Some Lacunary Difference Sequence Spaces defined by Musielak-Orlicz Functions

Bipan Hazarika

Department of Mathematics, Rajiv Gandhi University,
Rono Hills, Doimukh, Itanagar-791 112, Arunachal Pradesh, India
e-mail : bhrgu@yahoo.co.in

Abstract : In this article we are introduced the lacunary sequence spaces defined by Musielak-Orlicz functions and study their algebraic and topological properties. Also we obtain some relations related to these spaces.

Keywords : Difference sequence space; Lacunary sequence; Musielak-Orlicz function.

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1 Introduction

Throughout the article \( w, c, c_0, \ell_\infty, \ell_1 \) denote the spaces of all, convergent, null, bounded and absolutely summable sequences of complex numbers, respectively. The zero sequence is denoted by \( \theta \). Also \( \mathbb{N} \) and \( \mathbb{R} \) denote the set of all positive integers and set of real numbers respectively.

The difference sequence space was initially introduced by Kizmaz [1] and it was generalized by Et and Colak [2] defined in the following way:

\[
Z(\Delta^m) = \{(x_k) \in w : \Delta^m x_k \in Z\},
\]

for \( Z = c, c_0, \ell_\infty \), where \( m \in \mathbb{N} \); \( \Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1} \) and \( \Delta^0 x_k = x_k \), for all \( k \in \mathbb{N} \). The generalized difference operator is equivalent to the following binomial representation:

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\[ \Delta^m x_k = \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^\nu x_{k+\nu}. \]

A lacunary sequence is an increasing integer sequence \( \xi = (k_r), r = 1, 2, 3, \ldots \) where \( k_0 = 0 \) with \( h_r = k_r - k_{r-1} \to \infty \), as \( r \to \infty \). We denote \( I_r = (k_{r-1}, k_r) \) and \( \eta_r = \frac{k_r}{k_{r-1}}, \) for \( r = 1, 2, 3, \ldots \).

The lacunary strongly convergent sequence space \( N_\xi \) was defined by Freedman et al. [3] in the following way:

\[
N_\xi = \left\{ (x_k) : \lim_{r \to \infty} h_{r-1}^{-1} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.
\]

The space \( N_\xi \) is a BK-space with respect to the norm

\[
|| (x_k) ||_\xi = \sup_r h_{r-1}^{-1} \sum_{k \in I_r} |x_k|.
\]

\( N^0_\xi \) denotes the subset of these sequences in \( N_\xi \) for which \( L = 0 \). \( (N^0_\xi, ||.||_\xi) \) is also a BK-space. There is a relation between \( N_\xi \) and \( |\sigma_1| \) of strongly Cesàro summable sequences (see Freedman et al. [3]). The space \( |\sigma_1| \) is defined by

\[
|\sigma_1| = \left\{ (x_k) \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \text{ for some } L \right\}.
\]

For \( \xi = (2^r) \), we have a relation between the spaces \( |\sigma_1| \) and \( N_\xi \), i.e. \( |\sigma_1| = N_\xi \).

An Orlicz function is a function \( M : [0, \infty) \to [0, \infty) \), which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \), for \( x > 0 \) and \( M(x) \to \infty \), as \( x \to \infty \). An Orlicz function \( M \) is said to satisfy \( \Delta_2 \) condition for small \( x \) or at \( 0 \) if for each \( k > 0 \) there exist \( R_k > 0 \) and \( \rho_k > 0 \) such that \( M(kx) \leq R_k M(x) \), for all \( x \in (0, x_k] \). Moreover, an Orlicz function \( M \) is said to satisfy the \( \Delta_2 \)-condition if and only if

\[
\lim_{x \to \infty} \sup_{x \geq 0} \frac{M(2x)}{M(x)} < \infty.
\]

Two Orlicz functions \( M_1 \) and \( M_2 \) are said to be equivalent if there are positive constants \( \alpha, \beta \) and \( x_0 \) such that

\[
M_1(\alpha x) \leq M_2(x) \leq M_1(\beta x),
\]

for all \( x \) with \( 0 \leq x < x_0 \).

