta \, dy \, d\theta
\]

\[
= \frac{\Omega_y \Omega_q}{4k(2\pi)^{n/2}} \int_0^{\pi/2} \int_0^\infty \frac{e^{-y}}{c^{n/2k} \epsilon n^{2k}} \sin \theta \cos \theta \, dy \, d\theta
\]

\[
= \frac{\Omega_y \Omega_q}{4(2\pi)^{n/2} \epsilon n^{2k} c^{n/2k}} \Gamma \left( \frac{n}{2k} \right) \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta
\]

\[
|\Psi_{\epsilon}'(x)| \leq \frac{\Omega_y \Omega_q}{8(2\pi)^{n/2} \epsilon n^{2k} c^{n/2k}} \Gamma \left( \frac{n}{2k} \right) \Gamma \left( \frac{1-n}{2} \right) N.
\]

Set

\[
\omega_{\epsilon}(x, t) = f(x) * \Psi_{\epsilon}(x) + g(x) * \Phi_{\epsilon}(x). \tag{2.19}
\]

By \( \epsilon \)-approximation of \( \omega(x, t) \) in (2.15) for \( \epsilon \to 0 \), \( \omega_{\epsilon}(x, t) \to \omega(x, t) \) uniformly. Now

\[
\omega_{\epsilon}(x, t) = \int_{\mathbb{R}^n} f(r) \Psi_{\epsilon}(x - r) \, dr + \int_{\mathbb{R}^n} g(r) \Phi_{\epsilon}(x - r) \, dr.
\]

Thus

\[
|\omega_{\epsilon}(x) - r| \leq |\Psi_{\epsilon}(x - r)| \int_{\mathbb{R}^n} |f(r)| \, dr + |\Phi_{\epsilon}(x - r)| \int_{\mathbb{R}^n} |g(r)| \, dr
\]

\[
\leq \frac{\Omega_y \Omega_q}{8(2\pi)^{n/2} \epsilon n^{2k} c^{n/2k} \Gamma(n/2k)} \Gamma \left( \frac{1-n}{2} \right) M + \frac{\Omega_y \Omega_q}{8(2\pi)^{n/2} \epsilon n^{2k} c^{n/2k} \Gamma(n/2k - 1)} \Gamma \left( \frac{1-n}{2} \right) N,
\]

\[
e^n^{2k} |\omega_{\epsilon}(x, t)| \leq \frac{\Omega_y \Omega_q}{8(2\pi)^{n/2} \epsilon n^{2k} c^{n/2k}} \Gamma \left( \frac{n}{2k} \right) \Gamma \left( \frac{1-n}{2} \right) M + \frac{\Omega_y \Omega_q}{8(2\pi)^{n/2} \epsilon n^{2k} c^{n/2k}} \Gamma \left( \frac{n}{2k - 1} \right) \Gamma \left( \frac{1-n}{2} \right) N. \tag{2.20}
\]

where \( M = \int_{\mathbb{R}^n} |f(r)| \, dr \) and \( N = \int_{\mathbb{R}^n} |g(r)| \, dr \), since \( f \) and \( g \) are absolutely integrable. We consider the boundness of (2.21) in four cases:

Case 1: \( p \) odd and \( q \) even (\( n \) odd). If \( q = 8 \) then

\[
\frac{\Gamma \left( \frac{1-n}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} = \frac{\Gamma \left( 1 - \frac{n}{4} \right)}{\Gamma \left( 0 \right)} = 0. \tag{2.21}
\]

Putting (2.22) into (2.21), we obtain

\[
e^n^{2k} |\omega_{\epsilon}(x, t)| \leq K(\text{K is constant})
\]
Case 2: $p$ even and $q$ odd ($n$ odd).