Lindenstrauss and Tzafriri [4] used the idea of the Orlicz function to construct the sequence space:

\[
\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.
\]
The space $\ell_M$ becomes a Banach space, with respect to the norm

$$|| x || = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}$$

which is called an Orlicz sequence space. The space $\ell_M$ is closely related to the space $\ell_p$ which is an Orlicz sequence space with $M(t) = |t|^p$, for $1 \leq p < \infty$.

Later on, Orlicz sequence spaces were investigated by Parashar and Choudhary [5], Maddox [6], Tripathy et al. [7–10] and many others.

## 2 Definitions and Notations

A sequence $M = (M_k)$ of Orlicz functions is called a **Musielak-Orlicz function** (for details see [11, 12]). Also a Musielak-Orlicz function $\phi = (\phi_k)$ is called a **complementary function** of a Musielak-Orlicz function $M$ if

$$\phi_k(t) = \sup \{|t|s - M_k(s) : s \geq 0\}, \text{ for } k = 1, 2, 3, ....$$

For a given Musielak-Orlicz function $M$, the Musielak-Orlicz sequence space $l_M$ and its subspace $h_M$ are defined as follows:

$$l_M = \{ x = (x_k) \in w : I_M(cx) < \infty, \text{ for some } c > 0 \};$$

$$h_M = \{ x = (x_k) \in w : I_M(cx) < \infty, \text{ for all } c > 0 \},$$

where $I_M$ is a convex modular defined by

$$I_M = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in l_M.$$ 

We consider $l_M$ equipped with the Luxemburg norm

$$|| x || = \inf \left\{ k > 0 : I_M \left( \frac{x}{k} \right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$|| x ||^0 = \inf \left\{ \frac{1}{k} (1 + I_M(kx)) : k > 0 \right\}.$$ 

The main aim of this article is to introduce the following sequence spaces and examine some properties of the resulting sequence spaces. Let $p = (p_k)$ denote the sequences of positive real numbers, for all $k \in \mathbb{N}$. Let $M = (M_k)$ be a Musielak-Orlicz function and $v = (v_k)$ be any sequence of non-zero complex numbers. Let $X$ be a seminormed space over the field of complex numbers with the semi norm.
$q$ and $w(X)$ denotes the space of all sequences $x = (x_k)$, where $x_k \in X$. Then we define the following sequence spaces:

$$[N_\xi, M, \Delta^m, p, q, v]_1 = \left\{ (x_k) \in w(X) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k - L}{\rho} \right) \right) \right]^{p_k} \to 0, \right\}$$

for some $\rho > 0$ and $L \in \mathbb{C}$;

$$[N_\xi, M, \Delta^m, p, q, v]_0 = \left\{ (x_k) \in w(X) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k}{\rho} \right) \right) \right]^{p_k} \to 0, \right\}$$

for some $\rho > 0$;

$$[N_\xi, M, \Delta^m, p, q, v]_\infty = \left\{ (x_k) \in w(X) : \sup_{r} h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k}{\rho} \right) \right) \right]^{p_k} < \infty, \right\}$$

for some $\rho > 0$.

**Definition 2.1.** A sequence space $E$ is said to be **solid** (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

**Definition 2.2.** A sequence space $E$ is said to be **symmetric** if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi$ is a permutation of $\mathbb{N}$.

**Definition 2.3.** A sequence space $E$ is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Let $K = \{k_1 < k_2 < \cdots \} \subset \mathbb{N}$ and $E$ be a sequence space. A **$K$-step space** of $E$ is a sequence space $\lambda^E_K = \{(x_{k_n}) \in w : (k_n) \in E\}$. A canonical preimage of a sequence $(x_{k_n}) \in \lambda^E_K$ is a sequence $(y_n) \in w$ defined by

$$y_n = \begin{cases} x_n, & \text{if } n \in K; \\ 0, & \text{otherwise}. \end{cases}$$

A canonical preimage of a step space $\lambda^E_K$ is a set of canonical preimages of all elements in $\lambda^E_K$, i.e. $y$ is in canonical preimage of $\lambda^E_K$ if and only if $y$ is canonical preimage of some $x \in \lambda^E_K$.