In this case
\[ \frac{\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} \neq \infty \]  
(2.22)

Thus
\[ e^{n/2k}|u^*(x,t)| \leq K \text{ (K is constant)} \]

Case 3: $p$ and $q$ are both odd ($n$ even and $n \neq 4k$)

For $n \neq 8k$, $k = 1, 2, 3, \ldots$ Therefore
\[ \frac{\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} \neq \infty \]  
(2.23)

Thus
\[ e^{n/2k}|\omega^*(x,t)| \leq K \text{ (K is constant)} \]

Case 4: $p$ and $q$ are both even ($n$ even). In this case using the formula
\[ \Gamma(z) \Gamma(z-m) = (-1)^m \Gamma(-z+m+1) \Gamma(1-z), \quad m = 1, 2, 3, \ldots \]

We have
\[ \frac{\Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} = \frac{\Gamma\left(\frac{4-q}{4} - \frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} = \frac{\Gamma(1 - \frac{4-n}{4})}{(-1)^m \Gamma(-\frac{4-n}{4} + m + 1)} = \frac{\Gamma\left(\frac{4}{4} + m\right)}{(-1)^m \Gamma\left(\frac{4}{4} + m\right)} \]
(2.24)

Putting (2.18) into (2.16), we obtain
\[ e^{n/2k}|u^*(x,t)| \leq \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2}k e^{n/2k}} \frac{\Gamma\left(\frac{n}{4}\right) \Gamma\left(\frac{q}{4}\right) \Gamma\left(\frac{4}{4} - \frac{4-n}{4}\right)}{(-1)^m \Gamma\left(\frac{4}{4} + m\right)} M + \]
\[ \frac{\Omega_p \Omega_q \epsilon}{8(2\pi)^{n/2}k e^{n/2k}} \frac{\Gamma\left(\frac{n}{4} - 1\right) \Gamma\left(\frac{q}{4}\right) \Gamma\left(\frac{4}{4} - \frac{4-n}{4}\right)}{(-1)^m \Gamma\left(\frac{4}{4} + m\right)} N, \]
(2.25)

By (2.3)-(2.26) we conclude (2.21) is bounded.

By (2.16) we have
\[ e^{n/2k}|\omega^*(x,t)| \leq \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2}k e^{n/2k}} \frac{\Gamma\left(\frac{n}{4}\right) \Gamma\left(\frac{q}{4}\right) \Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} M + \]
\[ \frac{\Omega_p \Omega_q \epsilon}{8(2\pi)^{n/2}k e^{n/2k}} \frac{\Gamma\left(\frac{n}{4} - 1\right) \Gamma\left(\frac{q}{4}\right) \Gamma\left(\frac{4-n}{4}\right)}{\Gamma\left(\frac{4-q}{4}\right)} N, \]

and
\[
\lim_{\epsilon \to 0} \epsilon^{n/2k} |\omega'(x,t)| \leq \frac{\Omega_p \Omega_q}{8(2\pi)^{n/2} k e^{n/2k}} \frac{\Gamma \left( \frac{n}{k} \right) \Gamma \left( \frac{4-n}{4} \right)}{\Gamma \left( \frac{4}{4} \right)} M.
\]
By definition (2.2) we obtain the asymptotic solution of (2.15) in the form
\[
\omega(x,t) = O \left( \epsilon^{-n/2k} \right) \tag{2.26}
\]
for \( n \neq k \) as \( \epsilon \to 0 \).
Thus we obtain an asymptotic elementary solution of the operator by (2.10)
\[
E(x,t) = H(t) O(\epsilon^{-n/2k}) = O(\epsilon^{-n/2k}) , \quad t > 0 \tag{2.27}
\]

3 Main Results

**Theorem 3.1.** Given the equation
\[
\frac{\partial^2}{\partial t^2} u(x,t) + c^2 (-\nabla^4)^k u(x,t) = f(x,t,u(x,t)) \tag{3.1}
\]
for \((x,t) \in \mathbb{R}^n \times (0, \infty)\), \(k\) is a positive number and with the following conditions on \(u\) and \(f\) as follows

1. \(u(x,t)\) is the space of continuous function on \(\mathbb{R}^n \times (0, \infty)\).
2. \(f\) satisfies the Lipschitz condition,
\[
|f(x,t,u) - f(x,t,w)| \leq A|u - w|
\]
where \(A\) is constant with \(0 < A < 1\).
3. \(\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))| \, dx \, dt < \infty \) for \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\), \(0 < t < \infty\) and \(u(x,t)\) is a continuous function on \(\mathbb{R}^n \times (0, \infty)\).