**Definition 2.4.** A sequence space $E$ is said to be **monotone** if it contains the canonical preimages of its step spaces.

The following results will be used for establishing some results of this article.
Lemma 2.5 (Kamthan and Gupta [13, p. 53]). A sequence space \( E \) is solid implies \( E \) is monotone.

Lemma 2.6 (Freedman et al. [3, Lemma 2.1]). In order to \( |\sigma_1| \subseteq N_\xi \) it is necessary and sufficient that \( \lim r \inf \eta_r > 1 \).

Lemma 2.7 (Freedman et al. [3, Lemma 2.2]). In order to \( N_\xi \subseteq |\sigma_1| \) it is necessary and sufficient that \( \lim r \sup \eta_r < \infty \).

Lemma 2.8 (Et and Nuray [14, Theorem 2.2]). If \( X \) is a Banach space normed by \( ||.|| \), then \( \Delta^m(X) \) is also a Banach space normed by
\[
||x||_\Delta = \sum_{i=1}^{m} |x_i| + f(\Delta^m x).
\]

3 Main Results

Theorem 3.1. Let \( p = (p_k) \) in \( \ell_\infty \) of strictly positive real numbers and \( \xi = (k_r) \) be a lacunary sequence. Then \([N_\xi, M, \Delta^m, p, q, v], [N_\xi, M, \Delta^m, p, q, v]_1\) and \([N_\xi, M, \Delta^m, p, q, v]\) are linear spaces.

Proof. The proof of the theorem is easy, so omitted.

Theorem 3.2. Let \( M = (M_k) \) be a Musielak-Orlicz function and \( p = (p_k) \) in \( \ell_\infty \) of strictly positive real numbers and \( \xi = (k_r) \) be a lacunary sequence. Then \([N_\xi, M, \Delta^m, p, q, v]_0\) is a paranormed space (not totally paranormed) with the paranorm
\[
g_\Delta(x) = \sum_{i=1}^{m} |x_i| + \inf \left\{ \rho^{\frac{n_r}{n_r}} : \sup_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k}{\rho} \right) \right) \right] \leq 1, \text{ for some } \rho > 0 \text{ and } r = 1, 2, 3, \ldots \right\},
\]
where \( H = \max \{1, \sup p_k\} \).

Proof. Clearly \( g_\Delta(x) = g_\Delta(-x) \). Since \( M_k(0) = 0 \), for all \( k \in N \), we get \( g_\Delta(\bar{0}) = 0 \), for \( x = \bar{x} \). Let \( x = (x_k) \) and \( y = (y_k) \) be two elements in \([N_\xi, M, \Delta^m, p, q, v]_0\) and let us choose \( \rho_1 > 0 \) and \( \rho_2 > 0 \) be such that
\[
\sup_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k}{\rho_1} \right) \right) \right] \leq 1, \ r = 1, 2, 3, \ldots
\]
and
\[
\sup_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m y_k}{\rho_2} \right) \right) \right] \leq 1, \ r = 1, 2, 3, \ldots.
\]
Let $\rho = \rho_1 + \rho_2$, then we have

$$\sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m(x_k + y_k)}{\rho} \right) \right) \right] \leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k}{\rho_1} \right) \right) \right] + \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m y_k}{\rho_2} \right) \right) \right] \leq 1.$$ 