Then we obtain the convolution
\[
u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{3.2}
\]
as a unique solution of (2.21) for \(x \in \Omega\) where \(\Omega\) is a compact subset of \(\mathbb{R}^n\) and \(0 \leq t \leq T\) with \(T\) as constant and \(E(x,t)\) as an elementary solution defined by (2.8) and also \(u(x,t)\) is bounded for any fixed \(t > 0\). In particular, if we put \(n = 1, p = 0, q = 1\) and \(k = 1\) in (3.1), then (3.1) reduces to the nonlinear beam equation
\[
\frac{\partial^2}{\partial t^2} u(x,t) + c^2 \frac{\partial^4}{\partial x^4} u(x,t) = f(x,t,u(x,t)) \tag{3.3}
\]
and we obtain \(u(x,t) = \epsilon^{-\frac{1}{2}} * f(x,t,u(x,t))\) as an asymptotic solution of (3.3).
Proof. Convolving both sides of (3.1) with $E(x, t)$, that is:

$$E(x, t) \ast \left[ \frac{\partial^2}{\partial t^2}u(x, t) + c^2(-\nabla)^k u(x, t) \right] = E(x, t) \ast f(x, t, u(x, t))$$

or

$$\left[ \frac{\partial^2}{\partial t^2}E(x, t) + c^2(-\nabla)^k E(x, t) \right] \ast u(x, t) = E(x, t) \ast f(x, t, u(x, t)),$$

so

$$\delta(x, t) \ast u(x, t) = E(x, t) \ast f(x, t, u(x, t)).$$

Thus

$$u(x, t) = E(x, t) \ast f(x, t, u(x, t))$$

$$= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r, s)f(x - r, t - s, u(x - r, t - s))drds,$$

where $E(r, s)$ is given by definition (3.1). We next show that $u(x, t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$. We have

$$|u(x, t)| \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r, s)||f(x - r, t - s, u(x - r, t - s))|drds \leq |E(r, s)|N$$

where $N = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |f(x - r, t - s, u(x - r, t - s))|drds$. By condition (3) in Theorem 3.1 and (2.27) we obtain $u(x, t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$.

To show that $u(x, t)$ is unique. Suppose there is another solution $w(x, t)$ of (3.1). We next to show that $u(x, t)$ is unique. Let $w(x, t)$ be another solution of (2.1). Let the operator be

$$L = \frac{\partial^2}{\partial t^2} + c^2(-\nabla)^k$$

then (3.1) can be written in the form

$$Lu(x, t) = f(x, t, u(x, t))$$

Thus

$$Lu(x, t) - Lw(x, t) = f(x, t, u(x, t)) - f(x, t, w(x, t)).$$

By the condition (2) of the theorem 3.1,

$$|L(u(x, t) - w(x, t))| \leq A|u(x, t) - w(x, t)|.$$  (3.4)

Let $\Omega_0 \times (0, T]$ the compact subset of $\mathbb{R}^n \times [0, \infty)$ and $L : C^{(4k)}(\Omega_0) \to C^{(4k)}(\Omega_0)$ for $0 \leq t \leq T$.

Now $(C^{(4k)}(\Omega_0), ||.||)$ is a Banach space where $u(x, t) \in C^{(4k)}(\Omega_0)$ for $0 \leq t \leq T$ and $||.||$ is given by

$$||u(x, t)|| = \sup_{x \in \Omega_0, 0 \leq t \leq T} |u(x, t)|.$$
Then, from (2) with $0 < A < 1$, the operator $L$ is a contraction mapping on $C^{(4k)}(\Omega_0)$. Since $(C^{(4k)}(\Omega_0), \|\|)$ is a Banach space and $L : C^{(4k)}(\Omega_0) \to C^{(4k)}(\Omega_0)$ is a contraction mapping on $C^{(4k)}(\Omega_0)$, by Contraction Theorem 3, we obtain the operator $L$ which has a fixed point and has uniqueness property. Thus $u(x, t) = w(x, t)$.

In particular, if we put $n = 1, p = 0, q = 1$ and $k = 1$ then (3.5) reduces to the nonlinear beam equation

$$
\frac{\partial^2}{\partial t^2} u(x, t) + c^2 \left( \frac{\partial^2}{\partial x^2} \right)^4 u(x, t) = f(x, t, u(x, t)).
$$

Thus we obtain $u(x, t) = O(e^{-t/A})*f(x, t, u(x, t))$ is an asymptotic solution of (3.5). That complete the proof.