Since $\rho > 0$, we have

$$g_\Delta(x + y) = \sum_{i=1}^m |x_i + y_i| + \inf \left\{ \rho \mathbb{P} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m(x_k + y_k)}{\rho} \right) \right) \right] \leq 1, \quad r = 1, 2, 3, \ldots \right\}$$

$$\leq \sum_{i=1}^m |x_i| + \inf \left\{ \rho \mathbb{P} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k}{\rho_1} \right) \right) \right] \leq 1, \quad \text{for some } \rho_1 > 0 \text{ and } r = 1, 2, 3, \ldots \right\} + \sum_{i=1}^m |y_i| + \inf \left\{ \rho \mathbb{P} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m y_k}{\rho_2} \right) \right) \right] \leq 1, \quad \text{for some } \rho_2 > 0 \text{ and } r = 1, 2, 3, \ldots \right\}$$

$$= g_\Delta(x) + g_\Delta(y),$$

i.e. $g_\Delta(x + y) \leq g_\Delta(x) + g_\Delta(y)$.

Finally, let $\lambda$ be a given non-zero scalar in $\mathbb{C}$. Then the continuity of the product follows from the following expression.

$$g_\Delta(\lambda x) = \sum_{i=1}^m |\lambda x_i| + \inf \left\{ \rho \mathbb{P} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m(\lambda x_k)}{\rho} \right) \right) \right] \leq 1, \quad \text{for some } \rho > 0 \text{ and } r = 1, 2, 3, \ldots \right\}$$

$$= \lambda \sum_{i=1}^m |x_i| + \inf \left\{ \left| \frac{\lambda}{\eta} \right| \mathbb{P} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^m x_k}{\eta} \right) \right) \right] \leq 1, \quad \text{for some } \rho > 0 \text{ and } r = 1, 2, 3, \ldots \right\}$$

where $\eta = \frac{\rho}{|\lambda|} > 0$. This completes the proof of the theorem. \qed

The proof of the following theorem is easy, so omitted.
Theorem 3.3. Let $M = (M_k)$ and $\phi = (\phi_k)$ be two Musielak-Orlicz functions and $p = (p_k) \in \ell_\infty$ of strictly positive real numbers. Then

(i) $[N_\xi, M, \Delta^m, p, q, v]_Z \subseteq [N_\xi, \phi, M, \Delta^m, p, q, v]_Z$

(ii) $[N_\xi, M, \Delta^m, p, q, v]_Z \cap [N_\xi, \phi, \Delta^m, p, q, v]_Z \subseteq [N_\xi, \phi + M, \Delta^m, p, q, v]_Z,$

where $Z = 0, 1, \infty.$

Theorem 3.4. The inclusion $[N_\xi, M, \Delta^{m-1}, q]_Z \subseteq [N_\xi, M, \Delta^m, q]_Z$ holds, for $m \geq 1.$ In general $[N_\xi, M, \Delta^i, q]_Z \subseteq [N_\xi, M, \Delta^m, q]_Z,$ for $i = 0, 1, 2, \ldots, m - 1$ and the inclusions are strict, where $Z = 0, 1, \infty.$

Proof. Let $(x_k) \in [N_\xi, M, \Delta^{m-1}, q]_0.$ Then there exists $\rho > 0$ such that

$$\lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_k}{2 \rho} \right) \right) \right] = 0.$$

Since $M$ is nondecreasing and convex, we have

$$h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}}{2 \rho} \right) \right) \right]$$

$$= h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_k}{2 \rho} \right) \right) \right]$$

$$\leq h_r^{-1} \left\{ \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_k}{2 \rho} \right) \right) \right] + \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_{k+1}}{2 \rho} \right) \right) \right] \right\}$$

$$\leq h_r^{-1} \sum_{k \in I_r} \frac{1}{2} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_k}{\rho} \right) \right) \right] + h_r^{-1} \sum_{k \in I_r} \frac{1}{2} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_{k+1}}{\rho} \right) \right) \right]$$

$$< h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_k}{\rho} \right) \right) \right] + h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m-1} x_{k+1}}{\rho} \right) \right) \right].$$

Taking limit $r \to \infty,$ we have

$$h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{\Delta^{m} x_k}{\rho} \right) \right) \right] \to 0,$$

i.e. $(x_k) \in [N_\xi, M, \Delta^m, q]_0.$ The rest of the cases can be proved in the similar way. By using induction, we have $[N_\xi, M, \Delta^i, q]_Z \subseteq [N_\xi, M, \Delta^m, q]_Z,$ for $i = 0, 1, 2, \ldots, m - 1.$

The above inclusion is strict follows from the following example.