**Theorem 3.2.** A boundness of the elementary solution in Sobolev space.

Let the condition (1.8) of $f$ and $g$ be $f \in H_s(\mathbb{R}^n)$ and $g \in H_{s-1}(\mathbb{R}^n)$ then $E(x, t) \in H_s(\mathbb{R}^n \times (0, \infty))$ where $H_s(\mathbb{R}^n)$ is a Sobolev space of order $s$ defined by definition 2.3.

**Proof.** By the Plancherel theorem, $f \in H_s(\mathbb{R}^n)$ if and only if $(1 + \sqrt{s^2 - r^2})^s \hat{f}(\xi) \in L^2(\mathbb{R}^n)$. Now $(\partial_\alpha f)(\xi) = (i\xi)^\alpha \hat{f}(\xi)$ where

$$
\partial^\alpha = \frac{\partial^{\alpha_1 + \alpha_2 + \ldots + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}
$$

for $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \leq s$ and $s$ is a nonnegative integer. We have $(i\xi)^\alpha \hat{f}(\xi) \in L^2(\mathbb{R}^n)$ or equivalent $(1 + \sqrt{s^2 - r^2})^s \hat{f}(\xi) \in L^2(\mathbb{R}^n)$. We now show that $E(x, t) \in H_s(\mathbb{R}^n \times (0, \infty))$ with the Sobolev norm

$$
\|E(x, t)\|_s^2 = \int_{\mathbb{R}^n} \left| E f(\xi, t) \right|^2 (1 + \sqrt{s^2 - r^2})^k d\xi < \infty
$$

for any given $t \in (0, \infty)$. Now consider $(\partial^o \partial^j_t E)(\xi, t)$ where $\hat{E}(\xi, t)$ is an elementary solution is given by (2.8),

$$
\partial^o = \frac{\partial^{\alpha_1 + \alpha_2 + \ldots + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}
$$

and $\partial_t^j = \frac{\partial^j}{\partial t^j}$, $j$ is a nonnegative integer. We have

$$
(\partial^o \partial_t^j E)(\xi, t) = (i\xi)^\alpha \frac{\partial^j}{\partial t^j} \hat{E}(\xi, t) \text{ for } |\alpha| + j \leq s
$$

$$
= (i\xi)^\alpha \frac{\partial^j}{\partial t^j} \left( \hat{f}(\xi) \cos(c \left( \sqrt{s^2 - r^2} \right)^k t) + \hat{g}(\xi) \frac{\sin(c \left( \sqrt{s^2 - r^2} \right)^k t)}{c \sqrt{s^2 - r^2}} \right)
$$

$$
= (i\xi)^\alpha \left( c \left( \sqrt{s^2 - r^2} \right)^k \right)^j \text{ trig}(c \left( \sqrt{s^2 - r^2} \right)^k t) +
$$

$$
(i\xi)^\alpha \left( c \left( \sqrt{s^2 - r^2} \right)^k \right)^{j-1} \text{ trig}(c \left( \sqrt{s^2 - r^2} \right)^k t) \quad (3.6)
$$
where trig denotes one of the function $\pm \cos$ or $\pm \sin$. By the Plancherel theorem, if $f \in H_s(\mathbb{R}^n)$ and $g \in H_{s-1}(\mathbb{R}^n)$ then on the right hand side of (13) we have $(i\xi)^\alpha (c\sqrt{s^4-r^4})^j \tilde{f}(\xi) \in L^2(\mathbb{R}^n)$ and $(i\xi)^\alpha \left(c\left(\sqrt{s^4-r^4}\right)^k\right)^{j-1} \tilde{g}(\xi) \in L^2(\mathbb{R}^n)$. Thus $(\partial_\zeta \overline{\partial}_t u)(\xi, t) \in L^2(\mathbb{R}^n \times (0, \infty))$ and it follows that $\partial^{\alpha}_x \partial^j_t E(x, t) \in L^2(\mathbb{R}^n \times (0, \infty))$ with the Sobolev norm

$$ \|E(x, t)\|_s = \left( \int_{\mathbb{R}^n} |E(\xi, t)|^2 (1 + \sqrt{s^4-r^4})^k d\xi \right)^{1/2} $$

bounded independent of $t$ for $|\alpha| + j \leq s$. It follows that $E(x, t) \in H_s(\mathbb{R}^n \times (0, \infty))$. 

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**References**


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