Example 3.5. Let $M_k(x) = x^2,$ for all $x \in [0, \infty),$ $\xi = (2^r),$ for all $k \in \mathbb{N}$ and $q(x) = |x|.$ Consider a sequence $(x_k)$ defined by

$$(x_k) = (k^{m-1}, k^{m-1}, k^{m-1}, \ldots).$$

Then $\Delta^m x_k = 0,$ but $\Delta^{m-1} x_k = (-1)^{m-1}(m - 1)!,$ for all $n \in \mathbb{N}.$ Thus $(x_k) \in [N_\xi, M, \Delta^m, q]_0,$ but $(x_k) \notin [N_\xi, M, \Delta^{m-1}, q]_0.$
Theorem 3.6. Let $\xi = (k_r)$ be a lacunary sequence and let $M = (M_k)$ be a Musielak-Orlicz function. Then

(i) $[N_\xi, M, \Delta^n, p, q, v]_0 \subseteq [N_\xi, M, \Delta^n, p, q, v]_1 \subseteq [N_\xi, M, \Delta^n, p, q, v]_\infty$, and the inclusion is strict.

(ii) If $|v_k| \leq 1$, then $[N_\xi, M, \Delta^n, p, q, v]_Z \subseteq [N_\xi, M, \Delta^n, p, q, v]_Z$, for $Z = 0, 1, \infty$.

Proof. (i) The inclusion $[N_\xi, M, \Delta^n, p, q, v]_0 \subseteq [N_\xi, M, \Delta^n, p, q, v]_1$ is obvious. Let $(x_k)$ be an element of $[N_\xi, M, \Delta^n, p, q, v]_1$. Then there exists $\rho > 0$ such that

$$\lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^n x_k - L}{\rho} \right) \right) \right]^{p_k} \to 0.$$ 

Since $M_k$ is non-decreasing and convex for all $k \in \mathbb{N}$, we have

$$h_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^n x_k - L}{\rho} \right) \right) \right]^{p_k} \leq Dh_r^{-1} \sum_{k \in I_r} \left[ M_k \left( q \left( \frac{v_k \Delta^n x_k - L}{\rho} \right) \right) \right]^{p_k}$$

$$+ D \max \left[ 1, M_k \left( q \left( \frac{L}{\rho} \right) \right) \right]^H,$$

where $G = \sup_k p_k, D = \max \{1, 2^G-1\}$.

Thus the sequence $(x_k)$ belongs to $[N_\xi, M, \Delta^n, p, q, v]_\infty$.

The inclusions are strict follows from the following example.

Example 3.7. Let

$$p_k = \begin{cases} 4, & \text{if } k \text{ is even;} \\ 5, & \text{if } k \text{ is odd.} \end{cases}$$

Let $m \geq 0$ be given. Let $v_k = k, M_k(x) = x^2$, for all $k \in \mathbb{N}$ and $q(x) = |x|$. Let $\xi = (2^r)$ be a lacunary sequence. Consider a sequence $(x_k)$ defined by

$$(x_k) = (k^m, k^m, k^m, \ldots).$$

Thus the sequence $(x_k)$ belongs to $[N_\xi, M, \Delta^n, p, q, v]_1$, but $(x_k)$ does not belong to $[N_\xi, M, \Delta^n, p, q, v]_0$.

The proof of the part (ii) is easy, so omitted.

Theorem 3.8. Let $M = (M_k)$ and $\phi = (\phi_k)$ be two Musielak-Orlicz functions. If $M_k$ and $\phi_k$ are equivalent for each $k \in \mathbb{N}$ and $\xi = (k_r)$ be a lacunary sequence. Then

$$[N_\xi, M, \Delta^n, p, q, v]_Z = [N_\xi, \phi, \Delta^n, p, q, v]_Z,$$

where $Z = 0, 1, \infty$.

Proof. The proof of the theorem is easy, so omitted.

Theorem 3.9. Let $M = (M_k)$ be any Musielak-Orlicz function and let $q_1$ and $q_2$ be two semi norms. Then
(i) \([N_\xi, M, \Delta^m, p, q_1, v] \cap [N_\xi, M, \Delta^m, p, q_2, v] \subseteq [N_\xi, M, \Delta^m, p, q_1 + q_2, v] \cap Z\);

(ii) if \(q_1\) is stronger than \(q_2\), \([N_\xi, M, \Delta^m, p, q_1, v] \subseteq [N_\xi, M, \Delta^m, p, q_2, v] \cap Z\), where \(Z = 0, 1, \infty\).

\[\text{Proof.}\] The proof of the theorem is easy, so omitted. \(\square\)

We give the following two propositions without proof.

**Proposition 3.10.** Let \(\xi = (k_r)\) be a lacunary sequence. Then the followings hold:

(i) If \(\liminf_{r} \eta_r > 1\), then for any Musielak-Orlicz function \(M = (M_k)\), for all \(k \in \mathbb{N}\),

\([W, M, \Delta^m, p, q, v]_0 \subseteq [N_\xi, M, \Delta^m, p, q, v]_0\),

where

\([W, M, \Delta^m, p, q, v]_0 = \{(x_k) \in w(X) : \lim_{n \to \infty} \sum_{k=1}^{n} [M_k \left(q\left(\frac{v_k \Delta^m x_k}{\rho}\right)\right)]^{p_k} \to 0,\]

\(\text{for some } \rho > 0\}\).

(ii) If \(\limsup_{r} \eta_r < \infty\), then for any Musielak-Orlicz function \(M = (M_k)\), for all \(k \in \mathbb{N}\),

\([N_\xi, M, \Delta^m, p, q, v]_0 \subseteq [W, M, \Delta^m, p, q, v]_0\).

**Proposition 3.11.** Let \(\xi = (k_r)\) be a lacunary sequence, with \(0 < \liminf_{r} \eta_r \leq \limsup_{r} \eta_r < \infty\), then for any Musielak-Orlicz function \(M = (M_k)\), for all \(k \in \mathbb{N}\),

\([N_\xi, M, \Delta^m, p, q, v]_0 = [W, M, \Delta^m, p, q, v]_0\).

**Property 3.12.** The spaces \([N_\xi, M, p, q, v]_0\) and \([N_\xi, M, p, q, v]_\infty\) are solid as well as monotone. The spaces \([N_\xi, M, \Delta^m, p, q, v]_Z\) are not solid in general, for \(Z = 0, 1, \infty\).

\[\text{Proof.}\] Let \((x_k) \in [N_\xi, M, p, q, v]_0\). Then there exists \(\rho > 0\) such that

\[\lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q\left(\frac{v_k x_k}{\rho}\right)\right)\right]^{p_k} \to 0.\]

Let \((\alpha_k)\) be a sequence of scalars such that \(|\alpha_k| \leq 1\), for all \(k \in \mathbb{N}\). Since

\(|\alpha_k| \leq \max(1, |\alpha_k|^G) \leq 1, \text{ for all } k \in \mathbb{N}, \text{ where } G = \sup_{k} p_k < \infty.\)

Then for each \(r\), we have

\[h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q\left(\frac{v_k x_k}{\rho}\right)\right)\right]^{p_k} \leq h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q\left(\frac{v_k x_k}{\rho}\right)\right)\right]^{p_k}. \quad (3.1)\]
Therefore \((\alpha_k x_k) \in [N_\xi, M, p, q, v]_0\). Hence \([N_\xi, M, p, q, v]_0\) is solid.

By the Lemma 2.5, it follows that the space \([N_\xi, M, p, q, v]_0\) is monotone. Again by the inequality (3.1) and the Lemma 2.5, we can proved that the space \([N_\xi, M, \Delta^m, p, q, v]_\infty\) is solid as well as monotone. In order to prove that the spaces \([N_\xi, M, \Delta^m, p, q, v]_0\) and \([N_\xi, M, \Delta^m, p, q, v]_\infty\) are not solid in general, we consider the following example.

**Example 3.13.** Let \(M_k(x) = x^t\), for all \(k \in \mathbb{N}\) and \(t \geq 1\). Let \(p_k = \frac{1}{k^t}, v_k = k\), for all \(k \in \mathbb{N}\) and \(q(x) = |x|\). Let \(\xi = (2^t)\) be a lacunary sequence, for all \(k \in \mathbb{N}\).

Consider a sequence \((x_k)\) defined by
\[
x_k = k^2, \text{ for all } k \in \mathbb{N}.
\]
Then \((x_k)\) belongs to \([N_\xi, M, \Delta^m, p, q, v]_1\) and \([N_\xi, M, \Delta^m, p, q, v]_\infty\), for \(m = 1\). Let \((\alpha_k) = (-1)^k\), for all \(k \in \mathbb{N}\). Then \((\alpha_k x_k)\) does not belong to the spaces \([N_\xi, M, \Delta^m, p, q, v]_1\) and \([N_\xi, M, \Delta^m, p, q, v]_\infty\). Hence the spaces \([N_\xi, M, \Delta^m, p, q, v]_1\) and \([N_\xi, M, \Delta^m, p, q, v]_\infty\) are not solid.

Therefore by the Lemma 2.5, it follows that the spaces \([N_\xi, M, \Delta^m, p, q, v]_0\) and \([N_\xi, M, \Delta^m, p, q, v]_\infty\) are not monotone.

Next to show that the space \([N_\xi, M, \Delta^m, p, q, v]_0\) is not solid in general. We consider the following example.

**Example 3.14.** Under the restrictions on \(M, p, v, m, q\) and \(\xi\) as in Example 3.7. We consider a sequence \((x_k)\) defined by
\[
x_k = 2^k, \text{ for all } k \in \mathbb{N}.
\]
Let \((\alpha_k) = (-1)^k\), for all \(k \in \mathbb{N}\). Then \((\alpha_k x_k)\) does not belong to \([N_\xi, M, \Delta^m, p, q, v]_0\). Hence the space \([N_\xi, M, \Delta^m, p, q, v]_0\) is not solid.

Therefore by the Lemma 2.5, it follows that the space \([N_\xi, M, \Delta^m, p, q, v]_0\) is not monotone.

**Property 3.15.** The space \([N_\xi, M, p, q, v]_1\) is neither solid nor monotone.

**Proof.** The space \([N_\xi, M, p, q, v]_1\) is not monotone follows from the following example.

**Example 3.16.** Let \(p_k = 1 + \frac{1}{k^2}\) and \(v_k = k\), for all \(k \in \mathbb{N}\). Let \(M_k(x) = x^t\), for all \(k \in \mathbb{N}\) and \(t \geq 1\) and \(q(x) = |x|\). Let \(\xi = (2^t)\) be a lacunary sequence for all \(k \in \mathbb{N}\). Consider a sequence \((x_k)\) defined by
\[
(x_k) = (2, 2^2, \ldots), \text{ for all } k \in \mathbb{N}.
\]
Consider the \(K^{th}\)-step space \(E_K\) for a sequence space \(E\) and defined a sequence \((y_k)\) as follows:
\[
y_k = \begin{cases} x_k, & \text{if } k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}
\]
Then \((y_k)\) does not belong to the \(K^{th}\)-step space \(E_K\) of the sequence space \(E\). Hence the space \([N_\xi, M, p, q, v]_1\) is not monotone.
Therefore by the Lemma 2.5, it follows that the space \([N_\xi, M, p, q, v]_1\) is not solid.

**Property 3.17.** The spaces \([N_\xi, M, \Delta^m, p, q, v]_Z\) are not monotone in general, for \(Z = 0, 1, \infty\).

**Proof.** The proof of the result follows from the Examples 3.13 and 3.14, by considering the \(K^{th}\) step space \(E_K\) for a sequence space \(E\) and defined a sequence \((y_k)\) as follows:

\[
y_k = \begin{cases} 
x_k, & \text{if } k \text{ is even;} \\
0, & \text{otherwise.}
\end{cases}
\]

Then the sequence \((x_k)\) defined in the Example 3.13 belongs to \([N_\xi, M, \Delta^m, p, q, v]_Z\), but \((y_k)\) does not belong to \([N_\xi, M, \Delta^m, p, q, v]_Z\), for \(Z = 1, \infty\).

Similarly, \((x_k)\) defined in the Example 3.14 belongs to \([N_\xi, M, \Delta^m, p, q, v]_0\), but \((y_k)\) does not belong to \([N_\xi, M, \Delta^m, p, q, v]_0\). Hence the spaces \([N_\xi, M, \Delta^m, p, q, v]_Z\) are not monotone, for \(Z = 0, 1, \infty\).

**Property 3.18.** The spaces \([N_\xi, M, \Delta^m, p, q, v]_Z\) are not symmetric in general, for \(Z = 0, 1, \infty\).

**Proof.** The proof of the result follows from the following example.

**Example 3.19.** Let \(M_k(x) = x^2, p_k = k\) and \(v_k = k^2\), for all \(k \in \mathbb{N}\). and \(q(x) = |x|\). Let \(\xi = (2^r)\) be a lacunary sequence for all \(k \in \mathbb{N}\). Consider a sequence \((x_k)\) defined by

\[
x_k = k^3, \quad \text{for all } k \in \mathbb{N}.
\]

Then \((x_k)\) belongs to \([N_\xi, M, \Delta^m, p, q, v]_0\), for \(m = 1\). Consider the sequence \((y_k)\) which is the rearrangement of the sequence \((x_k)\) defined by

\[
y_k = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, \ldots).
\]

Then \((y_k)\) does not belong to \([N_\xi, M, \Delta^m, p, q, v]_Z\).

Hence the spaces \([N_\xi, M, \Delta^m, p, q, v]_Z\) are not symmetric in general.

**Property 3.20.** The space \([N_\xi, M, \Delta^m, p, q, v]_0\) is not convergence free.

**Proof.** The proof of the result follows from the following example.

**Example 3.21.** Let \(M_k(x) = x, p_k = k, v_k = k\), for all \(k \in \mathbb{N}\). and \(q(x) = |x|\). Let \(\xi = (2^r)\) be a lacunary sequence for all \(k \in \mathbb{N}\). Consider a sequence \((x_k)\) defined by

\[
x_k = \begin{cases} 
2, & \text{if } k \text{ is even;} \\
0, & \text{if } k \text{ is odd.}
\end{cases}
\]

Then \((x_k)\) belongs to \([N_\xi, M, \Delta^m, p, q, v]_0\), for \(m = 2\). Consider the sequence \((y_k)\) defined by

\[
y_k = \begin{cases} 
k^2, & \text{if } k \text{ is even;} \\
0, & \text{if } k \text{ is odd.}
\end{cases}
\]
Then \((y_k)\) does not belong to \([N_\xi, M, \Delta^m, p, q, v]_0\).

Hence the space \([N_\xi, M, \Delta^m, p, q, v]_0\) is not convergence free. □

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References


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